ON THE PULSATION OF A STAR IN WHICH THERE IS A PREVALENT MAGNETIC FIELD*

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Received August 24, 1953

ABSTRACT

In this paper a simple approximate formula is obtained for the frequency (σ) of radial pulsation of a gaseous star in which there is a prevalent magnetic field. The formula is

$$\sigma^2 I = -(3\gamma - 4)(\Omega + \mathfrak{M}),$$

where γ is the ratio of the specific heats,

$$I = \int_0^M r^2 dm(r),$$

and Ω and $\mathfrak{M}$ denote the gravitational potential energy and the magnetic energy of the star, respectively. The formula is derived from the virial theorem in the form recently established by Chandrasekhar and Fermi; and it supports their conclusion that the period of pulsation can be made as long as one may desire by letting the magnetic energy approach the upper limit (namely, $|\Omega|$) set by the virial theorem.

1. *Introduction.*—The pulsation of a star in which there is a prevalent magnetic field has been the subject of several investigations.[1] But so far no simple formula has been derived in terms of which one may easily visualize the effect of the magnetic field on the period. In this paper we shall show how such a formula can be derived by making use of the virial theorem in the form recently established by Chandrasekhar and Fermi.[2]

2. *An integral formula for the frequency of oscillation.*—We start with the statement of the virial theorem in the form (Paper I, eq. [11])

$$\frac{1}{2}\frac{d^2 I}{dt^2} = 2T + 3(\gamma - 1)\mathfrak{U} + \Omega + \mathfrak{M}, \tag{1}$$

where Ω denotes the gravitational potential energy,

$$I = \int_0^M r^2 dm, \qquad T = \frac{1}{2}\int_0^M |u|^2 dm, \tag{2}$$

$$(\gamma - 1)\mathfrak{U} = \int_0^M \frac{p}{\rho} dm, \qquad \text{and} \qquad \mathfrak{M} = \int_0^M \frac{|H|^2}{8\pi\rho} dm. \tag{2a}$$

In equations (2),

$$dm = \rho dx_1 dx_2 dx_3 \tag{3}$$

denotes the element of mass, and the integrations are effected over the entire mass, M,

* The research reported in this paper has been supported in part by the Geophysics Research Directorate of the Air Force Cambridge Research Center, Air Research and Development Command, under Contract AF 19(604)-299 with the University of Chicago.

[1] M. Schwarzschild, *Ann. d'ap.*, **12**, 148, 1949; G. Gjellestad, *Rep. No. 1. Inst. Theor. Ap.* (Oslo, 1950) and *Ann. d'ap.*, **15**, 276, 1952; V. C. A. Ferraro and D. J. Memory, *M.N.*, **112**, 361, 1952; T. G. Cowling, *M.N.*, **112**, 527, 1952.

[2] S. Chandrasekhar and E. Fermi, *Ap. J.*, **118**, 116, 1952. This paper will be referred to hereafter as "Paper I."

of the configuration. The remaining symbols have their usual meanings (they are all defined in Paper I).

We shall apply equation (1) to the adiabatic pulsation of a gaseous star in which there is a prevalent magnetic field. In analyzing this problem, we shall adopt the Lagrangian mode of description and follow each element of mass, dm, as it moves. The advantage of the Lagrangian over the Eulerian mode in our present context arises from the fact that dm remains constant during the motion; this is required by the conservation of mass.

Considering periodic oscillations with a frequency σ, we shall let $\boldsymbol{\xi} e^{i\sigma t}$ denote the displacement of an element of mass, dm, from its equilibrium position, $\boldsymbol{r}$. Similarly, we shall let $\delta p e^{i\sigma t}$, $\delta \rho e^{i\sigma t}$, and $\delta \boldsymbol{H} e^{i\sigma t}$ denote the corresponding changes in the other physical variables as we follow the element during its motion. The assumption that the oscillation takes place adiabatically requires that the changes in pressure and density, as we follow the motion, are related by

$$\frac{\delta p}{p}=\gamma \frac{\delta \rho}{\rho}, \tag{4}$$

while the equation of continuity requires that

$$\frac{\delta \rho}{\rho}=-\frac{\partial \xi_i}{\partial x_i} \tag{5}$$

(as in Paper I, we are adopting the summation convention).

Returning to equation (1) and letting $\delta I e^{i\sigma t}$, $\delta \mathfrak{U} e^{i\sigma t}$, $\delta \Omega e^{i\sigma t}$, and $\delta \mathfrak{M} e^{i\sigma t}$ denote the changes in I, $\mathfrak{U}$, Ω, and $\mathfrak{M}$, we can write

$$-\sigma^2 \int_0^M \xi_i x_i d m=3(\gamma-1)\, \delta \mathfrak{U}+\delta \Omega+\delta \mathfrak{M}, \tag{6}$$

since, to the first order in the displacement, the term in T will not make any contribution.

Considering the first of the three terms on the right-hand side of equation (6), we have (cf. eqs. [2], [4], and [5])

$$\begin{aligned} 3(\gamma-1)\, \delta \mathfrak{U} &= 3 \int_0^M \delta\left(\frac{p}{\rho}\right) d m=3(\gamma-1) \int_0^M \frac{p}{\rho} \frac{\delta \rho}{\rho}\, d m \\ &= -3(\gamma-1) \iiint p \frac{\partial \xi_i}{\partial x_i}\, d x_1 d x_2 d x_3, \end{aligned} \tag{7}$$

where the integration over x_1, x_2, and x_3 is over the entire volume[3] of the configuration. Letting

$$P=p+\frac{|H|^2}{8 \pi} \tag{8}$$

denote the total pressure, we shall rewrite equation (7) in the form

$$\begin{aligned} 3(\gamma-1)\, \delta \mathfrak{U}= & -3(\gamma-1) \iiint P \frac{\partial \xi_i}{\partial x_i}\, d x_1 d x_2 d x_3 \\ & +3(\gamma-1) \iiint \frac{|H|^2}{8 \pi} \frac{\partial \xi_i}{\partial x_i}\, d x_1 d x_2 d x_3 . \end{aligned} \tag{9}$$

Integrating by parts the first of the two integrals on the right-hand side of equation (9) and making use of the equation (cf. Paper I, eq. [4])

$$\frac{\partial P}{\partial x_i}=\rho \frac{\partial V}{\partial x_i}+\frac{1}{4 \pi} \frac{\partial}{\partial x_j} H_i H_j \tag{10}$$

[3] For a star with a prevailing magnetic field, the effective boundary may have to be placed quite outside the conventional photospheric surface. We return to this question presently.

(where V denotes the gravitational potential), which obtains in the equilibrium state, we find

$$\iiint P \frac{\partial \xi_i}{\partial x_i} dx_1 dx_2 dx_3 = \int_{\text{surface}} P \boldsymbol{\xi} \cdot d\boldsymbol{S} - \iiint \xi_i \left(\rho \frac{\partial V}{\partial x_i} + \frac{1}{4\pi} \frac{\partial}{\partial x_j} H_i H_j \right) dx_1 dx_2 dx_3 . \tag{11}$$

In equation (11) the surface integral is over the bounding surface (S) of the configuration. We shall now assume that P vanishes on $\boldsymbol{S}$. This assumption requires that not only p but also $\boldsymbol{H}$ vanish on the bounding surface. Accordingly, we must suppose that the bounding surface, $\boldsymbol{S}$, is placed at a sufficient distance from the photospheric surface that $\boldsymbol{H}$ may indeed be considered negligible here. It would appear that in most astrophysical contexts we can accomplish this without violating the assumption of the infinite electrical conductivity of the stellar material which underlies this whole development.

Assuming, then, that P vanishes on the bounding surface, we have (cf. eq. [11])

$$\iiint P \frac{\partial \xi_i}{\partial x_i} dx_1 dx_2 dx_3 = -\iiint \rho \xi_i \frac{\partial V}{\partial x_i} dx_1 dx_2 dx_3 - \frac{1}{4\pi} \iiint \xi_i \frac{\partial}{\partial x_j} H_i H_j \, dx_1 dx_2 dx_3 . \tag{12}$$

After a further integration by parts (of the second integral on the right-hand side), we obtain

$$\iiint P \frac{\partial \xi_i}{\partial x_i} dx_1 dx_2 dx_3 = -\iiint \rho \xi_i \frac{\partial V}{\partial x_i} dx_1 dx_2 dx_3 + \frac{1}{4\pi} \iiint H_i H_j \frac{\partial \xi_i}{\partial x_j} dx_1 dx_2 dx_3 , \tag{13}$$

the integrated part again making no contribution, since $\boldsymbol{H}$ has been assumed to vanish on the bounding surface.

Now, combining equations (9), (11), (12), and (13), we have

$$3(\gamma - 1)\,\delta \mathfrak{U} = 3(\gamma - 1) \left\{ \int_0^M \xi_i \frac{\partial V}{\partial x_i} dm + \frac{1}{8\pi} \iiint |\boldsymbol{H}|^2 \frac{\partial \xi_i}{\partial x_i} dx_1 dx_2 dx_3 - \frac{1}{4\pi} \iiint H_i H_j \frac{\partial \xi_i}{\partial x_j} dx_1 dx_2 dx_3 \right\} . \tag{14}$$

Turning next to the terms $\delta\Omega$ and $\delta\mathfrak{M}$ in equation (6), we first observe that

$$\delta\Omega = -\int^M \xi_i \frac{\partial V}{\partial x_i} dm . \tag{15}$$

This follows from entirely elementary considerations. It remains only to evaluate $\delta\mathfrak{M}$. We have (cf. eq. [2*a*])

$$\delta\mathfrak{M} = \frac{1}{4\pi} \iiint H_i \delta H_i dx_1 dx_2 dx_3 - \frac{1}{8\pi} \iiint |\boldsymbol{H}|^2 \frac{\delta \rho}{\rho} dx_1 dx_2 dx_3 ; \tag{16}$$

or, using equation (5), we have

$$\delta\mathfrak{M} = \frac{1}{4\pi} \iiint H_i \delta H_i dx_1 dx_2 dx_3 + \frac{1}{8\pi} \iiint |\boldsymbol{H}|^2 \frac{\partial \xi_i}{\partial x_i} dx_1 dx_2 dx_3 . \tag{17}$$

Now the change in the magnetic field $\delta \boldsymbol{H}$ as we follow the motion of an element of gas of infinite electrical conductivity is given by (cf. Paper I, eq. [36])

$$\delta \boldsymbol{H} = \text{curl}\,(\boldsymbol{\xi} \times \boldsymbol{H}) + (\boldsymbol{\xi} \cdot \text{grad})\, \boldsymbol{H}\,, \tag{18}$$

or, alternatively,

$$\delta H_i = H_j \frac{\partial \xi_i}{\partial x_j} - H_i \frac{\partial \xi_j}{\partial x_j}. \tag{19}$$

Substituting this last expression for δH_i in equation (17), we obtain

$$\delta \mathfrak{M} = \frac{1}{4\pi} \int\!\!\int\!\!\int H_i H_j \frac{\partial \xi_i}{\partial x_j} - \frac{1}{8\pi} \int\!\!\int\!\!\int |\boldsymbol{H}|^2 \frac{\partial \xi_i}{\partial x_i}\, dx_1 dx_2 dx_3\,. \tag{20}$$

Finally, inserting the expressions (14), (15), and (20) for $3(\gamma - 1)\delta\mathfrak{U}$, $\delta\Omega$, and $\delta\mathfrak{M}$ in equation (6), we obtain

$$\sigma^2 \int_0^M \xi_i x_i dm = -(3\gamma - 4) \left\{ \int_0^M \xi_i \frac{\partial V}{\partial x_i} dm + \frac{1}{8\pi} \int\!\!\int\!\!\int |\boldsymbol{H}|^2 \frac{\partial \xi_i}{\partial x_i}\, dx_1 dx_2 dx_3 - \frac{1}{4\pi} \int\!\!\int\!\!\int H_i H_j \frac{\partial \xi_i}{\partial x_j}\, dx_1 dx_2 dx_3 \right\}. \tag{21}$$

This is the required integral formula for σ^2.

3. *An approximate formula for the frequency of radial pulsation.*—It is known from an analysis by Ledoux[4] similar to the foregoing but relating to the radial pulsation of ordinary stars that a reliable estimate of the period of pulsation of such stars can be obtained by writing

$$\xi_i = \text{Constant}\; x_i\,, \tag{22}$$

in a formula for σ^2 analogous to equation (21); indeed, Ledoux's formula for σ^2 can be obtained by setting $\boldsymbol{H} = 0$ in equation (21). We shall assume that this will continue to be the case in our present problem. Therefore, making the substitution (22) in equation (21), we obtain

$$\sigma^2 \int_0^M r^2 dm = -(3\gamma - 4) \left\{ \int_0^M x_i \frac{\partial V}{\partial x_i} dm + \frac{1}{8\pi} \int\!\!\int\!\!\int |\boldsymbol{H}|^2 dx_1 dx_2 dx_3 \right\}, \tag{23}$$

or (cf. eqs. [2])

$$\sigma^2 = -(3\gamma - 4) \frac{(\Omega + \mathfrak{M})}{I}. \tag{24}$$

This formula for σ^2 confirms the conclusion reached by Chandrasekhar and Fermi (Paper I, p. 119) on general grounds that σ^2 must tend to zero as the total magnetic energy ($\mathfrak{M}$) of the star tends to the upper limit (namely, $|\Omega|$) set by the virial theorem.

[4] P. Ledoux, *Ap. J.*, **102**, 143, 1945.