THE GRAVITATIONAL INSTABILITY OF AN INFINITE HOMOGENEOUS MEDIUM WHEN CORIOLIS FORCE IS ACTING AND A MAGNETIC FIELD IS PRESENT*

S. CHANDRASEKHAR
Yerkes Observatory

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ABSTRACT

It is shown that Jeans's criterion for the gravitational instability of an infinite homogeneous medium is unaffected by the presence of a magnetic field, even if the system is partaking in rotation and Coriolis force is operative.

1. *Introduction.*—It has recently been shown that Jeans's criterion for the gravitational instability of an infinite homogeneous medium remains unaffected both when Coriolis force is acting[1] and when a magnetic field is present.[2] Generalizing these results, we shall show in this paper that Jeans's criterion remains valid also when both the agencies, Coriolis force and magnetic field, are simultaneously acting.

2. *The three modes of wave propagation and the condition for gravitational instability.*—Consider an extended homogeneous medium of infinite electrical conductivity partaking in rotation with an angular velocity $\boldsymbol{\Omega}$; and let there be present also a uniform magnetic field of intensity $\boldsymbol{H}$. Then the fluctuations in the density ($\delta\rho$), pressure (δp), magnetic field ($\boldsymbol{h}$), and gravitational potential (δV) are governed by the equations

$$\begin{gathered} \rho \frac{\partial \boldsymbol{u}}{\partial t} = \frac{1}{4\pi} \operatorname{curl} \boldsymbol{h} \times \boldsymbol{H} + 2\rho \boldsymbol{u} \times \boldsymbol{\Omega} - \operatorname{grad} \delta p + \rho \operatorname{grad} \delta V\,, \\ \frac{\partial \boldsymbol{h}}{\partial t} = \operatorname{curl}(\boldsymbol{u} \times \boldsymbol{H})\,; \qquad \operatorname{div} \boldsymbol{h} = 0\,, \\ \frac{\partial}{\partial t} \delta\rho = -\rho \operatorname{div} \boldsymbol{u} \qquad \text{and} \qquad \nabla^2 \delta V = -4\pi G \delta\rho\,. \end{gathered} \tag{1}$$

If the changes in pressure and density are assumed to take place adiabatically, then

$$\delta p = c^2 \delta\rho \tag{2}$$

where $c = \sqrt{(\gamma p/\rho)}$ denotes the conventional velocity of sound.

In the further discussion of equations (1) we shall suppose that the orientation of the co-ordinate axes is so chosen that

$$\boldsymbol{H} = (0, H_y, H_z)\,, \qquad \text{while} \qquad \boldsymbol{\Omega} = (\Omega_x, \Omega_y, \Omega_z)\,. \tag{3}$$

We shall seek solutions of equations (1) which correspond to the propagation of waves in the z-direction. Then $\partial/\partial z$ is the only nonvanishing component of the gradient; and it follows at once from the divergence condition, $\operatorname{div} \boldsymbol{h} = 0$, that $h_z = 0$. Equations (1)

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[1] S. Chandrasekhar, *Vistas in Astronomy* (London: Pergamon, 1954).

[2] S. Chandrasekhar and E. Fermi, *Ap. J.*, **118**, 116, 1953.

then take the forms

$$\frac{\partial h_x}{\partial t} - H_z \frac{\partial u_x}{\partial z} = 0 ,$$

$$\frac{\partial h_y}{\partial t} + H_y \frac{\partial u_z}{\partial z} - H_z \frac{\partial u_y}{\partial z} = 0 ,$$

$$\frac{\partial u_x}{\partial t} - \frac{H_z}{4\pi\rho} \frac{\partial h_x}{\partial z} + 2 (u_z \Omega_y - u_y \Omega_z) = 0 ,$$

$$\frac{\partial u_y}{\partial t} - \frac{H_z}{4\pi\rho} \frac{\partial h_y}{\partial z} + 2 (u_x \Omega_z - u_z \Omega_x) = 0 , \tag{4}$$

$$\frac{\partial u_z}{\partial t} + \frac{H_y}{4\pi\rho} \frac{\partial h_y}{\partial z} + \frac{c^2}{\rho} \frac{\partial}{\partial z} \delta\rho - \frac{\partial}{\partial z} \delta V + 2 (u_y \Omega_x - u_x \Omega_y) = 0 ,$$

$$\frac{\partial}{\partial t} \delta\rho + \rho \frac{\partial u_z}{\partial z} = 0 ,$$

and

$$\frac{\partial^2}{\partial z^2} \delta V + 4\pi G \delta\rho = 0 .$$

For a solution which represents the propagation of a wave in the z-direction

$$\frac{\partial}{\partial t} = i\omega \quad \text{and} \quad \frac{\partial}{\partial z} = -ik , \tag{5}$$

where ω denotes the frequency and k the wave number. Making substitutions (5) in equations (4), we obtain a system of linear homogeneous equations for the amplitudes of the various quantities which can be written in matrix notation in the following form:

$$\begin{Vmatrix} \omega & 0 & kH_z & 0 & 0 & 0 & 0 \\ 0 & \omega & 0 & kH_z & -kH_y & 0 & 0 \\ \frac{kH_z}{4\pi\rho} & 0 & \omega & +2i\Omega_z & -2i\Omega_y & 0 & 0 \\ 0 & \frac{kH_z}{4\pi\rho} & -2i\Omega_z & \omega & +2i\Omega_x & 0 & 0 \\ 0 & -\frac{kH_y}{4\pi\rho} & +2i\Omega_y & -2i\Omega_x & \omega & -\frac{c^2 k}{\rho} & k \\ 0 & 0 & 0 & 0 & -\rho k & \omega & 0 \\ 0 & 0 & 0 & 0 & 0 & 4\pi G & -k^2 \end{Vmatrix} \begin{Vmatrix} h_x \\ h_y \\ u_x \\ u_y \\ u_z \\ \delta\rho \\ \delta V \end{Vmatrix} = 0 . \tag{6}$$

The condition that equation (6) has a nontrivial solution is that the determinant of the matrix on the left-hand side should vanish. On expanding the determinant, we find that it can be reduced to the form

$$\begin{aligned} \omega^6 - \omega^4 \{ (4\Omega^2 + \Omega_A^2) + (\Omega_A^2 + \Omega_B^2 + \Omega_J^2) \} \\ + \omega^2 \{ \Omega_J^2 (4\Omega_z^2 + \Omega_A^2) + \Omega_A^2 (4\Omega_x^2 + \Omega_A^2 + \Omega_B^2 + \Omega_J^2) + 4 (\Omega_y \Omega_A - \Omega_z \Omega_B)^2 \} \\ - \Omega_A^4 \Omega_J^2 = 0 , \end{aligned} \tag{7}$$

where

$$\Omega_A^2 = \frac{k^2 H_z^2}{4\pi\rho}, \qquad \Omega_B^2 = \frac{k^2 H_y^2}{4\pi\rho}, \tag{8}$$ [3]

$$\Omega_J^2 = c^2 k^2 - 4\pi G\rho, \qquad \text{and} \qquad \Omega^2 = |\boldsymbol{\Omega}|^2 .$$

From equation (7) it follows that there are, in general, three modes in which a wave can be propagated through the medium; and that if ω_1, ω_2, and ω_3 denote the frequencies of these three modes, then

$$\omega_1^2 + \omega_2^2 + \omega_3^2 = 4\Omega^2 + 2\Omega_A^2 + \Omega_B^2 + \Omega_J^2 , \tag{9}$$

and

$$\omega_1 \omega_2 \omega_3 = \Omega_A^2 \Omega_J . \tag{10}$$

Hence *if* Ω_J *is imaginary, then either* ω_1, ω_2, *or* ω_3 *must be imaginary.* In all cases, there will therefore be a mode of wave propagation which will become unstable when Ω_J becomes imaginary. Now the condition for Ω_J to become imaginary is

$$c^2 k^2 < 4\pi G\rho ; \tag{11}$$

but this is precisely Jeans's condition. The condition for gravitational instability is therefore unaffected by Coriolis and magnetic forces acting jointly or separately.

[3] It can be readily verified that, when $\boldsymbol{H} = 0$, this equation reduces to eq. (14), given in reference 1, while, when $\boldsymbol{\Omega} = 0$, it reduces to eq. (169), given in reference 2.