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Stability of Nonparallel Flows: 'Minimal Composite' Theories

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Abstract

The theory of linear stability of shear flows has been studied extensively over much of the last century. Most studies have been based on the Orr-Sommerfeld equation for parallel flows, but in recent decades there have been several attempts at more general theories, including the use of parabolized stability equations. As shear flows tend in general to be nonparallel, the question has remained about the formulation of a proper theory accounting for flow nonparallelism. Introducing the concept of minimal composite equations, with the use of similarity coordinates, it has been possible during the last ten years to develop a hierarchy of stability equations ranging from an ordinary differential equation like the Orr-Sommerfeld (but not identical to it) to partial differential equations like the PSE. The approach through minimal composite equations has now been extended to include effects of wing sweep and compressibility, and we present a review of these developments and their implications.

Introduction

Tollmien [25] achieved his famous demonstration of boundary-layer instability in 1929 through approximate solutions of the Orr-Sommerfeld equation, which is strictly valid only for parallel flow, such as that in a plane channel. The assumption that the Orr-Sommerfeld (O-S) is adequate for spatially evolving flows like the boundary layer has long been questioned. Beginning in the 1970s, a series of non-parallel flow theories emerged [2, 3, 8, 22]. Among these, the results provided by Gaster [8] have stood the test of time, and have become a widely accepted standard of comparison. Today the most frequently used approach is probably that of Bertolotti *et al.* [2], who formulate a parabolized stability equation (PSE) to handle the problem. This is a partial differential equation, which can be solved numerically without much difficulty on a computer of modest power.

One important question about the PSE approach concerns the various terms with a factor R^{-1} that appear therein, apart from those already present in O-S. (Here R is a local Reynolds number, based on say the thickness of the boundary layer and the velocity at its edge.) Now if all R^{-1} terms are included in the stability analysis, one must necessarily take account of higher order (i.e. R^{-1}) effects on the mean flow as well, as these will make contributions comparable to those of some of the retained terms. It cannot however be argued that non-parallel flow effects can be consistently included only if the mean flow is obtained from higher order boundary layer theory (of the kind that Van Dyke [26] has described). This consideration suggests that it must be possible to formulate equations for non-parallel flow stability to lower orders than PSE.

We review here the basic ideas governing the present approach ([9, 10, 11, 20], the first three of which are referred to respectively as GN95, 97, 99) in formulating such lower order theories, and summarize the results obtained in a variety of problems.

Minimal Composite Equations

The material here is condensed from Narasimha and Govindarajan [20], which may be consulted for a more detailed account.

The approach we have pursued recognizes the fact that the equation formulated will eventually be solved numerically on a computer, but uses the idea of matched asymptotic expansions [27] in a rather unusual way. In classical applications of this method the objective is to find uniformly valid solutions to a given 'primitive' equation in the limit as some small parameter (say ε) in the problem goes to zero. This is accomplished by identifying the distinguished limits to the primitive equation (Lagerstrom and Casten [17]), solving the simpler equations that arise in each such limit, and matching neighbouring solutions in an asymptotic sense. From such matched solutions a uniformly valid composite solution can be constructed by methods that have been widely described by Van Dyke [27], Kevorkian and Cole [16] and others.

This approach is of value in stability theory as well. However, our present objective is different, because we propose to solve the equations numerically: writing down uniformly valid solutions is feasible but involves (in this view) unnecessary and unrewarding trouble. We do not therefore attempt to construct here uniformly valid solutions to the equation, but rather to derive an asymptotically consistent equation that, to an appropriate order, contains just those terms necessary to obtain uniformly valid solutions. Thus, once the necessary distinguished limits are identified, it must first be ensured that solutions of neighbouring limit equations can match each other in the asymptotic sense. Then a minimal composite *equation* is constructed; this equation includes *all* – and *only* – those terms that are necessary to ensure the existence of matched uniformly valid solutions to some prescribed order. The minimal composite equation so constructed may be seen as a consistent, lower order, reduced primitive; it includes every term that is important *somewhere*, and none that is important *nowhere* – 'important' to some prescribed order in the distinguished limits.

We shall find it convenient to speak of an equation as 'nominally' valid to some order in the small parameter R^{-1}, meaning thereby that the equation contains all the terms required to construct the relevant distinguished limits necessary to obtain uniformly valid solutions to that or to lower orders. Our objective then is to find the *minimal* composite equation that includes just those terms that are necessary to obtain uniformly valid solutions to the problem by numerical methods.

This approach leads to a systematic and 'rational' way of deriving new low-order equations that are simpler than those in current use seeking to include the effects of non-parallelism in the flow.

2D boundary layers

Formulation

In the classical linear stability analysis of the flow over a flat plate (described for example in Drazin and Reid [5]), the disturbance stream function is broken up into normal modes of the form

$$\hat{\phi}(x,y) = \phi(y)e^{i(\alpha x-\omega t)}$$

where α and ω are the wave number and frequency of the disturbance respectively, x is downstream distance and t is time. Only two-dimensional disturbances need be considered since, for a two-dimensional mean flow, they become unstable at a lower Reynolds number than three-dimensional disturbances (by Squire's theorem [5]). Since the disturbances are assumed small, their products may be neglected. If it is further assumed that the boundary layer is locally parallel, i.e. does not vary with x (so $\partial/\partial x = 0$ and the normal velocity is zero), ϕ satisfies the Orr-Sommerfeld equation

$$\{\mathrm{OS}\}\phi \equiv \left[i\left(\omega - \alpha\Phi'\right)\left(D^2-\alpha^2\right) + i\alpha\Phi''' + \frac{1}{R}\left(D^4 - 2\alpha^2D^2 + \alpha^4\right)\right]\phi = 0$$

which defines the Orr-Sommerfeld operator $\{\mathrm{OS}\}$. This equation has been nondimensionalised using the free stream velocity U and a boundary-layer thickness (in the sequel the momentum thickness θ) as scales; R is the Reynolds number based on the same scales,

$$D^k \equiv \frac{d^k}{dy^k}, \quad k = 1,2,\ldots,$$

and primes on the mean stream function $\Phi(y)$ denote differentiation with respect to y. In spatial stability analysis, $\alpha = \alpha_r + i\alpha_i$ is taken to be complex and ω to be real, the boundary layer being unstable to a given disturbance if α_i is negative.

To see how the assumption of parallelism in the Orr-Sommerfeld equation may be relaxed, we return to the incompressible Navier-Stokes equations in two-dimensional flow, which may be written in terms of the stream function ψ_d as

$$\frac{\partial}{\partial t_d}\nabla_d^2\psi_d + \frac{\partial\psi_d}{\partial y_d}\frac{\partial}{\partial x_d}\nabla_d^2\psi_d - \frac{\partial\psi_d}{\partial x_d}\frac{\partial}{\partial y_d}\nabla_d^2\psi_d - \nu\nabla_d^4\psi_d = 0 \tag{1}$$

where the subscript d indicates a dimensional quantity. The stream function may be expressed as the sum of a steady mean and a time-dependent perturbation,

$$\psi_d(x_d,y_t,t) = \Phi_d(x_d,y_d) + \hat{\phi}_d(x_d,y_d,t_d).$$

First the following nondimensionalization is used (GN95):

$$\psi_d = U(x_d)\theta(x_d)\psi, \quad dx = \frac{dx_d}{\theta(x_d)}, \quad y = \frac{y_d}{\theta(x_d)}, t = \frac{Ut_d}{\theta_d},$$
$$\alpha = \alpha_d\theta, \quad \omega = \frac{\omega_d\theta}{U};$$

we then have

$$\psi = \Phi(x,y) + \phi(x,y)\exp\left(i\left[\int\alpha(x)dx - \omega t\right]\right). \tag{2}$$

Note incidentally that $\omega t \equiv \omega_d t_d$. Further, as θ is permitted to be a function of x, the variable y, here and in all subsequent equations, is proportional to what is usually written as η in similarity solutions of the boundary layer equations. For a Falkner-Skan profile, $U \propto x_d^m$ where m is a constant. Therefore

$$x = \frac{2x_d}{(1+m)\theta} = \frac{R}{p}, \quad \frac{d\theta}{dx_d} = \frac{q}{R}, \quad \frac{d(U\theta)}{dx_d} = \frac{Up}{R}, \tag{3}$$

where p and q are constants given by

$$p = \theta^{*^2}, q = \theta^{*^2}\frac{(1-m)}{(1+m)}, \theta^* \equiv \sqrt{\frac{(1+m)U}{2\nu x_d}}\int_0^\infty \Phi'(1-\Phi')dy_d.$$

We note that $d\theta/dx_d = O\left(R^{-1}\right)$, and assume that α and ϕ cannot vary faster (in x) than does θ, i.e. that their first derivatives with respect to x_d are at most of order R^{-1}, and that their second derivatives are $o(R^{-1})$ and can therefore be neglected in comparison.

Furthermore, while α is permitted to vary with x, the disturbance field at any station is assumed to vary harmonically in time with the same frequency ω_d. This makes it feasible to handle experiments where a wave-maker imposes a disturbance of given frequency on the flow. (A more general disturbance can always be handled through a suitable Fourier decomposition.) However, as $\omega_d t_d = \omega t$, constant ω_d, as in wave-maker experiments, does not correspond to constant ω. When one is interested in following the downstream evolution of a disturbance of given frequency, it can be done in one of several ways. If a stability loop is presented in the (ω,R) plane, constant ω_d will correspond to a suitably curved trajectory in the (ω,R) plane, and it is along such a trajectory that the disturbance would have to be tracked. An alternative procedure, followed in GN95, is to present the stability loop in terms of a transformed non-dimensional frequency variable F that is directly proportional to ω_d at all R (i.e. x), so that a straight line in the (F,R) plane can represent the wave-maker experiment and a constant ω_d trajectory.

One now substitutes (2) in (1) and expands Φ as

$$\Phi(x,y) = \Phi_0(y) + \frac{1}{R}\Phi_1(x,y) + \cdots,$$

where Φ_0 represents the classical 'Prandtl' solution and Φ_1 comes from higher-order boundary layer theory [26]. It will then be seen that the lowest order mean flow is given by

$$\Phi_0^{\mathrm{iv}} + p\Phi_0\Phi_0''' + (2q-p)\Phi_0'\Phi_0'' = 0, \tag{4}$$

which is the classical Falkner-Skan similarity equation differentiated once with respect to y. Unlike in the traditional Orr-Sommerfeld approach, the correct mean flow equation emerges naturally here. The disturbance stream function is given by

$$\{\mathrm{NP}\}\phi = 0,$$

where the operator, including all terms nominally of $O\left(R^{-1}\right)$, is (from GN95)

$$\{\mathrm{NP}\} \equiv \Bigg\{ i(\omega-\alpha\Phi_0')[D^2-\alpha^2] + i\alpha\Phi_0''' + \frac{1}{R}\Bigg(D^4 - 2\alpha^2D^2 + \alpha^4 + \Big\{p\Phi_0D^3 + \left[(2q-p)\Phi_0'\right]D^2 + \left[2yq\alpha(\omega-\alpha\Phi_0') - p\alpha^2\Phi_0 + (2q-p)\Phi_0''\right]D + \left[(q-2p)\alpha\omega + p\Phi_0''' + 3(p-q)\alpha^2\Phi_0'\right]$$

$$+\left(-\omega+3\alpha\Phi_0'\right)R\alpha'+\left[\Phi_0'''+3\alpha^2\Phi_0'-2\alpha\omega-\Phi_0'D^2\right]$$
$$\times R\frac{\partial}{\partial x}\Bigg)\Bigg\}+\frac{1}{R}[-i\alpha\Phi_1'(D^2-\alpha^2)+\Phi_1'''].\qquad(5)$$

Terms of $O(y/R^2)$ have been neglected here. The boundary conditions are

$$\phi=D\phi=0\quad\text{at}\quad y=0\quad\text{and}\qquad(6)$$

$$\phi\to 0, D\phi\to 0\quad\text{as}\quad y\to\infty.\qquad(7)$$

The behaviour of ϕ at large y has been discussed by GN95.

Equation (5), which may be called the 'full non-parallel equation', has the form

$$\{\mathrm{OS}\}\phi+\frac{1}{R}\{\mathrm{NP}_1+\mathrm{NP_h}\}\phi=O\left[\frac{1}{R^2}\right],\qquad(8)$$

with the Orr-Sommerfeld operator $\{\mathrm{OS}\}$ containing certain terms of $O(1)$ and others with a factor R^{-1}. The operator $\{\mathrm{NP}_1\}$, contained within curly brackets in (5), consists of non-parallel terms due to the change in the boundary layer thickness, streamwise variations in the free stream velocity as well as the x-dependence of the disturbance. The operator $\{\mathrm{NP_h}\}$, the last term in (8), accounts for higher order corrections to the mean flow (the effect of displacement thickness on the mean flow for Falkner-Skan wedge flows was considered by GN95). Equation (8) includes all terms with the factor R^{-1} in the primitive variables, and will be termed the primitive 'nominally' correct to $O\left(R^{-1}\right)$ in the following.

Now a stability analysis conducted using a full non-parallel equation including all terms of $O\left(R^{-1}\right)$ would be rational only if the mean flow were correct upto this order. (The flow over an infinitesimally thin semi-infinite flat plate is a special case in which the $O(R^{-1})$ contribution happens to vanish.) Apart from it being not feasible always for the mean flow to be prescribed to this degree of accuracy, it would seem obvious that non-parallel effects must exist even when only the lowest order contributions to the mean flow are known or given. This question has been considered in GN97.

The lowest order theory

At first glance, it might appear from (5) that the Rayleigh equation

$$\left\{(\omega-\alpha\Phi_0')(D^2-\alpha^2)+\alpha\Phi_0'''\right\}\phi=0,\qquad(9)$$

which is the result of omitting all terms containing the factor R^{-1} in (5), is a valid lowest order equation. It is however well known that the solution of (9) has a singularity at the critical point $y=y_c$, and that in the associated 'critical layer' it is necessary to invoke viscosity. Similarly, near the wall satisfaction of the no-slip boundary condition also demands that viscous effects be taken into account. At large R, the thicknesses of the critical and wall layers are respectively of $O(R^{-1/3})$ and $O(R^{-1/2})$ [5]. On the lower branch of the Orr-Sommerfeld stability boundary the phase velocity c_r of the wave, and correspondingly also y_c, are so small that the two layers may even merge into each other. Without loss of generality, however, we can proceed by first considering the two separately in the present approach.

Thus, we can say that there are three distinguished limits to consider:

(i) the bulk of the flow (outside layers (ii) and (iii) below), governed by the outer inviscid ('Rayleigh') solutions, defined by y fixed, $R^{-1}\to 0$;

(ii) the critical layer, given by

$$\eta_c\equiv(y-y_c)/\varepsilon_1\quad\text{fixed},\quad\varepsilon_1\equiv(\alpha R)^{-1/3}\to 0;$$

(iii) the wall layer, given by

$$\eta_w\equiv y/\varepsilon_2\quad\text{fixed},\quad\varepsilon_2=(\alpha R)^{-1/2}\to 0.$$

We illustrate the present approach by examining in detail the critical layer. Here ϕ may be expressed as the asymptotic expansion

$$\phi(y)\equiv\chi(\eta_c)=\chi_0(\eta_c)+\varepsilon_1\chi_1(\eta_c)+\cdots,\qquad(10)$$

and Φ_0 expanded in a Taylor series around y_c,

$$\Phi_0=\Phi_{0c}+\Phi_{0c}'(y-y_c)+\Phi_{0c}''(y-y_c)^2/2+\ldots\qquad(11)$$

On substituting (10) and (11) into the full non-parallel equation (8), we get, to the leading two orders in ε_1, the equations

$$\chi_0^{(\mathrm{iv})}-\mathrm{i}\eta_c\Phi_{0c}''\chi_0''=0,\qquad(12)$$

$$\chi_1^{(\mathrm{iv})}-\mathrm{i}\eta_c\Phi_{0c}''\chi_1''=\mathrm{i}\Phi_{0c}'''\left(\frac{1}{2}\eta_c^2\chi_0''-\chi_0\right)-p\Phi_{0c}\chi_0'''.\qquad(13)$$

Compared to the well known inner viscous layer equations in Orr-Sommerfeld theory [5], we see that the only difference is the presence of the additional term $p\Phi_{0c}\chi_0'''$ in (13), which is in general comparable to the other terms in the equation. (It would become negligible only if y_c, and hence Φ_{0c} also, become small.)

Now it is known from Orr-Sommerfeld theory that, to match the logarithmic behaviour of the Rayleigh solution near y_c, it is necessary to consider the two leading terms in the expansion (10). Using (both of) them, we can now 'compose' the lowest order equations at the critical layer as follows. From the full non-parallel terms in (5), we select just those that yield, on the use of the expansions (10) and (11), the terms that appear in (12) and (13). This gives us the 'minimal' subset of (5) that adequately represents the critical layer as

$$\left\{\mathrm{i}(\omega-\alpha\Phi_0')D^2+R^{-1}D^4+[\mathrm{i}\alpha\Phi_0'''+R^{-1}p\Phi_0D^3]\right\}\phi=0\qquad(14)$$

where the last two terms, within square brackets, are $O(R^{-1/3})$ relative to the first two. Recalling the definition of p from (3), we see that the term $p\Phi_{0c}\chi_0'''$ in (13), and the corresponding term in (14), are a direct result of flow non-parallelism, and we shall return to its significance presently.

An exactly similar analysis can be carried out for the wall layer. Expanding Φ_0 around the wall $y=0$ and noting the wall boundary conditions $\Phi_0(0)=0$, $\Phi_0'(0)=0$, the minimal composite equation for the wall layer is found to be

$$\left\{\mathrm{i}\left(\omega-\alpha\Phi'\right)D^2+R^{-1}D^4\right\}\phi=0,\qquad(15)$$

which is already contained in (14).

In the 'bulk' of the flow we have of course the Rayleigh equation (9). As the idea is to treat the problem numerically, GN97 do not handle these distinguished limits separately but instead construct the minimal 'composite' equation

$$\left\{\mathrm{i}(\omega-\alpha\Phi_0')\left(D^2-\alpha^2\right)+\mathrm{i}\alpha\Phi_0'''+\frac{1}{R}\left(D^4+p\Phi_0D^3\right)\right\}\phi=0,\qquad(16)$$

which contains all terms that are of order $R^{-1/2}$ or lower anywhere in the boundary layer, and is therefore (in particular) the rational equation upto that order. A numerical solution of (16), with the boundary conditions (6,7), can therefore yield the lowest order stability boundaries for the (non-parallel) flow in a Falkner-Skan boundary layer.

The implic]ation is that the simplest approximation to the stability characteristics of a (non-parallel flow) Falkner-Skan boundary layer is given by the ordinary differential equation (16); the Orr-Sommerfeld is in principle not appropriate, because it considers only parallel flow. Furthermore, the effects of non-parallelism appear in two different ways. The first is purely geometric, and is taken care of by the introduction of local coordinates through the transformation leading to (2). The second is dynamic, and appears (in the lowest order) solely through the term involving p in (16). As shown by GN97, this dynamic effect is the transport of disturbance vorticity at the critical layer by the mean wall-normal velocity of the (non-parallel) boundary layer.

Now it so happens that the effect of the non-parallel-flow dynamics (i.e. the term containing p in (16)) is quite small in the flat-plate boundary layer. However the effect becomes appreciable as the pressure gradient becomes adverse, for then the critical Reynolds numbers drop, the critical layer moves further away from the surface, and the mean normal velocity is higher. In favourable pressure gradients, on the other hand, the effect is even smaller than on a flat plate.

A higher order treatment

A legitimate question about a theory of this type is the following: if an ordinary differential equation in y (like (16)) has a solution $\phi(y)$, an arbitrary function of x times $\phi(y)$ is also a solution; so how does the x-dependence get determined? In practice this question has been answered, e.g. in e^n-type calculations, by noting that an o.d.e. like (16) or the Orr-Sommerfeld equation, through the dependence of R on x, carries x as a parameter. A satisfactory answer to this question must however proceed from a primitive equation in which the x-dependence is explicit.

Following arguments similar to those advanced above but including only the next round of higher order terms, GN99 show that, to $O(R^{-2/3})$, the equation governing stability is

$$\Bigg[(\omega-\alpha\Phi_0')\left(\mathrm{D}^2-\alpha^2\right)+\alpha\Phi_0'''+ \frac{1}{\mathrm{i}R}\left\{\mathrm{D}^4+p\Phi_0\mathrm{D}^3+\left(-2\alpha^2+\Phi_0'\left(2q-p-\frac{\partial}{\partial x}\right)\right)\mathrm{D}^2\right\}\Bigg]\phi=0. \tag{17}$$

It may be noticed that the last term here contains the streamwise derivative of the disturbance eigenfunction, which was absent in the lowest order equation (16), i.e., the effects of the *parabolic* nature of the flow on its stability first appear in this equation. It is therefore appropriate to call it the 'Lowest-order Parabolic Stability Equation' (LOP for short). The boundary conditions in y remain the same as in (6,7), but need to be supplemented by an initial condition at some x.

It is important to note that the higher order contributions to the mean flow, i.e. Φ_1 and so on, do not affect stability upto the order considered.

Comparing LOP (17) with O-S, we see that the term $\alpha^4\phi$ of O-S still makes no appearance but the term $2\alpha^2D^2\phi$ does. The viscous term involving Φ_0' is new. Compared to (16), the new dynamical effect in (17) represents streamwise diffusion of the dominant term in disturbance vorticity.

The connection between iterative solutions of the kind adopted in GN97 and marching solutions like those obtained with PSE or LOP has been explored by Balachandran and Govindarajan [1]. They show that a marching procedure with n steps with an upstream condition that is the solution of an eigenvalue problem is equivalent to an n-dimensional local eigen solution. Furthermore, there is no need to employ an explicit normalization for the eigenfunction.

Results and Discussion

Various results that have come out of the present work have been published previously, but we would like to highlight two sets of results which are particularly revealing.

The first set concerns stability 'loops'. Since the work of Tollmien, such loops separating the stable and unstable regimes in the (ω,R) or (α,R) space have become very familiar. In non-parallel flows, however, it is now well known that stability characteristics (including the stability loop) depend on distance y normal to the surface. What is more, it can sometimes take surprising and unsuspected forms (GN97). It therefore becomes necessary to think of a stability *surface* in the space (y,ω,R) or (y,α,R). The nature of such a stability surface for Blasius flow is illustrated in figure 1, which shows several views of the surface. It is seen that the surface consists of two segments which are stuck to each other with almost a discontinuity located around the intermediate zero of the eigenfunction. At distances just above this location there is a little kink in the upper branch of the loop, of the kind shown in figure 2. The back of the surface has a marked valley as well as ridge.

What all this suggests in that attention should be turned away from stability loops to streamwise variation of disturbance amplitudes. Solutions of (5) show excellent agreement with the DNS results of Fasel and Konzelmann [7], as demonstrated by us [12]; the lower-order theories show some small deviations [11].

Other flows

The technique of minimal composite equations has now been applied to a variety of other problems, and we merely state results here.

Wakes

The plane far wake, with (constant) free-stream velocity U and a centre-line velocity defect w_0, has a mean streamwise velocity which, with appropriate non-dimensionalization, obeys the similarity solution

$$\Phi_0'(y)=\Lambda-g(y),$$

where $\Lambda(x)=U/w_0(x)$ is the reciprocal of the velocity defect ratio and $g(y)$ is the appropriate similarity function for the defect-velocity profile. In this far-field solution, the local Reynolds number is independent of x but Λ varies like $x^{1/2}$. It can then be shown [13] that the lowest order linear stability equation for this flow is

$$\left\{i[\omega-\alpha(\Lambda-g)](D^2-\alpha^2)-i\alpha g''+\frac{1}{R}\left(D^4+\frac{y}{2}D^3\right)\right\}\phi=0,$$

with the boundary conditions

$$D\phi=D^2\phi=0,\quad \text{at}\quad y=0,$$
$$\phi\to 0,\quad D\phi\to 0\quad \text{as}\quad y\to\infty.$$

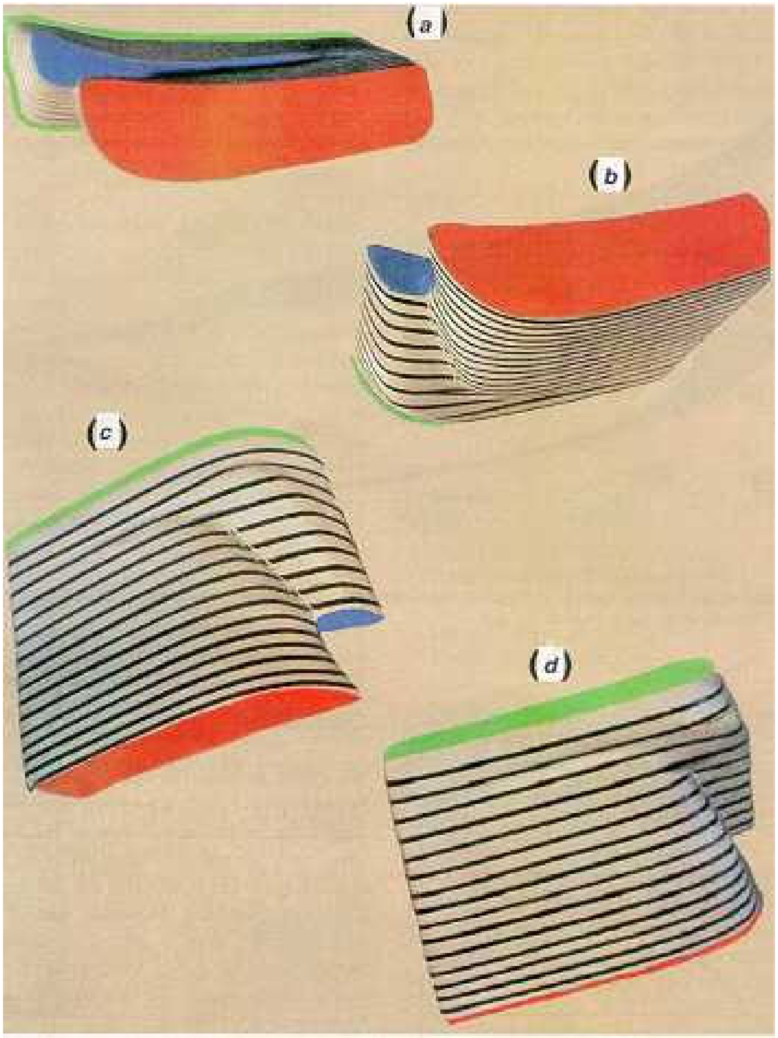

Figure 1: Four views of the stability surface for the Blasius boundary layer, in (y,ω,R) space. The surface is generated by stacking up, along the y-axis, stability loops generated at various values of y. (a) View with R to the right, ω towards the top and y into paper. (b) View from below, showing the lower branches of the stability loop stacked along y. (c,d) Other views, chiefly of the lower branches, showing the valley and ridge nature of the topography of the stability surface. From [20].

Swept wings

Following these ideas on 2D flows, a minimal composite theory can be formulated for the stability of Falkner-Skan-Cooke swept-wing boundary layers, including the effects of spatial development of the flow [14]. At order R^{-1} in the boundary layer Reynolds number (but ignoring the contributions of higher order boundary layer mean flow), the approach leads to two simultaneous partial differential equations in the velocity components parallel to the surface. At the lowest order, $O(R^{-1/2})$, the theory yields a single fourth order ordinary differential equation that involves both streamwise and cross-flow velocity profiles:

$$\mathrm{i}\left(\omega-\alpha\Phi'\right)\left(v''-\alpha^2 v-\xi^2\beta^2 v\right)+\mathrm{i}(\alpha f'''+\beta g'')v$$
$$+\frac{1}{R}\left(v^{\mathrm{iv}}+rfv'''\right)=0,$$

with the boundary conditions

$$v=v'=0 \quad \text{at} \quad y=0 \qquad \text{and} \qquad v,v'\to 0 \quad \text{for} \quad y\to\infty,$$

where v is the amplitude of the normal velocity fluctuation, f' and g are the stream-wise and cross-flow (similarity) velocity profiles, α,β are similarly the associated wavenumbers, ξ is a function of the sweep angle and r is a boundary layer growth parameter. This theory allows for wave propagation in an arbitrarily specified direction, and permits the marching direction for computing stability characteristics to be chosen at will. It is further possible to achieve the separation of the crossflow and TS modes at a particular spanwise wavenumber into distinct regions of instability in favorable pressure gradients (no such distinction occurs in adverse pressure gradient flow).

Compressible 2D boundary layers

Recently, several studies on the use of minimal composite theory for compressible flows have been undertaken. A preliminary attempt by Mitra *et al.*, [19] was followed by the discovery that, in the lowest order, a minimal composite theory for non-parallel compressible boundary layer stability yields a set of *three* ordinary differential equations (Seshadri *et al.*, [23]), one less than in the classical parallel flow theory as extensively studied by Mack [18]. This reduction is possible because, in the lowest order, the pressure disturbance can be eliminated. Furthermore, the bulk viscosity, often included in compressible stability theories, can be shown to be irrelevant at this order. The

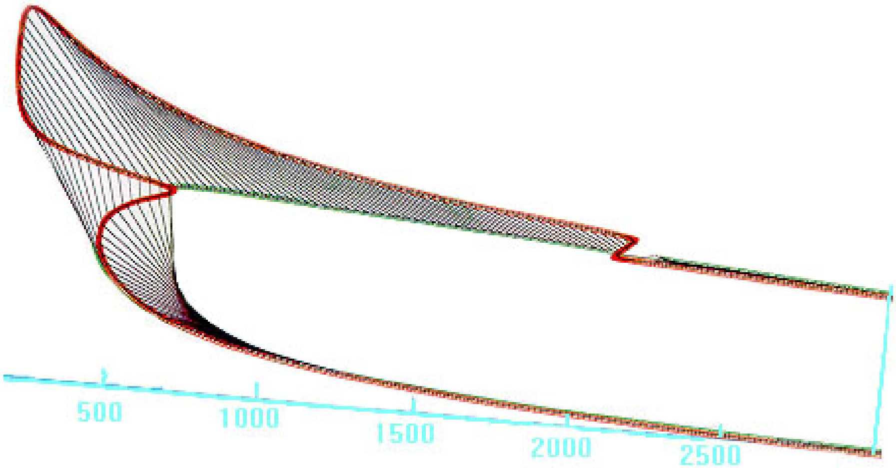

Figure 2: A slice of the stability surface of figure 1, taken around the near-discontinuity, bounded by $y = 0.69, 0.70$. The axis shown is R. Note the fold-back on the upper branch. From [20].

final lowest order stability equations from minimal composite theory are (Rao and Sashadri [21])

$$-i(\alpha U-\omega)(U-c)\gamma M^2\tilde{u}-\left((U-c)\gamma M^2 U'+T'\right)\tilde{v}$$

$$-i(\alpha U-\omega)\tilde{T}+T\tilde{z}=-\frac{T}{R}\Bigg[(U-c)\gamma M^2\mu\tilde{u}''$$

$$+\left(\frac{d\mu}{dT}T'+pF\right)\tilde{u}'+\frac{d\mu}{dT}U'\tilde{T}'+pF\tilde{T}'\Bigg],$$

$$-i(\alpha U-\omega)T'\tilde{u}+\left(\alpha(\alpha U-\omega)T+U''T-U'T'\right)\tilde{v}$$

$$+i(\alpha U-\omega)T\tilde{u}'+U'T\tilde{z}=\frac{T^2}{R}\Bigg[\mu\tilde{u}'''-i\alpha\mu\tilde{v}''$$

$$+\left(2\frac{d\mu}{dT}T'+pF\right)\tilde{u}''+\frac{d\mu}{dT}U'\tilde{T}''$$

$$+\left(2\frac{d^2\mu}{dT^2}U'T'+2frd\mu dTU''\right)\tilde{T}'\Bigg].$$

and

$$i(\alpha U-\omega)\tilde{T}+T'v'+(\gamma-1)T\tilde{z}=$$

$$\frac{T}{R}\Bigg[\frac{\gamma}{\sigma}\left(k\tilde{T}''+2\frac{dk}{dT}T'\tilde{T}'\right)+pF\tilde{T}'+2\mu\gamma(\gamma-1)M^2U'\tilde{u}'$$

$$+(\gamma-1)\left((p-q)(M^2-1)\tilde{u}+yq\tilde{u}'\right)\Bigg].$$

The notation here is largely standard: M is the free-stream Mach number, γ the ratio of specific heats, T is temperature, p is the pressure;

$$\tilde{z}=\mathrm{i}\alpha\tilde{u}+\tilde{v}'$$

is proportional to the divergence $\nabla\cdot\tilde{\mathbf{u}}$ which vanishes in the incompressible limit.

Compared to the well known parallel-flow stability equations of Mack [18], the present equations contain six additional terms arising out of the downstream growth of the boundary layer. However, ten terms contained in the parallel-flow equations are absent, since they are found to be of higher order everywhere.

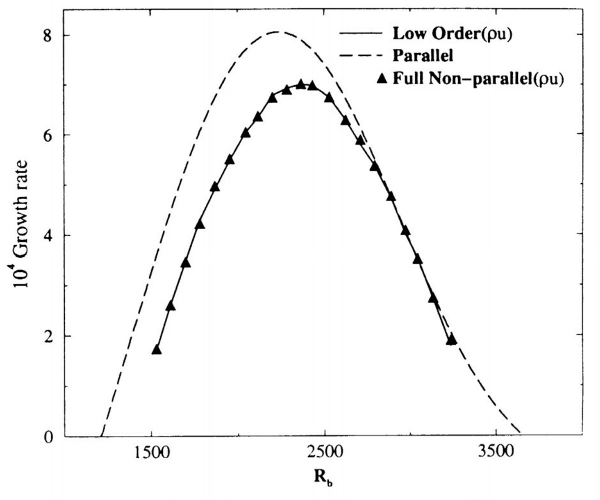

Figure 3: Comparisons of growth rates at $M = 1.6$ and $F = 6$, from [21]; full non-parallel results from [4].

In compressible flows, the mass flux disturbance (which is $\rho\tilde{u}+U\tilde{\rho}$) is more sensitive to the effect of compressibility than $\tilde{u}$, because of the factor ρ. Possibly for this reason, a neutral stability curve based on the mass flux disturbance in low-order theory is very close to that of the full non-parallel theory. This point is reinforced in figure 3, where the growth rate based on mass flux at Mach 1.6 at a low frequency of $F = 6$ is displayed. (Here $F = \omega\nu_e/U_e^2$, where ν_e is the kinematic viscosity and U_e the mean velocity at the edge of the boundary layer.) These parameters are chosen as the PSE computations of Chang and Malik [4] are available for comparison. Figure 3 shows that the low-order theory agress well with full non-parallel theory at all Reynolds numbers, and performs better than parallel flow theory everywhere. This is further confirmed in figure 4, where the growth rate of the second mode disturbances at the larger Mach number of 4.5, and a frequency of $F = 120$, is shown for a large interval of Reynolds numbers, using Horton's mean flow profile [15]. These growth rates are evaluated at the inner maximum of the mass flux. The corresponding results from the multiple scales approach of El-Hady [6] (not reproduced here) show many kinks, as can be seen in Chang and Malik [4]. Thus

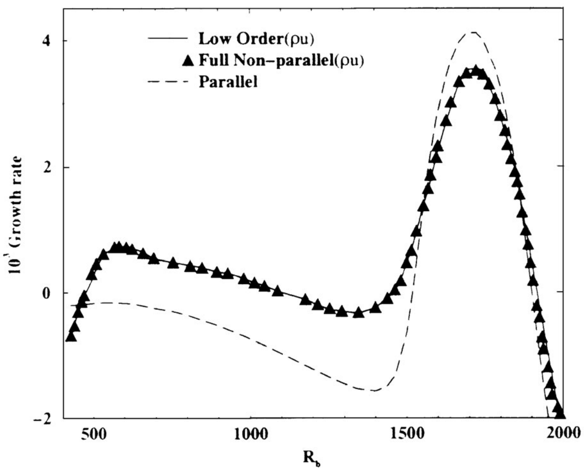

Figure 4: Comparisons of growth rate of second mode disturbances at $M = 4.5$ and $F = 120$, from [21]; full non-parallel results from [4].

the present low-order theory performs better than both multiple-scales and parallel-flow theories, and has an accuracy as good as PSE.

Conclusions

We can now summarize the position as follows. The Orr-Sommerfeld equations (we use the plural to include compressible flow), valid only for parallel flow, have the great (apparent) advantage that they are universal; i.e. they are valid for all parallel flows and all Reynolds numbers. However this universality is misleading and much less powerful than it seems, because the mean flow is supposed to be given independently, and does not always follow from the parallel flow assumption. The use of minimal composite equations, in the high Reynolds limit, leads in the lowest order to ordinary differential equations. In incompressible flow this equation is rather like the Orr-Sommerfeld, but is not the same. It already takes into account the non-parallelism in the flow; indeed there is one term which explicitly represents a dynamical effect involving advection of disturbance vorticity by the mean wall-normal flow. However the ordinary differential equation is valid only for similarity solutions of the boundary layer equation. In compressible flow the lowest-order equations are actually both simpler and more accurate than the standard parallel-flow Orr-Sommerfeld theories.

It is relevant to mention here that the present theory is different from the pioneering triple-deck approach of Smith [24], who also proposed the first rational theory for the nonparallel stability of boundary layers. The chief difference is that in the present work the frequency and wave number do not participate in the limiting process. Smith's equations are therefore simpler, but they are valid only for $R \gg R_{\rm cr}$, and can predict neither the critical Reynolds number nor the upper branch of the stability loop (although a separate five-deck theory can be formulated for the asymptotic part of the upper branch). On the other hand, Smith takes the problem all the way to the final solutions.

If the boundary layer is not similar, the present lowest order equations are strictly speaking not valid. In this case one can make approximate calculations either assuming local similarity or adopting a weakly non-similar approach (GN95). More importantly, the lowest order minimal composite equations are not universal. Thus for each type of flow the governing equation has to be specially derived.

The power of the present theory is most visible in compressible flow, where the minimal composite equations are both simpler and more accurate than Orr-Sommerfeld type parallel-flow theories, and are indistinguishable from PSE at the higher Mach numbers.

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