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The Hamilton-Jacobi equation revisited

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Abstract. A new analysis of the nature of the solutions of the Hamilton-Jacobi equation of classical dynamics is presented based on Caratheodory's theorem concerning canonical transformations. The special role of a principal set of solutions is stressed, and the existence of analogous results in quantum mechanics is outlined.

Keywords. Mechanics, classical; Hamilton-Jacobi theory.

Introduction

The equations of classical dynamics can be put into several mathematically equivalent forms, each of which has certain characteristic features associated with it. The Lagrangian form is associated with a configuration space statement of a principle of stationary action. The Hamiltonian form leads directly to a phase space description of dynamics, with the accompanying canonical transformation theory. Finally, we have the formulation *via* the Hamilton-Jacobi equation, whose importance lies in the fact that the solution of a dynamical problem involving several degrees of freedom is reduced to the determination of a single function of configuration and time.

It has been pointed out by Dirac (1951) that the Hamilton-Jacobi formulation of dynamics leads to an interesting structural element that is not at all apparent in the other two formulations. One is led to a grouping of distinct classical states of motion into families, each family corresponding to one solution of the Hamilton-Jacobi equation. The significance of this grouping lies not so much within classical dynamics as in the fact that the Hamilton-Jacobi equation is the classical analogue of the Schrödinger wave equation in quantum mechanics (Messiah 1970), making suitably constructed families of classical states of motion somehow analogous to single quantum states of motion.

Traditionally there have been two rather distinct ways of treating the Hamilton-Jacobi equation, which may for simplicity be called physical and mathematical. For a dynamical system described by $2n$ phase space variables $Q_1, Q_2, \ldots, Q_n, P_1, P_2, \ldots, P_n$, and a Hamiltonian $H(Q, P, t)$, the Hamiltonian equations are

$$\dot{Q}_r = \frac{\partial H(Q, P, t)}{\partial P_r}, \quad \dot{P}_r = -\frac{\partial H(Q, P, t)}{\partial Q_r}, \quad r = 1, 2, \ldots, n \tag{A}$$

while the Hamilton-Jacobi equation reads

$$H\left(Q, \frac{\partial S(Q, t)}{\partial Q}, t\right) + \frac{\partial S(Q, t)}{\partial t} = 0 \tag{B}$$

The latter is in general a nonlinear partial differential equation for a function S of the configuration space coordinates Q and time t. In the physical approach, one tries to find some solution to (B) which depends in an essential way on n independent constants of integration $\alpha_1, \alpha_2, \ldots, \alpha_n$ as parameters; using such a solution as a generating function, one then constructs a canonical transformation which is naturally time dependent and which takes one from the variables $Q_r(t)$, $P_r(t)$ to a time-independent canonical set α_r, β_r. (see, for instance, Goldstein 1950; Whittaker 1927 §142) (The α_r automatically have vanishing Poisson brackets with one another, the β_r are their canonical conjugates.) Such an approach to (B) is geared to the problem of producing the most general solution to Hamilton's equations (A) in an explicit form.

This physical treatment is to be contrasted with the mathematical approach to the Hamilton-Jacobi equation posed purely as a problem in the theory of partial differential equations, in fact as a boundary value problem in time. (Caratheodory 1965, Ch. 3). Such an analysis leads to the following result: Given any function $s(Q)$ on configuration space, there is a unique solution $S(Q, t)$ to (B) such that at a chosen initial time t_0 this solution reduces to the given $s(Q)$: $S(Q, t_0) = s(Q)$. One has therefore an infinity of solutions to the Hamilton-Jacobi equation, each solution uniquely determined by its value at $t = t_0$. This characterisation of the manifold of solutions is however somewhat unsatisfactory in the sense that no reference is made to the functional form of the Hamiltonian. On the other hand it is the analogue of what we know concerning the Schrödinger wave equation in quantum mechanics: here the knowledge of the wave function at one time t_0 suffices to determine it for all other times.

The aim of this paper is to present a new approach to the Hamilton-Jacobi equation, and to analyse the structure of its solutions from a novel point of view. Our main interest is in getting new insight into the manifold of solutions of (B), and not in discovering new ways of finding solutions to (B). Such an approach is intended both to reconcile and to clarify the relationship between the two traditional approaches described above. The main tool in the analysis is Caratheodory's theorem, which is an elegant characterisation of canonical transformations in classical dynamics (Caratheodory 1965, Ch. 6). What will be of primary importance is the fact that dynamical evolution *via* Hamilton's equations (A) is known in advance to be a one-parameter family of canonical transformations, the particular family depending on the particular Hamiltonian. The classification of the solutions that one gets in this approach will turn out to depend intimately on the functional form of the Hamiltonian; in this sense, therefore, it is more interesting than the result from the theory of partial differential equations quoted in the previous paragraph.

The material of this paper is arranged as follows. In section 1 we recapitulate briefly some features of the phase space description of classical dynamics, canonical transformation theory, and Caratheodory's theorem. Section 2 is devoted to the construction of certain special solutions of the Hamilton-Jacobi equation which we call a principal set of solutions; it is the enumeration and structure of these that are directly determined by the form of the Hamiltonian. Section 3 describes the construction of a much larger class of solutions starting from the principal set, and shows how the infinity of solutions promised by the mathematical approach is generated. In the concluding section, the relation of these results to analogous results in quantum mechanics is indicated. In an appendix we give an elementary but instructive demonstration of the fact that the generating functions of one-parameter families of canonical transformations, whose existence Caratheodory's theorem guarantees, can always be chosen so as to obey the Hamilton-Jacobi equation.

I. Canonical transformations and Caratheodory's theorem

For the phase space description of a classical dynamical system with $2n$ degrees of freedom, we may unite the position variables q_r and the conjugate momentum variables p_r, $r = 1, 2, \ldots, n$, into a single $2n$ component entity ω_μ: $\omega_1 = q_1$, $\omega_2 = p_1$, $\omega_3 = q_2$, $\omega_4 = p_2$, $\ldots$, $\omega_{2n-1} = q_n$, $\omega_{2n} = p_n$. The Poisson Bracket (PB) of two phase space functions $f(\omega)$ and $g(\omega)$, which is again a phase space function, can be compactly written as

$$\{f(\omega), g(\omega)\} = \epsilon_{\mu\nu} \frac{\partial f(\omega)}{\partial \omega_\mu} \frac{\partial g(\omega)}{\partial \omega_\nu} \tag{1.1}$$

with the help of the numerical antisymmetric nonsingular matrix $\|\epsilon_{\mu\nu}\|$ whose only nonvanishing elements are

$$\begin{aligned} \epsilon_{12} &= \epsilon_{34} = \ldots = \epsilon_{2n-1,\, 2n} = 1 \\ \epsilon_{21} &= \epsilon_{43} = \ldots = \epsilon_{2n,\, 2n-1} = -1 \end{aligned} \tag{1.2}$$

(unless explicitly stated otherwise, a summation from 1 to $2n$ over repeated Greek indices is understood). The inverse to the matrix $\|\epsilon_{\mu\nu}\|$ will be written $\|\epsilon^{\mu\nu}\|$.

A canonical transformation is the replacement of the phase space coordinates ω_μ by a new set ω'_μ, the latter being $2n$ independent functions of the former, such that all PB relationships are preserved. This condition can be expressed as

$$\epsilon_{\mu\nu} \frac{\partial \omega'_\rho}{\partial \omega_\mu} \frac{\partial \omega'_\sigma}{\partial \omega_\nu} = \epsilon_{\rho\sigma} \tag{1.3}$$

We are interested in the description of canonical transformations with the help of generating functions. Such descriptions are implicit rather than explicit ones in this sense: one expresses n of the old variables and n of the new ones in terms of the balance of n old and n new variables. Let $x = \{x_\mu\}$ denote a subset of n variables picked out of the complete set of $2n$ variables ω_μ in the following manner: from each canonical pair (q_r, p_r) we retain one variable and drop the other. It is clear that there are 2^n such subsets that one can form. In the set $\{x_\mu\}$ we place zeros in those positions that correspond to dropped variables. Given any one of these subsets x, the collection of variables not included in x will be written $\tilde{x} = \{\tilde{x}_\mu\}$; $\tilde{x}$ is also an allowed subset and we always have

$$x_\mu + \tilde{x}_\mu = \omega_\mu, \qquad \mu = 1, 2, \ldots, 2n \tag{1.4}$$

Let a canonical transformation $\omega \to \omega'$ be given, and choose any allowed subset x' from the new variables. The possibility of describing the transformation using a generating function is now based on Caratheodory's theorem which asserts that for each possible choice of the subset x' of ω', there exists at least one choice of a subset x of ω such that the $2n$ variables contained in x and x' together form a complete independent set of phase space variables. The PB condition (1.3) can be alternatively stated as the condition that the differential expression

$$\epsilon^{\mu\nu} (\tilde{x}'_\mu \, dx'_\nu - \tilde{x}_\mu dx_\nu)$$

be the differential of some phase space function, *i.e.*, be exact. Since x' and x together constitute a complete set of variables, this function can be explicitly expressed in terms of them as, say, $S(x'; x)$, and one then has

$$\epsilon^{\mu\nu} (\tilde{x}'_\mu dx'_\nu - \tilde{x}_\mu dx_\nu) = dS(x'; x) \tag{1.5}$$

The function $S(x'; x)$, which is determined up to an additive constant by the canonical transformation and by the choices of the two subsets x' and x, is called a generating function for the transformation; and the equations that implicitly determine the transformation follow on identifying the terms on the two sides of eq. (1.5):

$$\tilde{x}'_\mu = - \epsilon_{\mu\nu} \frac{\partial S(x'; x)}{\partial x'_\nu}, \quad \tilde{x}_\mu = \epsilon_{\mu\nu} \frac{\partial S(x'; x)}{\partial x_\nu} \tag{1.6}$$

We shall express the fact that the two subsets of variables x' and x together contain $2n$ *independent* variables by saying that they are compatible subsets. The real content of Caratheodory's theorem is therefore the statement that, given a canonical transformation $\omega \to \omega'$, for each choice of x' out of ω' there is at least one choice of x out of ω compatible with it. And associated with each such compatible pair (x', x) there is one generating function $S(x'; x)$ leading to equations of the form (1.6). Thus a canonical transformation can be described *via* generating functions in at least 2^n ways. The generating functions $S(x'; x)$ and $S'(y'; y)$ associated with two compatible sets for one and the same canonical transformation are connected essentially by a Legendre transformation:

$$S'(y'; y) = S(x'; x) + \epsilon^{\mu\nu} (x'_\mu y'_\nu - x_\mu y_\nu) \tag{1.7}$$

This equation, which will be used in the sequel, is of course valid up to the presence of additive constants.

Two canonical transformations can be multiplied, *i.e.*, applied one after the other, to yield a third such transformation. We need an equation that determines the way the corresponding generating functions combine. Let (x', x) be a compatible set of variables associated with the first canonical transformation $\omega \to \omega'$, and (x'', x') be similarly associated with the second transformation $\omega' \to \omega''$; the corresponding generating functions may be written $S_1(x'; x)$, $S_2(x''; x')$ respectively. In general, it is not true that the pair (x'', x) forms a compatible set with respect to the product transformation $\omega \to \omega''$. However, fiom Caratheodory's theorem we know that a subset y out of the ω can be found, which will be compatible with x''. By using equations such as (1.5) for $S_1(x'; x)$, $S_2(x''; x')$ and

$S_3(x''; y)$, the last of which is a generating function for the product canonical transformation $\omega \to \omega''$, and taking into account the replacement of x by y, we get the following composition rule for generating functions:

$$S_3(x''; y) = S_2(x''; x') + S_1(x'; x) + \epsilon^{\mu\nu} y_\mu x_\nu \tag{1.8}$$

This rule of combination corresponds to the multiplication of two canonical transformations to produce a third one. In interpreting it, one must realise that one is supposed to express the sets of variables x' and x, which occur explicitly on the right hand side, in terms of the complete set of variables (x'', y) before one obtains the function $S_3(x''; y)$; thus a process of substitution of variables is involved. With the help of eq. (1.7) one can generalise the combination rule (1.8) so that the subset x' of ω' is not necessarily repeated in the generating functions describing $\omega \to \omega'$ and $\omega' \to \omega''$, and similarly x'' is not repeated in the generating functions chosen for $\omega' \to \omega''$ and $\omega \to \omega''$; that is, one can easily express the most general generating function for $\omega \to \omega''$ in terms of general ones for $\omega \to \omega'$ and $\omega' \to \omega''$. However, we do not need this generalisation and so will omit it. As with eq. (1.7), eq. (1.8) is also valid up to the presence of additive constants.

To conclude this section, we would like to make a division of the set of all canonical transformations into two distinct types possessing different physical interpretations (*see* for instance, Sudarshan and Mukunda 1974). Certain canonical transformations can only be thought of as changes of coordinates in phase space; the points of phase space remain the same, but the labels attached to each point get changed both in nature and in value. We shall call such canonical transformations *passive* transformations. An elementary example is the passage from Cartesian coordinates and momentum variables for a particle to spherical polar ones. On the other hand, certain other canonical transformations arise basically from a mapping of phase space on to itself; in the transformation $\omega \to \omega'$, both ω and ω' are variables of the same nature, and we may picture ω' as being the coordinates of the point which is the image under the mapping of the point with coordinates ω. In this interpretation, the same underlying canonical coordinate system in phase space is used throughout, and ω and ω' are the coordinates of two distinct points; as ω runs over the whole of phase space, so does its image ω'. A canonical transformation susceptible to such an interpretation will be called an *active* transformation. Dynamical evolution *via* Hamilton's equations (A), expressing the canonical coordinates at one time in terms of those at another time, is an example of an active canonical transformation. Even with an active canonical transformation one may wish to undo the mapping and interpret the transformation as merely a change of coordinates in phase space; however what distinguishes an active from a passive transformation is the fact that for the latter no interpretation in terms of a phase space mapping is possible. This distinction between these two kinds of canonical transformations will be important for the analysis of the Hamilton-Jacobi equation.

Caratheodory's theorem and eqs (1.7, 1.8) for generating functions are applicable to all canonical transformations, whether active or passive. They are statements about the functional forms involved in a canonical transformation $\omega \to \omega'$, expressing ω' as functions of ω, independent of the nature of the transformation,

2. Principal solutions of the Hamilton-Jacobi equation

Let us consider a dynamical system following the Hamiltonian equations of motion (A) obtained from some definite Hamiltonian. We shall write Q_r, P_r for the phase space coordinates of the system at a general time t, and q_r, p_r for the coordinates at an initial time t_0. It is well known that the equations that express Q_r, P_r in terms of q_r, p_r (and t, t_0) are the equations of a canonical transformation; the solution to equations of motion in Hamiltonian form leads to a phase space mapping that is an active canonical transformation. Caratheodory's theorem then tells us that this transformation can be described *via* generating functions, one for each compatible pair of subsets of variables (X, x); here X is an allowed subset of the variables Q, P and x is then suitably chosen from q, p. The Hamilton-Jacobi equation (B) singles out the configuration space coordinates Q_r from out of the complete set of variables Q_r, P_r. We will therefore also make the special choice $X \equiv Q$, i.e., $X_1 = Q_1, X_3 = Q_2, X_5 = Q_3, \ldots, X_2 = X_4 = \ldots = 0$. There will then be certain subsets x, chosen out of q_r, p_r, which are each compatible with Q. A method for determining these in terms of the Hamiltonian is developed below. These subsets x may be labelled by an index α, say $x^{(\alpha)}$; there will be at least one such subset, and of course not more than 2^n. Choosing one such subset $x^{(\alpha)}$, we see that there exists a corresponding generating function $S_\alpha(Q, x^{(\alpha)}; t, t_0)$, and eqs (1.5, 1.6) take the form

$$\sum_{\epsilon=1}^{n} P_r\, \mathrm{d}Q_r - \epsilon^{\mu\nu}\, \tilde{x}_\mu{}^{(\alpha)}\, \mathrm{d}x_\nu{}^{(\alpha)} = \mathrm{d}S_\alpha(Q, x^{(\alpha)}; t, t_0);$$

$$P_r = \frac{\partial S_\alpha(Q, x^{(\alpha)}; t, t_0)}{\partial Q_r}, \quad \tilde{x}_\mu{}^{(\alpha)} = \epsilon_{\mu\nu} \frac{\partial S_\alpha(Q, x^{(\alpha)}; t, t_0)}{\partial x_\nu{}^{(\alpha)}} \tag{2.1}$$

(For the moment, the time t is treated purely as a parameter). The physical significance of the statement that $(Q, x^{(\alpha)})$ is a compatible pair, or that the $2n$ variables in Q_r and $x_\mu{}^{(\alpha)}$ are independent ones, is the following; on independently choosing any set of values for the n position coordinates Q_r at time t, and any set of values for the n quantities $x_\mu{}^{(\alpha)}$ at time t_0, one picks out an (in general) unique solution of the equations of motion obeying these boundary conditions.

Each of the generating functions $S_\alpha(Q, x^{(\alpha)}; t, t_0)$ is known up to an additive constant (constant over phase space) which can however be a function of t and t_0. A simple analysis given in the appendix shows that one can adjust this freedom so that each S_α is a solution of the Hamilton-Jacobi equation

$$H\left(Q, \frac{\partial S_\alpha}{\partial Q}, t\right) + \frac{\partial S_\alpha}{\partial t} = 0 \tag{2.2}$$

We obtain in this way a special set of solutions of the Hamilton-Jacobi equation, which we will call the *principal solutions*. The main point we wish to emphasize is that these are uniquely determined solutions, there being no ambiguities in the functional form of S_α expressed in terms of Q, $x^{(\alpha)}$, t and t_0. Further, each S_α has its own characteristic *explicit* time dependence, determined by the choice of $x^{(\alpha)}$. In particular, even if the Hamiltonian has no explicit time dependence (in which case S_α depends on $t - t_0$ alone), the dependence of S_α on t need not be linear. In eq. (2.1) the time t was treated as a parameter. We can now combine eqs (2.1, 2.2) to get a statement in which the $(2n+1)$ quantities Q_r, $x_\mu^{(\alpha)}$ and t are all capable of independent variation (t_0 continues to be a parameter):

$$\sum_{r=1}^{n} P_r \mathrm{d}Q_r - \epsilon^{\mu\nu} \tilde{x}_\mu^{(\alpha)} \, \mathrm{d}x_\nu^{(\alpha)} = H(Q, P, t)\, \mathrm{d}t + \mathrm{d}S_\alpha (Q, x^{(\alpha)}; t, t_0) \quad (2.3)$$

The variables P_r and $\tilde{x}_\mu^{(\alpha)}$ are then the dependent ones, being definite functions of Q, $x^{(\alpha)}$, t (and t_0). Equation (2.3) will be used in the next section in building up the general solution to the Hamilton-Jacobi equation.

We next develop a criterion to determine which subsets $x_{(\alpha)}$ are compatible with Q at time t. From the Hamiltonian equations of motion (A), we find that the phase space coordinates Q_r', P_r' at a time $t + \delta t$ are related to those at time t by

$$Q_r' = Q_r + \delta Q_r = Q_r + \frac{\partial H(Q, P, t)}{\partial P_r} \delta t,$$

$$P_r' = P_r + \delta P_r = P_r - \frac{\partial H(Q, P, t)}{\partial Q_r} \delta t \quad (2.4)$$

Let X_μ' be an allowed subset of the variables Q', P'. Since Q, P obviously form a complete independent set, it follows that (Q, X') form a compatible pair if and only if the Jacobian

$$\frac{\partial (Q, X')}{\partial (Q, P)}$$

(only the nonzero components of X' being retained) is non zero. This immediately simplifies to the condition that the Jacobian determinant

$$\frac{\partial (X')}{\partial (P)}$$

be non zero. It is enough to evaluate this determinant to order δt only. We will make the assumption that if for a small time interval δt, Q and X' are compatible then they remain compatible even for finite time intervals. This is a kind of stability condition on the Hamiltonian. The rule for picking up subsets $x^{(\alpha)}$ at t_0 compatible with Q at t can then be set up in this way. Construct a rectangular matrix $\| \Delta_{r\mu} \|$, $r = 1, 2, \ldots, n$, $\mu = 1, 2, \ldots, 2n$ with matrix elements

$$\Delta_{r\mu} = \delta_{2r,\,\mu} + \epsilon_{\mu\nu} \frac{\partial^2 H(q, p, t)}{\partial p_r \partial \omega_\nu} \delta t \quad (2.5)$$

Out of $\| \Delta \|$, form the 2^n square matrices with n rows and n columns by retaining one of the first two columns of Δ and dropping the other, retaining one out of the third and fourth columns and dropping the other, and so on. Corresponding to each *nonsingular* $n \times n$ matrix, $\| D \|$ say, obtained in this way, one has one subset x_μ of q, p compatible with Q at time t: included in x are those components of ω_μ $(= q_r, p_r)$ that correspond to values of μ that denote columns of $\| \Delta_{r\mu} \|$ retained in $\| D \|$. For a system with a finite number of degrees of freedom, one can introduce a natural order into the set of 2^n square matrices obtained from $\| \Delta \|$ in the manner described above; and each of the nonsingular ones, which may be labelled by an index α, leads to one set $x^{(\alpha)}$ compatible with Q, which then signals the existence of a corresponding principal solution of the Hamilton-Jacobi equation.

The qualitative behaviour of these principal solutions, as $t \to t_0$, can be easily described. Only with the choice $x^{(\alpha)} = p_r$ is the determinant $|D|$ of the form $1 + 0(\delta t)$; for every other choice of $x^{(\alpha)}$, some of the q_r will be present in $x^{(\alpha)}$ and $|D|$ will be directly proportional to δt. This just corresponds to the fact that only (Q, p) remains a complete and independent set as $t \to t_0$, and no other set $(Q, x^{(\alpha)})$ has this property. As a result, provided the stability assumption on the Hamiltonian is valid, the generating function $S(Q, p; t, t_0)$ is always a principal solution to the Hamilton-Jacobi equation, and it has a well-defined limit,

$$S(Q, p; t, t_0) \to \sum_{r=1}^{n} Q_r p_r \tag{2.6}$$

as $t \to t_0$; but every other principal solution $S_\alpha(Q, x^{(\alpha)}; t, t_0)$ is *necessarily singular* as $t \to t_0$.

In most discussions of the Hamilton-Jacobi equation, the particular solution $S(Q, q; t, t_0)$ plays a special role (Whittaker 1927, § 143). Our criterion shows that this solution exists only when the determinant of the matrix

$$\left\| \frac{\partial^2 H(q, p, t)}{\partial p_r \, \partial p_s} \right\|$$

is nonzero. This is also the case in which the Hamiltonian may be obtained from a Lagrangian starting point, without any constraints being present. In that situation, $S(Q, q; t, t_0)$ is in fact the time-integral of the Lagrangian along the trajectory of the system determined by the values of Q, q, t and t_0:

$$S(Q, q; t, t_0) = \int_{q,\, t_0}^{Q,\, t} L(q(t'), \dot{q}(t'), t')\, dt' \tag{2.7}$$

A principal solution differing from this one can then be expressed in terms of the Lagrangian by use of eq. (1.7):

$$S_\alpha(Q, x^{(\alpha)}; t, t_0) = \int_{x^{(\alpha)},\, t_0}^{Q,\, t} L(q(t'), \dot{q}(t'), t')\, dt' + \epsilon^{\mu\nu} x_\mu^{(\alpha)} q_\nu$$

$$= \int_{x^{(\alpha)},\, t_0}^{Q,\, t} L(q(t'), \dot{q}(t'), t')\, dt' + x_2^{(\alpha)} q_1 + x_4^{(\alpha)} q_2 + \ldots \tag{2.8}$$

It is understood that those q_r on the right hand side that do not occur in x^α must be expressed in terms of Q, x^α, t, t_0. While this discussion holds for a dynamical system with a Lagrangian basis, the general results concerning the principal solutions of the Hamilton-Jacobi equation are valid for any form of Hamiltonian.

To conclude this section, we shall illustrate the concepts introduced by means of two simple examples, namely, a free particle in one dimension, and a harmonic oscillator in one dimension. The former is described by the Hamiltonian $H = \frac{1}{2} p^2$; using the criterion developed above, we find that (Q, q) and (Q, p) are both compatible sets, as is physically reasonable; and the corresponding principal solutions to the Hamilton-Jacobi equation are

$$S(Q, q; t, t_0) = \frac{1}{2} \frac{(Q - q)^2}{(t - t_0)};$$

$$S'(Q, p; t, t_0) = Qp - \tfrac{1}{2} p^2 (t - t_0) \tag{2.9}$$

The behaviours as $t \to t_0$ are as expected. For the harmonic oscillator, the Hamiltonian is $H = \frac{1}{2}(q^2 + p^2)$; again, (Q, q) and (Q, p) are compatible sets; and the principal solutions are:

$$S(Q, q; t, t_0) = [(Q^2 + q^2)\cos(t - t_0) - 2Qq]/2\sin(t - t_0);$$

$$S'(Q, p; t, t_0) = \frac{Qp}{\cos(t - t_0)} - \frac{1}{2}(Q^2 + p^2)\tan(t - t_0) \tag{2.10}$$

3. General solutions of the Hamilton-Jacobi equation

For a given Hamiltonian, each principal solution of the Hamilton-Jacobi equation is one possible generating function with which to describe the active canonical transformation constituting dynamical evolution. These solutions are uniquely determined by the Hamiltonian. As stated in the introduction, however, there is actually an infinity of solutions to the Hamilton-Jacobi equation, one determined by each arbitrarily chosen boundary condition at time t_0. We shall analyse now how this infinity of solutions arises by combining the above-mentioned active canonical transformation (which, for a given Hamiltonian, is unambiguous) with an arbitrarily chosen (passive) canonical transformation $q, p \to u, v$ at time t_0. The generating functions combine according to eq. (1.8) and lead to new solutions of the Hamilton-Jacobi equation.

We shall for definiteness deal at first with the so-called complete solutions of the Hamilton-Jacobi equation (Caratheodory 1965, Ch. 3). A complete solution is one which reduces at time t_0 to a preassigned function of the n position variables Q_r and of n other parameters u_r, there being an essential dependence on the latter. Also, to start with, we consider solutions that remain finite as $t \to t_0$. The only principal solution with this property is $S(Q, p; t, t_0)$, its limiting behaviour being given by eq. (2.6); we also know that this solution exists for any choice of Hamiltonian. We therefore make use of it in constructing the complete solutions.

Let some function $s(q, u)$ of the q_r and n parameters u_r be given, obeying the condition

$$D(s) \equiv \det \left| \frac{\partial^2 s(q, u)}{\partial q_r \, \partial u_s} \right| \neq 0 \tag{3.1}$$

One can then convert the u_r into phase space functions by imposing the equations

$$p_r = \frac{\partial s(q, u)}{\partial q_r} \tag{3.2}$$

In fact, condition (3.1) is the necessary and sufficient condition under which one can turn eq. (3.2) "inside out" and express the u_r as functions of q, p. [Had the determinant $D(s)$ of (3.1) vanished, imposition of eq. (3.2) would have been inconsistent with the fact that the q_r and p_r are independent variables, since by elimination of the u's from eq. (3.2) one could have produced relations among the q's and p's by themselves.] Further, the u_r will be mutually independent functions on phase space, with their PB's with each other vanishing:

$$\{u_r, u_s\} = 0 \tag{3.3}$$

They can, in fact, be regarded as the "q's" of a new canonical coordinate system in phase space. We set up n other phase space variables v_r to be conjugate to the u's and complete this new coordinate system by the equations

$$v_r = -\frac{\partial s\,(q,\,u)}{\partial u_r} \tag{3.4}$$

Since the u_r are functions of q, p, so are the v_r. We now have, in addition to eq. (3.3), the following relationships:

$$\left.\begin{aligned} &\{v_r, v_s\} = 0, \quad \{u_r, v_s\} = \delta_{rs}; && (a) \\ &\sum_{r=1}^{n} (p_r \mathrm{d}q_r - v_r \mathrm{d}u_r) = \mathrm{d}s\,(q,\,u) && (b) \end{aligned}\right\} \tag{3.5}$$

Summarizing, we see that eqs (3.2), (3.4) lead directly to eq. (3.5 b); from eq. (3.5 b), as is well known, the PB conditions, eqs (3.3), (3.5 a) follow; and these PB conditions ensure that the u_r, and v_r, are $2n$ independent phase space functions. The passage $q, p \rightarrow u, v$ is therefore a canonical transformation which must in general be thought of as passive; it is such that (q, u) form a compatible pair, and $s\,(q, u)$ is the corresponding generating function.

We now have the two canonical transformations $q, p \rightarrow u, v$ and $q, p \rightarrow Q, P$, with the respective compatible pairs (q, u) and (p, Q); the former is passive and the latter is active and time dependent. The transformation $u, v \rightarrow Q, P$ is, therefore, also a time dependent canonical one, and from physical continuity we may assume that for each t, (Q, u) form a compatible pair. The dynamical significance of this last remark is that we may choose values for the variables Q_r, u_r, t independently (t_0 being fixed), and that then fixes a unique state of motion, *i.e.*, a solution of the Hamiltonian equations of motion. The truth of this statement can be seen by thinking through the various changes of variables, as follows. We start from the knowledge that on independently choosing values for p_r, Q_r, and t, a definite state of motion results. In particular, the values of q_r (and P_r) would be determined. Knowing both p_r and q_r (the latter in terms of p, Q, t), the values of u_r are determined. All in all, we may meaningfully write equations of the form

$$u_r = \phi_r\,(p, Q; t, t_0) \tag{3.6}$$

Now we know that for fixed t, u_r and Q_r form $2n$ independent phase space functions. It must therefore be the case that on choosing values for u_r and Q_r independently, at any t, values for p_r can be found so that eqs (3.6) are satisfied. This is the same as saying that the functions ϕ_r appearing in eq. (3.6) permit us to solve these equations for the p_r in terms of u, Q, t:

$$p_r = \psi_r\,(Q, u; t, t_0). \tag{3.7}$$

It is now clear that the values of Q, u and t do determine a state of motion uniquely. One can then think of the q_r also as functions of Q, u and t:

$$q_r = \chi_r\,(Q, u;\; t, t_0) \tag{3.8}$$

The χ's will then reduce at $t = t_0$ to the Q's:

$$\chi_r\,(Q, u;\; t_0, t_0) = Q_r \tag{3.9}$$

We will use this in the sequel.

As a particular case of eq. (2.3) we have

$$\sum_{r=1}^{n} (P_r \mathrm{d}Q_r + q_r \,\mathrm{d}p_r) = H(Q, P, t)\,dt + \mathrm{d}S(Q, p;\ t, t_0) \tag{3.10}$$

with Q, p and t independent. By combining this with eq. (3.5 *b*) we can get a generating function to describe the time-dependent canonical transformation $u, v \to Q, P$:

$$\sum_{r=1}^{n} (P_r \,\mathrm{d}Q_r - v_r \mathrm{d}u_r) = H(Q, P, t)\,\mathrm{d}t + \mathrm{d}\left[S(Q, p;\ t, t_0) + s(q, u) - \sum_{r=1}^{n} q_r p_r\right] \tag{3.11}$$

Since by eqs (3.7), (3.8), both q and p are expressible in terms of Q, u, t we see that the generating function of type (Q, u) for the transformation $u, v \to Q, P$ is given by

$$S'(Q, u;\ t, t_0) = S(Q, p;\ t, t_0) + s(q, u) - \sum_{r=1}^{n} q_r p_r \tag{3.12}$$

it being understood that on the right hand side one substitutes for q and p from eqs (3.7), (3.8). Equation (3.12) is a particular case of the composition law, eq. (1.8). This function S' is, as is clear from eq. (3.11), a solution to the Hamilton-Jacobi equation,

$$H\left(Q, \frac{\partial S'}{\partial Q}, t\right) + \frac{\partial S'}{\partial t} = 0 \tag{3.13}$$

and it reduces at time $t = t_0$ to the given s; use of eqs (2.6), (3.8), (3.9) immediately shows that

$$S'(Q, u;\ t_0, t_0) = s(Q, u) \tag{3.14}$$

Finally, the uniqueness of the solution S' given the boundary value s, follows from the fact that for a given canonical transformation (here $u, v \to Q, P$) and compatible pair [here (Q, u)] the generating function is unique up to an additive constant.

We see in this way how complete solutions to the Hamilton-Jacobi equation, obeying arbitrary initial conditions, are synthesized. The really important point in the construction of S' from S using eq. (3.12) is this: while S by itself is a solution, the process of substitution for q and p in terms of Q, u, t, though it introduces fresh dependences on Q and t in each term on the right hand side of eq. (3.12), does not prevent S' from also being a solution. Direct demonstration of this fact would have been somewhat involved; our derivation makes it essentially automatic, though no less important.

The above construction of an infinity of solutions of the Hamilton-Jacobi equation can be somewhat generalised by making use of a principal solution other than the particular one $S(Q, p;\ t, t_0)$. However, these new solutions will generally be singular as $t \to t_0$. We start with a principal solution $S(Q, x;\ t, t_0)$, say, of the Hamilton-Jacobi equation, omitting for simplicity the label α on x and S. Next we consider an arbitrary (passive) canonical transformation $q, p \to u, v$ for which (q, u) need not be a compatible pair. To describe this passive transformation, let the subset x of q, p (the one appearing in the principal solution) be compatible with a subset ξ' of u, v, the associated generating function being $s(x;\xi')$:

$$\epsilon^{\mu\nu}(\tilde{x}_\mu \mathrm{d}x_\nu - \tilde{\xi}_\mu{}' \mathrm{d}\xi_\nu{}') = \mathrm{d}s(x;\xi') \tag{3.15}$$

Finally, with respect to the (time-dependent) canonical transformation $u, v \to Q, P$, let the subset Q be compatible with the subset ξ of u, v. On working through the relationships among these variables, one can convince oneself that a choice of values for Q, ξ and t determines a state of motion uniquely; and the sets of variables x_μ, $\xi_\mu{}'$, for instance, can be expressed as functions of Q, ξ and t. Combining eqs (2.3) and (3.15), and allowing for the replacement of ξ' by ξ, one gets in a straightforward manner the result

$$\sum_{r=1}^{n} P_r \mathrm{d}Q_r - \epsilon^{\mu\nu}\tilde{\xi}_\mu \mathrm{d}\xi_\nu = H(Q, P, t)\,\mathrm{d}t$$
$$+ \mathrm{d}\,[S(Q, x; t, t_0) + s(x, \xi') - \epsilon^{\mu\nu}\xi_\mu{}'\xi_\nu] \tag{3.16}$$

We have thus produced a new solution $S'(Q, \xi; t, t_0)$ to the Hamilton-Jacobi equation: it is given as

$$S'(Q, \xi; t, t_0) = S(Q, x; t, t_0) + s(x, \xi') - \epsilon^{\mu\nu}\xi_\mu{}'\xi_\nu \tag{3.17}$$

with the understanding that on the right hand side one substitutes for x and ξ' in terms of Q, ξ and t. Once again, the really important point to appreciate is that this process of substitution of variables does not violate the Hamilton-Jacobi equation. (Of course, in the new solution one identifies P with $\partial S'/\partial Q$ while in the old solution one had $P = \partial S/\partial Q$, but this is automatic.)

The behaviour of $S'(Q, \xi; t, t_0)$ as $t \to t_0$ is qualitatively determined as follows: if the subsets q of q, p and ξ of u, v are compatible, the limiting form of $S'(Q, \xi, t, t_0)$ will be finite, otherwise it will be singular. Only the former kind of solution is already covered by our earlier discussion of complete solutions of the Hamilton-Jacobi equation with nonsingular behaviour at $t = t_0$. Naturally, in the nonsingular case, the form of S' at $t = t_0$ must be determined entirely by the passive transformation $q, p \to u, v$; we easily deduce:

$$S'(Q, \xi; t, t_0) \xrightarrow[t \to t_0]{} [s(x, \xi') - \epsilon^{\mu\nu}(\xi_\mu{}'\xi_\nu + q_\mu x_\nu)]_{q \to Q} \tag{3.18}$$

The meaning of the right hand side is this: we first use the equations of the transformation $q, p \to u, v$ to express x_μ and $\xi_\mu{}'$ as functions of q and ξ inside the square-bracketed expression, and then write Q for q everywhere. [The symbol q_μ of course stands for $(q_1, 0, q_2, 0, \ldots)$].

To conclude this section, we discuss briefly solutions of the Hamilton-Jacobi equation which are not as general as the complete solutions. Each such solution reduces at $t = t_0$ to a corresponding preassigned function $s(Q)$ of the position variables, but $s(Q)$ need not involve any parameters. (In fact, it is such solutions that were referred to in the introduction when discussing the mathematical approach to the Hamilton-Jacobi equation). There is then no possibility of first performing a passive canonical transformation at time t_0 with the help of $s(q)$, and then following the steps outlined earlier to obtain a complete solution. But one can prove the following result (details of the proof are omitted): First replace $s(q)$ by a function $\bar{s}(q, u)$ as follows:

$$\bar{s}(q, u) = s(q) + f(q, u) \tag{3.19}$$

$f(q, u)$ can be any function of q and n parameters u_r subject to only two conditions: (i) it should vanish at $u = 0$; (ii) $\bar{s}(q, u)$ (in fact $f(q, u)$) should obey the condition (3.1). With $s(q, u)$, a complete solution $\bar{S}'(Q, u; t, t_0)$ to the Hamilton-Jacobi equation can be constructed, unique for a given f. Setting $u = 0$ in this complete solution, we get a solution $\bar{S}'(Q, 0; t, t_0) = S'(Q; t, t_0)$ to the Hamilton-Jacobi equation, obeying the boundary condition

$$S'(Q; t_0, t_0) = s(Q), \tag{3.20}$$

and, most important, independent of the way s is extended to $\bar{s}$, *i.e.*, independent of the choice of f. The uniqueness of $S'(Q; t, t_0)$ for given $s(Q)$ follows. This $S'(Q; t, t_0)$ can also be directly described in a manner similar to eq. (3.12), without going through the intermediary function $\bar{s}(q, u)$. A given initial phase q_r, p_r at time t_0 determines one state of motion, hence one set of values for Q_r at time t. Initial phases of the particular form $(q_r, \partial s(q)/\partial q_r)$ lead to a special family of states of motion, one element in the family for each choice of q_r (see introduction and Dirac 1951). In general, only one member of the family will reproduce a desired set of values for Q_r at a desired time t. In this way, elements of the family can be labelled by, and the initial phase variables $(q_r, p_r) \equiv (q_r, \partial s(q)/\partial q_r)$ become functions of, Q_r and t. We then find that

$$S'(Q; t, t_0) = S(Q, p; t, t_0) + s(q) - \sum_{r=1}^{n} q_r p_r, \tag{3.21}$$

with the understanding that on the right hand side we substitute the above-derived dependences on Q_r and t for q and p. Such substitution again does not violates the Hamilton-Jacobi equation.

4. Concluding remarks

We have presented a new analysis of the Hamilton-Jacobi equation and its solutions, based on the theory of canonical transformations in classical dynamics. The structures of both the so-called complete solutions and less general particular solutions have been clarified. While we have not been concerned with finding new methods for solving this important equation, we have made clear what kind of solutions do in principle exist and what relationships different solutions bear to one another. It is true that the Hamilton-Jacobi equation always possesses infinitely many solutions, whatever the Hamiltonian. But at the core of this infinity is a " small " number of basic or principal solutions (in fact a finite number if the number of degrees of freedom is finite) which are completely determined by the Hamiltonian and which really characterise the particular dynamical system. By combining one of these principal solutions with an arbitrary canonical transformation on phase space, having nothing to do with the Hamiltonian, an infinity of solutions gets built up. The principle underlying this construction is the composition law for generating functions of canonical transformations, which involves the process of substitution of variables. The fact that this process does not violate the Hamilton-Jacobi equation is the most important element of our structure analysis.

In setting up the principal solutions to the Hamilton-Jacobi equation, we have limited ourselves to the descriptions of a canonical transformation by generating functions corresponding to compatible sets of observables. Existence of such sets is guaranteed by Caratheodory's theorem. The question arises: given a canonical transformation. $q, p \to Q, P$, suppose we pick the subset Q and then a subset x of q, p not compatible with Q *i.e.*, such that the $2n$ variables in x and Q are not an inaependent set; can the transformation still be described *via* a generating function depending on Q and x? The answer is that this can be done; however the generating function will depend on some additional variables, λ say, apart from Q and x; these extra variables are Lagrange multipliers, there being one of them for each independent relation existing among the Q and x (Whittaker 1937, §126). The equations for the canonical transformation do finally determine each λ also as a phase space function, such that in the totality of variables Q, x, λ there are precisely $2n$ independent ones. We are therefore led to consider new principal solutions $S(Q, x; \lambda; t, t_0)$ to the Hamilton-Jacobi equation, where Q and x are not compatible, and λ are new (time-independent) variables. However, we do not pursue the analysis of these solutions any further because, unlike the principal solutions we defined in section 2, these new ones have some ambiguity in functional form: this is caused by the freedom we have to state functional relations among Q and x in different ways. But one must not leave these solutions completely out of consideration, if only for the reason that they do possess quantum analogues (see below).

All the results of this paper have interesting analogues in quantum mechanics. Elsewhere we have proved an operator form of Caratheodory's theorem, applicable to unitary transformations in quantum mechanics (Mukunda 1974). If q_r, p_r, $r = 1, 2, \ldots, n$, are an irreducible set of hermitian operators obeying the usual basic Heisenberg-Dirac commutation relations and U is a unitary operator a new irreducible hermitain solution to the commutation relations is given by

$$Q_r = Uq_r U^{-1}, \qquad P_r = Up_r U^{-1}, \qquad r = 1, 2, \ldots, n \tag{4.1}$$

It then happens that to each complete commuting set X chosen from Q, P there corresponds at least one complete commuting set x from q, p such that X and x are together irreducible; we shall again say that such pairs (X, x) are compatible. (This word is not used here in the sense of quantum measurement theory). Choosing U to be the unitary operator of time development from t_0 to t for a given Hamiltonian operator $H(q, p, t)$, and setting $X = Q$, to each complete commuting set x from q, p compatible with $Q(t)$ there corresponds one basic solution to the Schrödinger wave equation. Following the notation for Feynman kernels (Feynman and Hibbs 1965), we write these soiutions as

$$K(Q', t; x', t_0) = \langle Q' t \mid x' t_0 \rangle \tag{4.2}$$

what appears here is the scalar product of an eigenvector of $Q(t)$ with eigenvalue Q' and an eigenvector of $x(t_0)$ with eigenvalue x'. Here the eigenvectors depend on time. The allowed complete commuting sets $x(t_0)$ and the corresponding kernels, are in principle determined by the Hamiltonian; it is justified to call these a principal set of kernels. The characteristic property of the kernel in eq. (4.2) following from the compatibility of $Q(t)$ and $x(t_0)$ is that it is nonzero for all values of Q' and x'. While the number of principal kernels is finite (for finite n), we know that the Schrodinger equation has infinitely many solutions, each determined by its boundary value at $t = t_0$. A particular time-dependent wave function $\psi(Q', t)$ describing a particular state of motion in quantum mechanics is obtained from its form *at* $t = t_0$ by means of

$$\psi(Q', t) = \int dx' \, K(Q', t; \, x', t_0) \, \tilde{\psi}(x', t_0) \tag{4.3}$$

On the right appears the wave function or representative of the state vector at time t_0 in the basis determined by $x(t_0)$. Equation (4.3) is analogous to eq. (3.21) in the classical case; one can say that the process of integrating over x' in eq. (4.3) which is the same as summing over a complete set of states, is the quantum analogue of the process of substitution of variables implicit in equations such as (3.21) (Dirac 1958). Just as the latter does not violate the Hamilton-Jacobi equation, the sum-over-states does not violate the Schrödinger equation because that equation is linear. However, in expressing this analogy in terms of states of motion, one must realise that whereas eq. (4.3) refers to one quantum state of motion, its analogue in eq. (3.21) refers not to one but a family of classical states of motion constructed in the special way indicated just before eq. (3.21).

In the context of quantum mechanics it is clear that all knowledge about the development in time of the system is contained in any kernel $K(Q', t; x', t_0)$ *even if* $Q(t)$ *and* $x(t_0)$ *are not compatible.* With the help of such a solution to the Schrödinger equation one can still follow the time development of any state, and eq. (4.3) remains valid. The property that $Q(t)$ and $x(t_0)$ do not form an irreducible set has the following consequences: (i) $K(Q', t; \, x', t_0)$ will be proportional to a product of delta functions whose arguments involve Q' and x'; (ii) if one insists on giving a generating-function like description of the unitary time evolution operator using the operators $Q(t)$ and $x(t_0)$, one is forced to introduce additional operators λ which are quantum analogues of classical Lagrange multipliers! (Dirac 1945). Such kernels are thus somewhat like the solutions $S(Q, x; \lambda; t, t_0)$ to the Hamilton-Jacobi equation that fall outside the principal set, though in the kernels $K(Q', t; \, x', t_0)$ there do not appear any variables other than those indicated.

A particular kernel $K(Q', t; \, x', t_0)$ may be based on a set of operators $Q(t)$, $x(t_0)$ which are compatible and irreducible for $t \neq t_0$, so that for unequal times K is non-singular and as stated earlier nonvanishing for all Q' and x'. However, if $Q(t_0)$ and $x(t_0)$ possess operators in common, which happens if $x(t_0)$ is not the set $p_r(t_0)$, then as $t \to t_0$ the kernel must become proportional to delta functions in the eigenvalue differences of these common operators. Such singular limiting behaviour is the exact parallel of the fact that a principal solution $S(Q, x; \, t, t_0)$ to the Hamilton-Jacobi equation must necessarily be singular as $t \to t_0$ if $x \neq p_r$. In fact, for simple linear systems, the Feynman kernel $K(Q', t; \, q', t_0)$ is very closely related to the exponential of the solution $S(Q, q; \, t, t_0)$ to the Hamilton-Jacobi equation; this happens for the free particle and the harmonic oscillator. (Feynman and Hibbs 1965 §3.5). And the singular behaviour of $S(Q, q; t, t_0)$ as $t \to t_0$ in eqs (2.9), (2.10) is indeed directly responsible for $K(Q', t; q', t_0)$ reducing to $\delta(Q' - q')$ as $t \to t_0$.

For simplicity, we have used the language of point mechanics throughout our discussion. However, formally there is no difficulty in extending our results to relativistic mechanics on the one hand, and to systems involving fields on the other. Equally interesting would be the extension of our considerations to constrained dynamical systems (Dirac 1950). We hope to examine these questions elsewhere.

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Appendix

We give here an elementary proof of the fact that the generating functions $S_a(Q, x^{(a)}; t, t_0)$, which were introduced in eq. (2.1) on the basis of Caratheodory's theorem can always be chosen so as to obey the Hamilton-Jacobi equation (2.2). We consider for definiteness the particular generating function $S(Q, p; t, t_0)$ corresponding to the choice $x = p$, since this exists for any Hamiltonian; other choices of x can be handled similarly. The following notation is useful: given any phase space function $f(Q, P, t)$, we shall denote by $(f(Q, P, t))_S$ the function on configuration space obtained by making the substitution

$$P_r \to \frac{\partial S(Q, p;\ t, t_0)}{\partial Q_r}$$

in f, *i.e.*,

$$(f(Q, P, t))_S = f\left(Q, \frac{\partial S}{\partial Q}, t\right) \tag{A.1}$$

Of course, both f and $(f)_S$ can have explicit time dependences, and the latter will also depend on those variables other than Q_r that are present in S, in the present instance p_r and t_0.

Equation (2.1) reads, in the present case,

$$\sum_{r=1}^{n} (P_r \mathrm{d}Q_r + q_r \mathrm{d}p_r) = \mathrm{d}S(Q, p;\ t, t_0). \tag{A.2}$$

Keeping t_0 fixed throughout, a choice of values for p_r, Q_r and t determines one definite state of motion, with the momenta at the time t having the values $\partial S(Q, p;\ t, t_0)/\partial Q_r$. A slightly altered state of motion is produced by considering p_r Q_r and $t + \Delta t$, where Δt is small; the coordinate values Q_r which were attained at time t in the previous state of motion, are attained at time $t + \Delta t$ in the altered state. The phase at time $t + \Delta t$ in the altered state is then given by the coordinate and momentum values Q_r, $\partial S(Q, p;\ t + \Delta t, t_0)/\partial Q_r$; by Hamilton's equations of motion, the phase at time t must then be

$$Q_r' = Q_r - \left(\frac{\partial H(Q, P, t)}{\partial P_r}\right)_S \Delta t,$$

$$P_r' = \frac{\partial S(Q, p;\, t + \Delta t, t_0)}{\partial Q_r} + \left(\frac{\partial H(Q, P, t)}{\partial Q_r}\right)_S \Delta t \tag{A.3}$$

But eq. (A.2) applied to the altered state implies

$$P_r' = \frac{(\partial S(Q', p;\, t, t_0)}{\partial Q_r'}\;; \tag{A.4}$$

combining eqs (A.3), (A.4), we see that we must have:

$$\frac{\partial S(Q', p;\, t, t_0)}{\partial Q_r'} = \frac{\partial S(Q, p;\, t + \Delta t, t_0)}{\partial Q_r} + \left(\frac{\partial H(Q, P, t)}{\partial Q_r}\right)_S \Delta t,$$

i.e.,

$$\sum_{S=1}^{n} \frac{\partial^2 S(Q, p;\, t, t_0)}{\partial Q_r \partial Q_s}(Q_S - Q_S') + \frac{\partial^2 S(Q, p;\, t, t_0)}{\partial Q_r \partial t}\Delta t$$

$$+ \left(\frac{\partial H(Q, P, t)}{\partial Q_r}\right)_S \Delta t = 0,$$

i.e.,

$$\left(\frac{\partial H(Q, P, t)}{\partial Q_r}\right)_S + \sum_{s=1}^{n}\left(\frac{\partial H(Q, P, t)}{\partial P_S}\right)_S \frac{\partial^2 S(Q, p;\, t, t_0)}{\partial Q_r\, \partial Q_S}$$

$$+ \frac{\partial^2 S(Q, p;\, t, t_0)}{\partial Q_r \partial t} = 0 \tag{A.5}$$

i.e.,

$$\frac{\partial}{\partial Q_r}\left[H\left(Q, \frac{\partial S}{\partial Q}, t\right) + \frac{\partial S}{\partial t}\right] = 0$$

Similarly, by imposing the condition that

$$(p_r, Q_r, t) \text{ and } (p_r, Q_r + (\partial H(Q, P, t)/\partial P_r)_S \Delta t, t + \Delta t)$$

determine the same state of motion so that there should be no net change in the initial position q_r, we obtain:

$$\frac{\partial^2 S(Q, p;\, t, t_0)}{\partial p_r \partial t}\Delta t + \frac{\partial^2 S(Q, p;\, t, t_0)}{\partial p_r \partial Q_S}\left(\frac{\partial H(Q, P, t)}{\partial P_S}\right)_S \Delta t = 0$$

i.e.,

$$\frac{\partial}{\partial p_r}\left[H\left(Q, \frac{\partial S}{\partial Q}, t\right) + \frac{\partial S}{\partial t}\right] = 0 \tag{A.6}$$

The last lines of eqs (A.5), (A.6) imply that the square-bracketed expression is constant over phase space and can only be a function of t and t_0. Since in any case eq. (A.2) leaves $S(Q, p;\, t, t_0)$ arbitrary up to an additive function of t and t_0, we see that the generating function S can be so chosen that it will obey the Hamilton-Jacobi equation (2.2). Once this is done, the only arbitrariness in S is that a time-independent constant may be added to it.

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