PROBLEMS OF GRAVITATIONAL STABILITY IN THE PRESENCE OF A MAGNETIC FIELD

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ABSTRACT

In this paper a number of problems are considered which are related to the gravitational stability of cosmical masses of infinite electrical conductivity in which there is a prevalent magnetic field. In Section I the virial theorem is extended to include the magnetic terms in the equations of motion, and it is shown that when the magnetic energy exceeds the numerical value of the gravitational potential energy, the configuration becomes dynamically unstable. It is suggested that the relatively long periods of the magnetic variables may be due to the magnetic energy of these stars approaching the limit set by the virial theorem. In Section II the adiabatic radial pulsations of an infinite cylinder along the axis of which a magnetic field is acting is considered. An explicit expression for the period is obtained. Section III is devoted to an investigation of the stability for transverse oscillations of an infinite cylinder of incompressible fluid when there is a uniform magnetic field acting in the direction of the axis. It is shown that the cylinder is unstable for all periodic deformations of the boundary with wave lengths exceeding a certain critical value, depending on the strength of the field. The wave length of maximum instability is also determined. It is found that the magnetic field has a stabilizing effect both in increasing the wave length of maximum instability and in prolonging the time needed for the instability to manifest itself. For a cylinder of radius $R = 250$ parsecs and $\rho = 2 \times 10^{-24}$ gm/cm^3 a magnetic field in excess of 7×10^{-6} gauss effectively removes the instability. In Section IV it is shown that a fluid sphere with a uniform magnetic field inside and a dipole field outside is not a configuration of equilibrium and that it will tend to become oblate by contracting in the direction of the field. Finally, in Section V the gravitational instability of an infinite homogeneous medium in the presence of a magnetic field is considered, and it is shown that Jeans's condition is unaffected by the presence of the field.

1. *Introduction.*—In this paper we shall consider a number of problems relating to the dynamical and gravitational stability of cosmical masses in which there is a prevalent magnetic field. In the discussion of these problems, the assumption will be made that the medium is effectively of infinite electrical conductivity. This latter assumption implies only that the conductivity is large enough for the magnetic lines of force to be considered as practically attached to the matter during the length of time under consideration; it has been known for some time that this is the case in most astronomical connections.[1]

The abstract gives an adequate summary of the paper.

I. THE VIRIAL THEOREM AND THE CONDITION FOR DYNAMICAL STABILITY

2. *The virial theorem.*—In a subject such as this it is perhaps best that we start by establishing theorems of the widest possible generality. The extension of the virial theorem to include the forces derived from the prevailing magnetic field provides such a starting point. We shall see that under conditions of equilibrium this extension of the virial theorem leads to the relation

$$2T + 3(\gamma - 1)\mathfrak{U} + \mathfrak{M} + \Omega = 0 \tag{1}$$

between the kinetic energy (T) of mass motion, the heat energy ($\mathfrak{U}$) of molecular motion, the magnetic energy ($\mathfrak{M}$) of the prevailing field, and the gravitational potential energy (Ω), where γ denotes the ratio of the specific heats. That a relation of the form (1) should exist is readily understood: For the balance between the pressures p_{kin}, p_{gas}, and p_{mag} due

[1] Cf. L. Biermann, *Annual Review of Nuclear Science*, **2** (Stanford: Annual Reviews, Inc., 1953), 349.

to the visible motions, the molecular motions, and the magnetic field, on the one hand, and the gravitational pressure, p_{grav}, on the other, requires

$$p_{\text{kin}} + p_{\text{gas}} + p_{\text{mag}} = p_{\text{grav}} , \tag{2}$$

while the order of magnitudes of these pressures are given by

$$p_{\text{kin}} = c_1 \frac{T}{V} , \qquad p_{\text{gas}} = c_2 \frac{\mathfrak{U}}{V} , \qquad p_{\text{mag}} = \frac{H^2}{8\pi} = c_3 \frac{\mathfrak{M}}{V} , \tag{3}$$

and

$$p_{\text{grav}} = \text{Density} \times \text{gravity} \times \text{linear dimension} = - c_4 \frac{\Omega}{V} , \tag{3a}$$

where V denotes the volume of the configuration and c_1, c_2, c_3, and c_4 are numerical constants. A relation of the form (1) is therefore clearly implied. We now proceed to establish the exact relation (1).

With the usual assumptions of hydromagnetics, the equations of motion governing an inviscid fluid can be written in the form

$$\rho \frac{d u_i}{d t} = - \frac{\partial}{\partial x_i} \left(p + \frac{|\boldsymbol{H}|^2}{8\pi} \right) + \rho \frac{\partial V}{\partial x_i} + \frac{1}{4\pi} \frac{\partial}{\partial x_j} H_i H_j , \tag{4}$$

where ρ denotes the density, p the pressure, V the gravitational potential, and $\boldsymbol{H}$ the intensity of the magnetic field. (In eq. [1] and in the sequel, summation over repeated indices is to be understood.)

Multiply equation (4) by x_i and integrate over the volume of the configuration. Reducing the left-hand side of the equation in the usual manner, we find

$$\begin{aligned} \iiint \rho x_i \frac{d u_i}{d t} d x_1 d x_2 d x_3 &= \int_0^M x_i \frac{d^2 x_i}{d t^2} d m \\ &= \frac{1}{2} \frac{d^2}{d t^2} \int_0^M r^2 d m - \int_0^M |\boldsymbol{u}|^2 d m , \end{aligned} \tag{5}$$

where $dm = \rho dx_1 dx_2 dx_3$ and the integration is effected over the entire mass, M, of the configuration. Letting

$$I = \int_0^M r^2 d m \qquad \text{and} \qquad T = \frac{1}{2} \int |\boldsymbol{u}|^2 d m \tag{6}$$

denote the moment of inertia and the kinetic energy of mass motion, respectively, we have

$$\begin{aligned} \frac{1}{2} \frac{d^2 I}{d t^2} - 2T = &- \iiint x_i \frac{\partial}{\partial x_i} \left(p + \frac{|\boldsymbol{H}|^2}{8\pi} \right) d x_1 d x_2 d x_3 \\ &+ \frac{1}{4\pi} \iiint x_i \frac{\partial}{\partial x_j} H_i H_j d x_1 d x_2 d x_3 + \int_0^M x_i \frac{\partial V}{\partial x_i} d m . \end{aligned} \tag{7}$$

The last of the three integrals on the right-hand side of this equation represents the gravitational potential energy, Ω, of the configuration. The remaining two volume integrals can be reduced by integration by parts. Thus the first of the two integrals gives

$$\begin{aligned} &- \iiint x_i \frac{\partial}{\partial x_i} \left(p + \frac{|\boldsymbol{H}|^2}{8\pi} \right) d x_1 d x_2 d x_3 \\ &\qquad = - \int \left(p + \frac{|\boldsymbol{H}|^2}{8\pi} \right) \boldsymbol{r} \cdot d\boldsymbol{S} + 3 \iiint \left(p + \frac{|\boldsymbol{H}|^2}{8\pi} \right) d x_1 d x_2 d x_3 . \end{aligned} \tag{8}$$

The surface integral (over dS) vanishes, since the pressure (including the magnetic pressure $|H|^2/8\pi$) must vanish on the boundary of the configuration; and the volume integral over p and $|H|^2/8\pi$ is readily expressible in terms of the internal energy ($\mathfrak{U}$) and the magnetic energy ($\mathfrak{M}$) of the configuration. Thus we have

$$-\iiint x_i \frac{\partial}{\partial x_i}\left(p+\frac{|H|^2}{8\pi}\right) dx_1 dx_2 dx_3 = 3(\gamma-1)\mathfrak{U}+3\mathfrak{M}, \tag{9}$$

where γ denotes the ratio of the specific heats. In the same way the second volume integral in equation (7) gives

$$\frac{1}{4\pi}\iiint x_i \frac{\partial}{\partial x_j} H_i H_j dx_1 dx_2 dx_3 = -2\mathfrak{M}. \tag{10}$$

Now, combining equations (7), (9), and (10), we have

$$\frac{1}{2}\frac{d^2 I}{dt^2} = 2T + 3(\gamma-1)\mathfrak{U}+\mathfrak{M}+\Omega. \tag{11}$$

This is the required generalization of the virial theorem; it differs from the usual one only in the appearance of $\mathfrak{M}+\Omega$ in place of Ω.

3. *The condition for dynamical stability.*—If the configuration is one of equilibrium, then it follows from the virial theorem that

$$3(\gamma-1)\mathfrak{U}+\mathfrak{M}+\Omega = 0. \tag{12}$$

On the other hand, the total energy, $\mathfrak{E}$, of the configuration is given by

$$\mathfrak{E} = \mathfrak{U}+\mathfrak{M}+\Omega. \tag{13}$$

Eliminating $\mathfrak{U}$ between equations (12) and (13), we obtain

$$\mathfrak{E} = -\frac{3\gamma-4}{3(\gamma-1)}(|\Omega|-\mathfrak{M}). \tag{14}$$

From this equation for the total energy it follows that *a necessary condition for the dynamical stability of an equilibrium configuration is*

$$(3\gamma-4)(|\Omega|-\mathfrak{M}) > 0. \tag{15}$$

Thus, even when $\gamma > \frac{4}{3}$ (the condition for dynamical stability in the absence of a magnetic field) a sufficiently strong internal magnetic field can induce dynamical instability in the configuration. In fact, according to formula (15), the condition for dynamical stability, when $\gamma > \frac{4}{3}$, is

$$\mathfrak{M} = \frac{1}{8\pi}\iiint |H|^2 dx_1 dx_2 dx_3 = \tfrac{1}{6}R^3 (H^2)_{\text{av}} < |\Omega|, \tag{16}$$

where $(H^2)_{\text{av}}$ denotes the mean square magnetic field.

For a spherical configuration of uniform density,

$$\Omega = -\frac{3}{5}\frac{GM^2}{R}, \tag{17}$$

where M is its mass, R is its radius, and G is the constant of gravitation. We can use this expression for Ω to estimate the limit imposed by the virial theorem on the magnetic fields which can prevail. On expressing M and R in units of the solar mass and the solar radius, we find from equations (16) and (17) that

$$\sqrt{(H^2)_{av}} < 2.0 \times 10^8 \frac{M}{R^2} \text{ gauss}. \tag{18}$$

For the peculiar A stars for which Babcock has found magnetic fields of the order of 10^4 gauss, we may estimate that

$$M \simeq 4\odot \quad \text{and} \quad R \simeq 5R_\odot \quad \text{(A star)}; \tag{19}$$

and expression (18) gives

$$\sqrt{(H^2)_{av}} < 3 \times 10^7 \text{ gauss} \quad \text{(A stars)}. \tag{19a}$$

Of greater interest is the limit set by expression (18) for an S-type star for which Babcock has found a variable magnetic field of the order of 1000 gauss. For an S-type star we can estimate that

$$M \simeq 15\odot \quad \text{and} \quad R \simeq 300R_\odot \quad \text{(S star)}; \tag{20}$$

and (18) now gives

$$\sqrt{(H^2)_{av}} < 3 \times 10^4 \text{ gauss} \quad \text{(S star)}. \tag{20a}$$

Thus the limit set by (18) is seen to be two to three orders of magnitude larger than the surface fields observed by Babcock. However, the fields in the interior may be much stronger than the surface fields; and it is even possible that the actual root-mean-square fields in these stars are near their maximum values. Indeed, from the fact that the periods of the magnetic variables are long compared with the adiabatic pulsation periods they would have if they were nonmagnetic, we may surmise that $\sqrt{(H^2)_{av}}$ is near the limit set by (18); for, as is well known, we may lengthen the period of the lowest mode of oscillation of a system by approaching the limit of dynamical stability; and we can accomplish this by letting $\mathfrak{M} \to |\Omega|$.

Note added June 17.—Since we wrote this paper, Dr. Babcock has informed us that he has measured a variable magnetic field (+2000 to −1200 gauss) for the star VV Cephei. It has been estimated that for this star $M = 100\odot$ and $R = 2600R_\odot$. With these values, inequality (18) gives $\sqrt{(H^2)_{av}} < 3000$ gauss. We may conclude that this star must be on the verge of dynamical stability and, anticipating the result established in Section IV, probably highly oblate.

4. *The virial theorem for an infinite cylindrical distribution of matter*.—Some care is needed in applying the results of §§ 2 and 3 to a distribution of matter which can be idealized as an infinite cylinder (such as, for example, a spiral arm); for the potential energy per unit length of an infinite cylinder is infinite. For this reason it is perhaps best that we consider the problem *de novo*.

We shall consider, then, an infinite cylinder in which the prevailing magnetic field is in the direction of the axis of the cylinder; and we shall suppose that all the variables are functions only of the distance r from the axis of the cylinder. Under these conditions the equations of motion reduce to the single one

$$\rho \frac{du_r}{dt} = -\frac{\partial}{\partial r}\left(p + \frac{H^2}{8\pi}\right) - \frac{2Gm(r)}{r}\rho, \tag{21}$$

where $m(r)$ is the mass per unit length interior to r.

Multiplying equation (21) by $2\pi r^2 dr$ and integrating over the entire range of r, we find in the usual manner that

$$\frac{1}{2}\frac{d^2}{dt^2}\int_0^M r^2 dm - \int_0^M \left(\frac{dr}{dt}\right)^2 dm = 2(\gamma - 1)\mathfrak{U} + 2\mathfrak{M} - GM^2, \tag{22}$$

where M denotes the mass per unit length of the cylinder and $\mathfrak{U}$ and $\mathfrak{M}$ are the kinetic and the magnetic energies per unit length of the cylinder, respectively.

From equation (22) it follows that, under equilibrium conditions, we should have

$$2(\gamma-1)\mathfrak{U}+2\mathfrak{M}-GM^2=0\ . \tag{23}$$

A necessary condition for equilibrium to obtain is, therefore,

$$\mathfrak{M}<\tfrac{1}{2}GM^2\ . \tag{24}$$

We can rewrite the condition (24) alternatively in the form

$$\mathfrak{M}=\frac{1}{4}\int_0^R H^2 r\,d r=\tfrac{1}{8}\,(H^2)_{\text{av}}R^2<\tfrac{1}{2}\pi^2R^4\bar{\rho}^2G\ , \tag{25}$$

or

$$\sqrt{(H^2)_{\text{av}}}<2\pi R\bar{\rho}G\ . \tag{25a}$$

This last condition on the root-mean-square field is essentially equivalent to one of the formulae used in the preceding paper[2] (eq. [13]) for estimating the magnetic field in the spiral arm; the difference between the two formulae arises from the fact that in that paper the gravitational attraction was not limited to the interstellar gas only; allowance was also made for the stars contributing to the gravitational force acting on the gas.

II. THE RADIAL PULSATIONS OF AN INFINITE CYLINDER

5. *The pulsation equation.*—In view of the inconclusive nature of the current treatments[3] of the adiabatic pulsations of magnetic stars, it is perhaps of interest to see how the corresponding problem in infinite cylinders can be fully solved. We consider, then, the radial pulsation of an infinite cylinder, along the axis of which there is a prevailing magnetic field.

Choosing the time t and the mass per unit length, $m(r)$, interior to r, as the independent variables, we can write the equations of continuity and motion in the forms

$$\frac{\partial}{\partial m}(\pi r^2)=\frac{1}{\rho} \tag{26}$$

and

$$\frac{\partial^2 r}{\partial t^2}=-2\pi r\,\frac{\partial P}{\partial m}-\frac{2Gm(r)}{r}\ , \tag{27}$$

where

$$P=p+\frac{H^2}{8\pi} \tag{28}$$

denotes the total pressure. Distinguishing the values of the various parameters for the equilibrium configuration by a subscript zero and writing

$$r=r_0+\delta r,\qquad P=P_0+\delta P,\qquad \rho=\rho_0+\delta\rho,\ \text{etc.} \tag{29}$$

we find that the equations governing radial oscillations of small amplitudes are

$$\frac{\partial}{\partial m}(2\pi r_0\,\delta r)=-\frac{\delta\rho}{\rho_0^2} \tag{30}$$

and

$$\frac{\partial^2}{\partial t^2}\,\delta r=-2\pi\,\delta r\,\frac{\partial P_0}{\partial m}-2\pi r_0\,\frac{\partial}{\partial m}\,\delta P+\frac{2Gm}{r_0^2}\,\delta r\ . \tag{31}$$

[2] *Ap. J.*, **118**, 113, 1953.

[3] M. Schwarzschild, *Ann. d'ap.*, **12**, 148, 1949; G. Gjellestad, *Rep. No. 1, Inst. Ther. Ap.* (Oslo, 1950), and *Ann. d'ap.*, **15**, 276, 1952; V. C. A. Ferraro and D. J. Memory, *M.N.*, **112**, 361, 1952; T. G. Cowling, *M.N.*, **112**, 527, 1952.

Using the equation

$$2\pi r_0 \frac{\partial P_0}{\partial m} = -\frac{2Gm}{r_0}, \tag{32}$$

which must obtain in equilibrium, we can rewrite equation (31) in the form

$$\frac{\partial^2}{\partial t^2}\delta r = -2\pi r_0 \frac{\partial}{\partial m}\delta P + \frac{4Gm}{r_0^2}\delta r. \tag{33}$$

We shall now evaluate δP. For an adiabatic pulsation,

$$\delta P = \delta p + \frac{\boldsymbol{H}_0 \cdot \delta \boldsymbol{H}}{4\pi} = \gamma \frac{p_0}{\rho_0}\delta\rho + \frac{\boldsymbol{H}_0 \cdot \delta \boldsymbol{H}}{4\pi}. \tag{34}$$

Now when the medium is of infinite electrical conductivity, the change, $\Delta \boldsymbol{H}$, *at a given point* in a prevailing magnetic field, $\boldsymbol{H}_0$, caused by a displacement $\delta \boldsymbol{r}$, is given quite generally by

$$\Delta \boldsymbol{H} = \operatorname{curl}(\delta \boldsymbol{r} \times \boldsymbol{H}_0). \tag{35}$$

This relation is derived in § 14 below (see eq. [130]); but we may note here that it merely expresses the fact that the changes in the magnetic field are simply a consequence of the lines of force being pushed aside. According to equation (35), the change in the magnetic field, $\delta \boldsymbol{H}$, *as we follow the motion*, is given by

$$\delta \boldsymbol{H} = \operatorname{curl}(\delta \boldsymbol{r} \times \boldsymbol{H}_0) + (\delta \boldsymbol{r} \cdot \operatorname{grad}) \boldsymbol{H}_0. \tag{36}$$

When $\boldsymbol{H}_0$ is in the z-direction and $\delta \boldsymbol{r}$ is radial, the only nonvanishing component of $\delta \boldsymbol{H}$ is

$$\delta H_z = -\frac{1}{r}\frac{\partial}{\partial r}(H_0 r \delta r) + \delta r \frac{\partial H_0}{\partial r} = -\frac{H_0}{r}\frac{\partial}{\partial r}(r\delta r), \tag{37}$$

in the z-direction. Hence in the case under consideration

$$\frac{\boldsymbol{H}_0 \cdot \delta \boldsymbol{H}}{4\pi} = -\frac{H_0^2 \rho_0}{4\pi}\frac{\partial}{\partial m}(2\pi r_0 \delta r), \tag{38}$$

and the expression for δP becomes

$$\delta P = -\left(\gamma p_0 + \frac{H_0^2}{4\pi}\right)\rho_0 \frac{\partial}{\partial m}(2\pi r_0 \delta r), \tag{39}$$

where we have substituted for $\delta\rho$ in equation (34) in accordance with equation (30).

With δP given by equation (38), equation (33) takes the form

$$\frac{\partial^2}{\partial t^2}\delta r = 4\pi^2 r \frac{\partial}{\partial m}\left\{\left(\gamma p + \frac{H^2}{4\pi}\right)\rho \frac{\partial}{\partial m}(r\delta r)\right\} + \frac{4Gm}{r^2}\delta r. \tag{40}$$

In writing equation (39), we have suppressed the subscripts zero distinguishing the equilibrium configuration, since there is no longer any cause for ambiguity.

When all the physical variables vary with time like $e^{i\sigma t}$, equation (39) reduces to

$$\left(\sigma^2 + \frac{4Gm}{r^2}\right)\delta r = -4\pi^2 r \frac{d}{dm}\left\{\left(\gamma p + \frac{H^2}{4\pi}\right)\rho \frac{d}{dm}(r\delta r)\right\}, \tag{41}$$

where δr has now the meaning of an amplitude.

The boundary conditions,

$$\delta r = 0 \quad \text{at} \quad m = 0 \quad \text{and} \quad \delta P = 0 \quad \text{at} \quad m = M\,, \tag{42}$$

in conjunction with the pulsation equation (40) will determine for σ^2 a sequence of possible characteristic values, σ_k^2. And it can be readily shown that the solutions, δr_k, belonging to the different characteristic values, are orthogonal:

$$\int_0^M \delta r_k \delta r_l dm = 0 \qquad (k \neq l)\,. \tag{43}$$

In view of this orthogonality of the functions δr_k, we should expect that the characteristic values themselves could be determined by a variational method. The basis for this method is developed in the following section.

6. *An integral formula for σ^2 and a variational method for determining it.*—Multiply equation (41) by δr and integrate over the range of m, i.e., from 0 to M. We obtain

$$\sigma^2 \int_0^M (\delta r)^2 dm + 4G \int_0^M \left(\frac{\delta r}{r}\right)^2 dm = -4\pi^2 \int_0^M r\,\delta r \frac{d}{dm}\left\{\left(\gamma p + \frac{H^2}{4\pi}\right) \rho \frac{d}{dm}(r\,\delta r)\right\} dm\,. \tag{44}$$

By integrating by parts the integral on the right-hand side, we obtain

$$\sigma^2 \int_0^M (\delta r)^2 dm + 4G \int_0^M \left(\frac{\delta r}{r}\right)^2 dm = 4\pi^2 \int_0^M \left(\gamma p + \frac{H^2}{4\pi}\right) \rho \left[\frac{d}{dm}(r\,\delta r)\right]^2 dm\,. \tag{45}$$

Writing $p = (P - H^2/8\pi)$ in equation (45), we obtain, after some elementary reductions,

$$\sigma^2 \int_0^M (\delta r)^2 dm = \gamma \int_0^M \frac{P}{\rho}\left[\frac{1}{r}\frac{d}{dr}(r\,\delta r)\right]^2 dm - 4G \int_0^M \left(\frac{\delta r}{r}\right)^2 m\,dm + \frac{1}{8\pi}(2-\gamma) \int_0^M \frac{H^2}{\rho}\left[\frac{1}{r}\frac{d}{dr}(r\,\delta r)\right]^2 dm\,. \tag{46}$$

It can be shown that the foregoing equations give a minimum value for σ^2 when the true solution δr belonging to the lowest characteristic value of the pulsation equation is substituted; and any other function δr (satisfying the boundary conditions) will give a larger value for σ^2. These facts can clearly be made the basis of a variational procedure for determining σ^2.

In the theory of the adiabatic pulsations of ordinary stars, it is known[4] that we get a very good estimate of σ^2 (for the fundamental mode) by setting

$$\delta r = \text{Constant } r\,, \tag{47}$$

in an integral formula for σ^2 similar to equation (46). We shall assume that this will continue to be the case in our present problem. Therefore, making the substitution (47) in equation (46), we obtain

$$\sigma^2 \int_0^M r^2 dm = 4\gamma \int_0^M \frac{P}{\rho} dm - 2GM^2 + (2-\gamma) \int_0^R H^2 r\,dr\,. \tag{48}$$

[4] P. Ledoux and C. L. Pekeris, *Ap. J.*, **94**, 124 1941.

On the other hand,

$$\int_0^M \frac{P}{\rho}\, d m = 2\pi \int_0^R P\, r\, d r = -\pi \int_0^R r^2 \frac{dP}{d r}\, d r = G \int_0^M m\, d m = \tfrac{1}{2} G M^2 . \tag{49}$$

Hence

$$\sigma^2 \int_0^M r^2 d m = 2\,(\gamma - 1)\, G M^2 + (2 - \gamma) \int_0^R H^2 r\, d r . \tag{50}$$

An alternative form of this equation is (cf. eq. [23])

$$\begin{aligned}\sigma^2 \int_0^M r^2 d m &= 4\,(\gamma - 1)\,[\tfrac{1}{2} G M^2 - \mathfrak{M}] + 4\,\mathfrak{M} \\ &= 4\,[\,(\gamma - 1)^2 \mathfrak{U} + \mathfrak{M}\,] .\end{aligned} \tag{51}$$

III. THE GRAVITATIONAL INSTABILITY OF AN INFINITELY LONG CYLINDER WHEN A CONSTANT MAGNETIC FIELD IS ACTING IN THE DIRECTION OF THE AXIS

7. *The formulation of the problem.*—In Section II we have seen that an infinitely long cylinder in which there is a prevalent magnetic field in the direction of the axis is stable for radial oscillations. But the question was left open as to whether the cylinder may not be unstable for transverse or for longitudinal oscillations. In Section III we shall take up the discussion of the transverse oscillations; however, in order not to complicate an already difficult problem, we shall restrict ourselves to the case when the medium is incompressible in addition to being an infinitely good electrical conductor.

We picture to ourselves, then, an infinite cylinder of uniform circular cross-section of radius R_0, along the axis of which a constant magnetic field of intensity H_0 is acting. Since any transverse perturbation can be expressed as a superposition of waves of different wave lengths, the question of stability can be investigated by considering, individually, perturbations of different wave lengths. We suppose, then, that the cylinder is subject to a perturbation, the result of which is to deform the boundary into

$$r = R + a \cos kz . \tag{52}$$

Since the fluid is assumed to be incompressible, the mass per unit length must be the same before and after the deformation; this, clearly, requires that

$$R_0^2 = R^2 + \tfrac{1}{2} a^2 . \tag{53}$$

We shall see that, as a result of the deformation, the mean field in the z-direction is also changed by an amount of order a^2 (see eq. [87] below).

The investigation of the stability of the cylinder consists of two parts. First, we must calculate the change in the potential energy, $\Delta\Omega$, and the magnetic energy, $\Delta\mathfrak{M}$, per unit length resulting from the perturbation. Then, depending on whether $\Delta\Omega + \Delta\mathfrak{M}$ is positive or negative, we shall have stability or instability. We shall see presently that $\Delta\Omega + \Delta\mathfrak{M} < 0$ for all k less than a certain determinate value depending on H_0. In other words, the cylinder is unstable for all wave lengths exceeding a certain critical value λ_*. The determination of λ_* is the first problem in the investigation of stability. The second problem concerns the specification of the wave number k_m (say) for which the instability will develop at the maximum rate. We can determine this mode of maximum instability by considering the amplitude of the deformation (cf. eq. [52]) as a function of time, constructing a Lagrangian for the cylinder and determining the manner of increase of the amplitudes of the unstable modes. We shall find that whenever $\lambda > \lambda_*$ (or $k < k_*$), the amplitude increases like e^{qt}, where q is a function of k (and H_0). The mode of maximum instability is clearly the one which makes q (for a given H) a maximum.

Before proceeding to the details of the calculations, we may state that the method we have described derives from an early investigation of Rayleigh's[5] on the stability of liquid jets.

8. *The change in the potential energy per unit length caused by the deformation.*—Following the outline given in § 7, we shall first calculate the change in the potential energy, $\Delta\Omega$, per unit length caused by the deformation which makes the cross-section change from one of a constant radius R_0 to one whose boundary is given by equation (52). Since the potential energy per unit length of an infinite cylinder is infinite, the evaluation of $\Delta\Omega$ requires some care. We proceed as follows:

Let U and V denote the external and the internal gravitational potentials of the deformed cylinder. They satisfy the equations

$$\nabla^2 U = 0 \qquad \text{and} \qquad \nabla^2 V = -4\pi G\rho . \tag{54}$$

We shall first solve these equations to the first order in the amplitude a appropriately for the problem on hand. The solutions must clearly be of the forms

$$U = -2\pi G\rho R^2 \log r + aAK_0(kr) \cos kz + c_0 \tag{55}$$

and

$$V = -\pi G\rho r^2 + aBI_0(kr) \cos kz , \tag{55a}$$

where c_0 is an additive constant (with which we need not further concern ourselves), A and B are constants to be determined, and I_n and K_n are the Bessel functions of order n for a purely imaginary argument, which have no singularity at the origin and at infinity, respectively.

The constants A and B in solutions (55) are to be determined by the condition that U and V and $\partial U/\partial r$ and $\partial V/\partial r$ must be continuous on the boundary (52). Carrying out the calculations consistently to the first order in a, we find that the continuity conditions require

$$A\, K_0\,(kR) = B\, I_0\,(kR) \tag{56}$$

and

$$A\, K_1\,(kR) + B\, I_1\,(kR) = \frac{4\pi G\rho}{k} . \tag{56a}$$

Solving these equations, we find

$$A = 4\pi G\rho R I_0(kR) \qquad \text{and} \qquad B = 4\pi G\rho R K_0(kR) . \tag{57}$$

The required solution for V is, therefore,

$$V = -\pi G\rho r^2 + 4\pi G\rho R\, aK_0(kR) I_0(kr) \cos kz + O(a^2). \tag{58}$$

Now suppose that the amplitude of the deformation is increased by an *infinitesimal* amount from a to $a + \delta a$. The change in the potential energy, $\delta\Delta\Omega$, consequent to this infinitesimal increase in the amplitude, can be determined by evaluating the work done in the redistribution of the matter required to increase the amplitude. For evaluating this latter work, it is necessary to specify in a quantitative manner the redistribution which takes place; and we shall now do this.

An arbitrary deformation of an incompressible fluid can be thought of as resulting from a displacement $\boldsymbol{\xi}$ applied to each point of the fluid. The assumed incompressibility of the medium requires that div $\boldsymbol{\xi} = 0$; and, since no loss of generality is implied by supposing that the displacement is irrotational, we shall write

$$\boldsymbol{\xi} = \text{grad}\ \psi, \tag{59}$$

[5] Lord Rayleigh, *Scientific Papers* (Cambridge: At the University Press, 1900), **2**, 361; also *Theory of Sound* ("Dover Reprints" [New York, 1945]), **2**, 350–362.

and require that

$$\nabla^2 \psi = 0 . \tag{60}$$

A solution of equation (60) which is suitable for considering the deformation of a uniform cylinder into one whose boundary is given by (52) is

$$\psi = A I_0(kr) \cos kz , \tag{61}$$

where A is a constant. The corresponding radial and z-components of $\boldsymbol{\xi}$ are

$$\xi_r = A k I_1(kr) \cos kz \quad \text{and} \quad \xi_z = -A k I_0(kr) \sin kz . \tag{62}$$

Since at $r = R$, ξ_r must reduce to $a \cos kz$ (cf. eq. [52]), we must have

$$A = \frac{a}{k I_1(kR)} . \tag{63}$$

The displacements,

$$\xi_r = a \frac{I_1(kr)}{I_1(kR)} \cos kz \quad \text{and} \quad \xi_z = -a \frac{I_0(kr)}{I_1(kR)} \sin kz , \tag{64}$$

applied to each point of the cylinder will deform it into the required shape. The displacement $\delta\boldsymbol{\xi}$, which must be applied to increase the amplitude from a to $a + \delta a$, is therefore,

$$\delta \xi_r = \delta a \frac{I_1(kr)}{I_1(kR)} \cos kz \quad \text{and} \quad \delta \xi_z = -\delta a \frac{I_0(kr)}{I_1(kR)} \sin kz . \tag{65}$$

The change in the potential energy, $\delta\Delta\Omega$, per unit length involved in the infinitesimal deformation (65) can be obtained by integrating over the whole cylinder the work done by the displacement $\delta\boldsymbol{\xi}$ in the force field specified by the gravitational potential (58). It is therefore given by

$$\delta\Delta\Omega = -2\pi\rho \left\{ \int_0^{R+a\cos kz} \delta\boldsymbol{\xi} \cdot \operatorname{grad} V \, r \, dr \right\}_{\mathrm{av}} , \tag{66}$$

where the averaging is to be done with respect to z. Substituting for V and $\delta\boldsymbol{\xi}$ from equations (58) and (65), we obtain

$$\begin{aligned} \delta\Delta\Omega = -2\pi\rho\,\delta a \Big\{ & \int_0^{R+a\cos kz} \cos kz \frac{I_1(kr)}{I_1(kR)} [-2\pi G\rho r \\ & + 4\pi\rho G R a k K_0(kR) I_1(kr) \cos kz] \, r \, dr \\ & + \int_0^{R+a\cos kz} \sin kz \frac{I_0(kr)}{I_1(kR)} [4\pi\rho G R a k K_0(kR) I_0(kr) \sin kz] \, r \, dr \Big\}_{\mathrm{av}} . \end{aligned} \tag{67}$$

Evaluating the foregoing expression consistently to the first order in a, we find

$$\delta\Delta\Omega = 2\pi^2\rho^2 G R^2 a\,\delta a - 4\pi^2\rho^2 G R a\,\delta a \frac{k K_0(kR)}{I_1(kR)} \int_0^R [I_1^2(kr) + I_0^2(kr)] \, r \, dr , \tag{68}$$

or, using the readily verifiable result,

$$\int_0^R [I_1^2(kr) + I_0^2(kr)] \, r \, dr = \frac{R}{k} I_0(kR) I_1(kR) , \tag{69}$$

we have

$$\delta\Delta\Omega = 4\pi^2\rho^2 G R^2 [\tfrac{1}{2} - I_0(x) K_0(x)] a\,\delta a , \tag{70}$$

where for the sake of brevity we have written

$$x = kR. \tag{71}$$

Finally, integrating equation (70) over a from 0 to a, we obtain

$$\Delta\Omega = 2\pi^2\rho^2 GR^2[\tfrac{1}{2} - I_0(x)K_0(x)]a^2. \tag{72}$$

This is the required expression for the change in the potential energy per unit length caused by the deformation.

9. *The change in the magnetic energy per unit length caused by the deformation.*—The changes in the magnetic field inside the cylinder can best be determined from the condition that the magnetic induction across any section normal to the axis of the cylinder must remain unaffected by the deformation. This condition follows from the assumed infinite electrical conductivity of the medium. Thus, if

$$H\mathbf{1}_z + \boldsymbol{h} \tag{73}$$

represents the magnetic field inside the cylinder (where $\mathbf{1}_z$ is a unit vector in the z-direction, $\boldsymbol{h}$ is a field, of order a, varying periodically with z, and H is the mean field), we must require that

$$N = \int_0^R H_0\, r\, d\, r = \int_0^{R+a\cos kz} (H + h_z)\, r\, d\, r = \text{Constant}. \tag{74}$$

Turning to the determination of H and $\boldsymbol{h}$, we may first observe that $\boldsymbol{h}$ can be derived from a magnetostatic potential ϕ satisfying the equation $\nabla^2\phi = 0$. For the problem on hand we can represent ϕ as a series in powers of a of the form

$$\phi = \sum_{n=1}^{\infty} \frac{a^n A_n}{n k} I_0(n k r) \sin n k z, \tag{75}$$

where the A_n's are constants to be determined. Retaining terms up to the second order in a, we have

$$h_r = aA_1 I_1(kr) \sin kz + a^2 A_2 I_1(2kr) \sin 2kz \tag{76}$$

and

$$h_z = aA_1 I_0(kr) \cos kz + a^2 A_2 I_0(2kr) \cos 2kz, \tag{77}$$

for the components of $\boldsymbol{h}$.

With h_z given by equation (77), the magnetic induction across a normal section of the cylinder is given by

$$N = \int_0^{R+a\cos kz} \{H + aA_1 I_0(kr) \cos kz + a^2 A_2 I_0(2kr) \cos 2kz\}\, r\, d\, r. \tag{78}$$

Evaluating N correct to the second order in a, we obtain

$$\begin{aligned} N = \tfrac{1}{2}H(R^2 + \tfrac{1}{2}a^2) + \tfrac{1}{2}a^2 A_1 R I_0(kR) + a\left[HR + \frac{A_1}{k} I_1(kR) R\right] \cos kz \\ + a^2\left[\tfrac{1}{4}H + \tfrac{1}{2}A_1 R I_0(kR) + \frac{A_2}{2k} R I_1(2kR)\right] \cos 2kz; \end{aligned} \tag{79}$$

and according to equation (74) this must be identically equal to (cf. eq. [53])

$$\tfrac{1}{2}H_0 R_0^2 = \tfrac{1}{2}H_0(R^2 + \tfrac{1}{2}a^2). \tag{80}$$

Hence we must require that

$$\tfrac{1}{2}H(R^2+\tfrac{1}{2}a^2)+\tfrac{1}{2}a^2A_1RI_0(kR)=\tfrac{1}{2}H_0(R^2+\tfrac{1}{2}a^2)\,, \tag{81}$$

$$HR+\frac{A_1}{k}I_1(kR)R=0\,, \tag{82}$$

and

$$\tfrac{1}{4}H+\tfrac{1}{2}A_1RI_0(kR)+\frac{A_2}{2k}RI_1(2kR)=0\,. \tag{83}$$

From equations (82) and (83) we find:

$$A_1=-\frac{H}{R}\frac{x}{I_1(x)} \tag{84}$$

and

$$A_2=\frac{H}{R^2}\frac{x}{I_1(2x)}\left\{\frac{xI_0(x)}{I_1(x)}-\frac{1}{2}\right\}, \tag{85}$$

where $x=kR$ (cf. eq. [71]).

With A_1 given by equation (84), equation (81) gives (correct to order a^2)

$$H_0=H\left\{1-\frac{a^2}{R^2}\frac{xI_0(x)}{I_1(x)}\right\}, \tag{86}$$

or, equivalently,

$$H=H_0\left\{1+\frac{a^2}{R^2}\frac{xI_0(x)}{I_1(x)}\right\}. \tag{87}$$

This equation shows that the mean field inside the deformed cylinder is larger than that in the undeformed cylinder; the difference is of order a^2 and depends on the wave number of the deformation.

Equations (76), (77), (84), (85), and (87) determine the field inside the cylinder correct to the second order in a. It may be noted here that the same solution can also be derived from the alternative (but equivalent) condition that the magnetic lines of force follow the boundary of the cylinder (52).

With the field inside the cylinder determined, we can now evaluate the magnetic energy, $\mathfrak{M}$, per unit length. We have

$$\begin{aligned}\mathfrak{M}&=\frac{1}{4}\left\{\int_0^{R+a\cos kz}|\boldsymbol{H}|^2r\,dr\right\}_{\text{av}}\\&=\frac{1}{4}\left\{\int_0^{R+a\cos kz}(H^2+2Hh_z+h_z^2+h_r^2)\,r\,dr\right\}_{\text{av}},\end{aligned} \tag{88}$$

where the averaging is to be done with respect to z. Substituting for h_r and h_z from equations (76) and (77) and evaluating $\mathfrak{M}$ correct to the second order in a, we obtain (cf. eq. [69])

$$\begin{aligned}\mathfrak{M}&=\tfrac{1}{8}H^2(R^2+\tfrac{1}{2}a^2)+\tfrac{1}{2}aH\left\{\cos kz\int_0^{R+a\cos kz}A_1I_0(kr)\,r\,dr\right\}_{\text{av}}\\&\quad+\tfrac{1}{2}a^2HA_2\left\{\cos 2kz\int_0^{R+a\cos kz}I_0(2kr)\,r\,dr\right\}_{\text{av}}\\&\quad+\tfrac{1}{8}a^2A_1^2\int_0^R[I_0^2(kr)+I_1^2(kr)]\,r\,dr\\&=\tfrac{1}{8}H^2(R^2+\tfrac{1}{2}a^2)+\tfrac{1}{4}a^2HA_1RI_0(kR)+\tfrac{1}{8}a^2A_1^2\frac{R}{k}I_0(kR)I_1(kR)\,.\end{aligned} \tag{89}$$

On making further use of equations (53), (84), and (87), we can reduce this last expression for $\mathfrak{M}$ to the form

$$\mathfrak{M} = \tfrac{1}{8} H_0^2 R_0^2 + \tfrac{1}{8} a^2 H^2 \frac{x I_0(x)}{I_1(x)}. \tag{90}$$

But the magnetic energy per unit length of the undeformed cylinder is $\frac{1}{8} H_0^2 R_0^2$. Hence

$$\Delta \mathfrak{M} = \tfrac{1}{8} a^2 H^2 \frac{x I_0(x)}{I_1(x)}. \tag{91}$$

10. *The modes of deformation which are unstable.*—Combining the results of §§ 8 and 9, we have

$$\Delta\Omega + \Delta\mathfrak{M} = \left\{ 2\pi^2 \rho^2 G R^2 \left[\tfrac{1}{2} - I_0(x) K_0(x)\right] + \tfrac{1}{8} H^2 \frac{x I_0(x)}{I_1(x)} \right\} a^2. \tag{92}$$

Letting

$$H_s^2 = 16\pi^2 \rho^2 R^2 G \qquad \text{or} \qquad H_s = 4\pi\rho R\sqrt{G}, \tag{93}$$

we can rewrite equation (92) more conveniently in the form

$$\Delta\Omega + \Delta\mathfrak{M} = 2\pi^2 \rho^2 R^2 G \left\{ \left[\tfrac{1}{2} - I_0(x) K_0(x)\right] + \frac{x I_0(x)}{I_1(x)} \left(\frac{H}{H_s}\right)^2 \right\} a^2. \tag{94}$$

Whether the mode of deformation considered is stable or unstable will depend upon the sign of the quantity in braces in the foregoing expression.

Now the asymptotic behaviors of the Bessel functions which appear in equation (94) are:

$$I_0(x) \to 1, \quad I_1(x) \to \tfrac{1}{2}x, \quad \text{and} \quad K_0(x) \to -(\gamma + \log \tfrac{1}{2}x) \quad (x \to 0), \tag{95}$$

where γ (not to be confused with the ratio of the specific heats) is Euler's constant 0.5772 . . . , and

$$I_0(x) \to \frac{e^x}{(2\pi x)^{1/2}}, \quad I_1(x) \to \frac{e^x}{(2\pi x)^{1/2}}, \quad \text{and} \quad K_0(x) \to \left(\frac{\pi}{2x}\right)^{1/2} e^{-x} \quad (x \to \infty). \tag{96}$$

Hence $\Delta\Omega$ (cf. eq. [72]) tends to minus infinity logarithmically as $x \to 0$ and tends monotonically to the positive limit $\pi^2\rho^2 R^2 G$ as $x \to \infty$, while $\Delta\mathfrak{M}$ (cf. eq. [91]) tends to the positive limit $\frac{1}{4}a^2 H^2$ as $x \to 0$ and increases monotonically to infinity (linearly) as $x \to \infty$. These behaviors of $\Delta\Omega$ and $\Delta\mathfrak{M}$ are illustrated in Figure 1, in which the functions $[\frac{1}{2} - I_0(x) K_0(x)]$ and $x I_0(x)/I_1(x)$ are plotted.

From the asymptotic behaviors of $\Delta\Omega$ and $\Delta\mathfrak{M}$ it follows that the equation

$$\tfrac{1}{2} - I_0(x) K_0(x) + \frac{x I_0(x)}{I_1(x)} \left(\frac{H}{H_s}\right)^2 = 0 \tag{97}$$

allows a single positive root. Let $x = x_*$ denote this root. Then

$$\Delta\Omega + \Delta\mathfrak{M} > 0 \qquad \text{for} \qquad x > x_*, \tag{98}$$

and

$$\Delta\Omega + \Delta\mathfrak{M} < 0 \qquad \text{for} \qquad x < x_*. \tag{98a}$$

Hence *all modes of deformation with $x < x_*$ are unstable.* Since $x = kR$, x_* specifies the minimum wave number (in units of $1/R$) for a stable deformation; alternatively, we could also say that *all modes of deformation with wave lengths exceeding*

$$\lambda_* = \frac{2\pi R}{x_*} \tag{99}$$

are unstable.

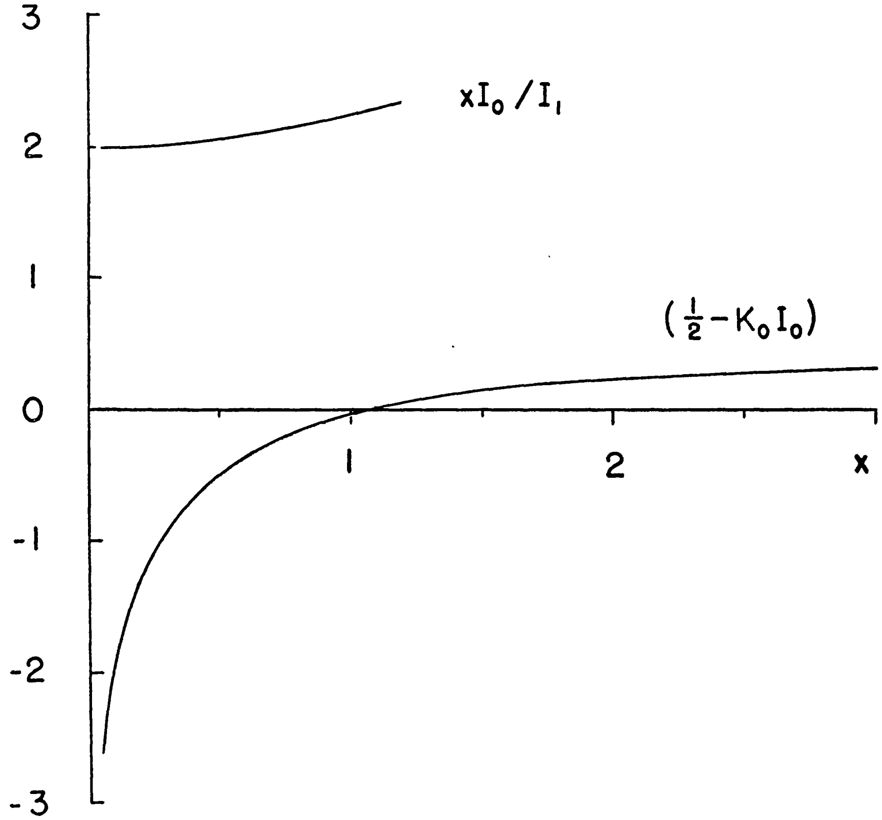

FIG. 1.—The dependence of the changes in the potential energy, $\Delta\Omega$, and magnetic energy, $\Delta\mathfrak{M}$, per unit length of an infinitely long cylinder on the wave number of the deformation; $\Delta\Omega$ is proportional to $[\frac{1}{2} - I_0(x)K_0(x)]$, while $\Delta\mathfrak{M}$ is proportional to $xI_0(x)/I_1(x)$, where x is the wave number measured in the unit $1/R$.

TABLE 1

DEPENDENCE OF WAVE NUMBERS x_* AND x_m AT WHICH INSTABILITY FIRST SETS IN AND AT WHICH IT IS MAXIMUM, ON PREVAILING MAGNETIC FIELD

H/H_s	x_*	x_m	$q_m/(4\pi G\rho)^{1/2}$
0.........	1.067	0.58	0.246
0.25.........	0.832	.47	.208
0.50.........	0.480	.28	.133
0.75.........	0.232	.14	.0685
1.00.........	0.092	.057	.0281
1.25.........	0.0299	.0182	.0091
1.50.........	0.00757	.00459	.00229
2.00.........	0.000228	0.000139	0.0000693

In Table 1 we have listed x_* for a few values of H/H_s. This table exhibits the strong stabilizing effect of the magnetic field: this is shown in the present connection by the very rapid increase, with increasing H, of the wave length at which instability sets in. In fact, for $H > H_s$ this increase becomes exponential; this can be shown in the following way:

Since $x_* = 0.092$ already for $H = H_s$, for $H > H_s$ we may replace the Bessel functions which occur in equation (97) by their dominant terms for $x \to 0$; thus,

$$\tfrac{1}{2} + \gamma + \log \tfrac{1}{2} x_* + 2\left(\frac{H}{H_s}\right)^2 = 0 \qquad (H > H_s) \,. \tag{100}$$

Hence

$$x_* = 2 \exp\left\{-\left[\gamma + \tfrac{1}{2} + 2\left(\frac{H}{H_s}\right)^2\right]\right\} \qquad (H > H_s) \,, \tag{101}$$

or, numerically,

$$x_* = 0.6811\, e^{-2(H/H_s)^2} \qquad (H > H_s) \,. \tag{102}$$

11. *The mode of maximum instability.*—In the preceding section we have seen that an infinite cylinder is gravitationally unstable for all modes of deformation with wave lengths exceeding a certain critical value. We shall now show that there exists a wave length for which the instability is a maximum. For this purpose we shall suppose that the amplitude, a, of the deformation is a function of time and seek an equation of motion for it.

We have already seen that the potential energy (gravitational plus magnetic) per unit length of the cylinder measured from the equilibrium state is

$$\mathfrak{V} = \Delta\mathfrak{M} + \Delta\Omega = -2\pi^2\rho^2 R^2 G F(x) a^2 \,, \tag{103}$$

where

$$F(x) = I_0(x) K_0(x) - \tfrac{1}{2} - \left(\frac{H}{H_s}\right)^2 \frac{x I_0(x)}{I_1(x)} \,. \tag{104}$$

Defined in this manner, $F(x) > 0$ for $x < x_*$, i.e., it is positive for all unstable modes and negative for all stable modes.

To obtain the Lagrangian function for the cylinder, we must find the kinetic energy of the motion resulting from the varying amplitude. Since we have assumed that the fluid is incompressible, a velocity potential, ψ, exists which satisfies Laplace's equation. And the solution for the velocity potential appropriate to the problem on hand is

$$\psi = B I_0(kr) \cos kz \,, \tag{105}$$

where B is a constant to be determined. The components of the velocity derived from the foregoing potential are

$$u_r = \frac{\partial \psi}{\partial r} = +B k I_1(kr) \cos kz \tag{106}$$

and

$$u_z = \frac{\partial \psi}{\partial z} = -B k I_0(kr) \sin kz \,. \tag{106a}$$

The constant of proportionality, B, in the foregoing equations must be determined from the condition that the radial velocity, u_r, at $r = R$ must agree with that implied by equation (52); i.e., we should have

$$B k I_1(kR) \cos kz = \frac{da}{dt} \cos kz \,. \tag{107}$$

Hence

$$B = \frac{1}{k I_1(x)} \frac{da}{dt} \,. \tag{108}$$

From equations (106) we obtain, for the kinetic energy per unit length, the expression (cf. eq. [69])

$$\begin{aligned}\mathfrak{T} &= \tfrac{1}{2}\pi\rho B^2 k^2 \int_0^R [I_0^2(kr) + I_1^2(kr)]\, r\, dr \\ &= \tfrac{1}{2}\pi\rho B^2 k^2 \frac{R}{k} I_0(x) I_1(x)\,,\end{aligned} \tag{109}$$

or, substituting for B from equation (108), we have

$$\mathfrak{T} = \tfrac{1}{2}\pi\rho R^2 \frac{I_0(x)}{x I_1(x)} \left(\frac{da}{dt}\right)^2. \tag{110}$$

The Lagrangian function (per unit length) for the infinite cylinder is therefore given by

$$\mathfrak{L} = \mathfrak{T} - \mathfrak{B} = \tfrac{1}{2}\pi\rho R^2 \frac{I_0(x)}{x I_1(x)} \left(\frac{da}{dt}\right)^2 + 2\pi^2\rho^2 R^2 G F(x)\, a^2. \tag{111}$$

The equation of motion for a derived from the Lagrangian (111) is

$$\pi\rho R^2 \frac{I_0(x)}{x I_1(x)} \frac{d^2 a}{dt^2} = 4\pi^2\rho^2 R^2 G F(x)\, a\,, \tag{112}$$

or, alternatively,

$$\frac{d^2 a}{dt^2} = 4\pi G\rho \left\{\frac{x I_1(x)}{I_0(x)} [I_0(x) K_0(x) - \tfrac{1}{2}] - \left(\frac{H}{H_s}\right)^2 x^2\right\} a\,, \tag{113}$$

where we have substituted for $F(x)$ in accordance with equation (104). The solution for a is therefore of the form

$$a = \text{Constant } e^{\pm qt}\,, \tag{114}$$

where

$$q^2 = 4\pi G\rho \left\{\frac{x I_1(x)}{I_0(x)} [I_0(x) K_0(x) - \tfrac{1}{2}] - \left(\frac{H}{H_s}\right)^2 x^2\right\}. \tag{115}$$

Accordingly, q is purely imaginary for $x > x_*$ and is real for $x < x_*$; this is in agreement with the fact that all modes with $x > x_*$ are stable, while all modes with $x < x_*$ are unstable.

As defined by equation (115), $q = 0$ both for $x = 0$ and for $x = x_*$. There is, therefore, a determinate intermediate value of x—say, x_m—for which q attains a maximum—say, q_m. *The wave number x_m clearly represents the mode of maximum instability;* for it is the mode for which the amplitude of the deformation increases most rapidly. The wave length

$$\lambda_m = \frac{2\pi R}{x_m}\,, \tag{116}$$

corresponding to the wave number x_m, gives approximately the length of the "pieces" into which the cylinder will ultimately break up: for the component with the wave length λ_m, in the Fourier analysis of an arbitrary perturbation, is the one whose amplitude will increase most rapidly with time and, therefore, represents the mode in which the instability will first assert itself. Finally, it is clear that $1/q_m$ gives a measure of the time needed for the instability to make itself manifest.

In Table 1 the values of x_m and $q_m/(4\pi G\rho)^{1/2}$ are also listed. As in the case of x_* (§ 10), we can give explicit formulae for x_m and q_m for $H > H_s$. Since for $H > H_s$ we are concerned only with values of $x \ll 1$, we may replace the Bessel functions which occur in the expression for q^2 by their dominant terms for $x \to 0$. Thus

$$q^2 = 4\pi G\rho \left\{ -\tfrac{1}{2}x^2 (\gamma + \tfrac{1}{2} + \log \tfrac{1}{2}x) - x^2 \left(\frac{H}{H_s}\right)^2 \right\} \qquad (H > H_s) . \tag{117}$$

The expression on the right-hand side attains its maximum when

$$(\gamma + \tfrac{1}{2} + \log \tfrac{1}{2}x) + \tfrac{1}{2} + 2 \left(\frac{H}{H_s}\right)^2 = 0 . \tag{118}$$

Hence

$$x_m = 2 \exp \left\{ -(\gamma + 1) - 2 \left(\frac{H}{H_s}\right)^2 \right\} = 0.4131\, e^{-2(H/H_s)^2} \qquad (H > H_s) . \tag{119}$$

The corresponding expression for q_m is

$$q_m = \tfrac{1}{2}x_m (4\pi G\rho)^{1/2} \qquad (H > H_s) . \tag{120}$$

These formulae emphasize the fact, apparent from an examination of Table 1, that, as the strength of the magnetic field increases, not only does the wave length of the mode of maximum instability increase exponentially, but the time needed for the instability to manifest itself also increases exponentially.

TABLE 2

WAVE LENGTHS λ_* AND λ_m AT WHICH INSTABILITY SETS IN AND AT WHICH IT IS MAXIMUM AND CHARACTERISTIC TIME, q_m^{-1}, NEEDED FOR INSTABILITY TO MANIFEST ITSELF FOR CASE $R = 250$ PARSECS AND $\rho = 2 \times 10^{-24}$ GM/CM^3

H (Gauss)	λ_* (Parsecs)	λ_m (Parsecs)	q_m^{-1} (Years)
0	1.5×10^3	2.7×10^3	1.0×10^8
1.25×10^{-6}	1.9×10^3	3.3×10^3	1.2×10^8
2.5×10^{-6}	3.3×10^3	5.6×10^3	1.8×10^8
3.75×10^{-6}	6.8×10^3	1.1×10^4	3.6×10^8
5.0×10^{-6}	1.7×10^4	2.8×10^4	8.7×10^8
6.25×10^{-6}	5.2×10^4	8.6×10^4	2.7×10^9
7.5×10^{-6}	2.1×10^5	3.4×10^5	1.1×10^{10}
10.0×10^{-6}	6.9×10^6	1.1×10^7	3.5×10^{11}

12. *Numerical illustrations*.—To illustrate the theory developed in the preceding sections we shall take, as typical of a spiral arm of a galaxy,

$$R = 250 \text{ parsecs} \quad \text{and} \quad \rho = 2 \times 10^{-24} \text{ gm/cm}^3 . \tag{121}$$

The corresponding value of H_s is (cf. eq. [93])

$$H_s = 5.0 \times 10^{-6} \text{ gauss} . \tag{122}$$

For these values of the physical parameters, the nondimensional results given in Table 1 can be converted into astronomical measures; they are given in Table 2. From the values given in this table it follows that between $H = H_s$ and $H = 2H_s$ the characteristic time of the instability becomes so long that, for all practical purposes, the instability is effectively removed by the presence of the magnetic field.

IV. THE FLATTENING OF A GRAVITATING FLUID SPHERE UNDER THE INFLUENCE OF A MAGNETIC FIELD

13. *The formulation of the problem.*—In this section we shall consider the gravitational equilibrium of an incompressible fluid sphere with a uniform magnetic field inside and a dipole field outside. We shall show that under these circumstances the sphere is not a configuration of equilibrium and that it will become oblate by contracting along the axis of symmetry.

We suppose, then, that initially the magnetic field in the interior of the sphere is uniform and of intensity H in the z-direction. In spherical polar co-ordinates (r, θ, φ) the components of $\boldsymbol{H}$ in the radial (r) and the transverse (θ) directions are

$$H_r^{(i)} = H\mu \quad \text{and} \quad H_\theta^{(i)} = -H(1-\mu^2)^{1/2} \qquad (r < R), \tag{123}$$

where $\mu = \cos\theta$ and the superscript i indicates that these are the components of the field *inside* the sphere.

When the field inside the sphere is uniform, that outside the sphere must be a dipole field given by

$$H_r^{(e)} = H\left(\frac{R}{r}\right)^3 \mu \quad \text{and} \quad H_\theta^{(e)} = \tfrac{1}{2} H\left(\frac{R}{r}\right)^3 (1-\mu^2)^{1/2}, \tag{124}$$

where R denotes the radius of the sphere.

The energy, $\mathfrak{M}$, of the magnetic field specified by equations (123) and (124) is given by

$$\begin{aligned}\mathfrak{M} &= \frac{H^2}{8\pi}\left(\tfrac{4}{3}\pi R^3\right) + \tfrac{1}{4} H^2 \int_R^\infty \int_{-1}^{+1} \left(\frac{R}{r}\right)^6 \left\{\mu^2 + \tfrac{1}{4}(1-\mu^2)\right\} r^2 dr\, d\mu \\ &= \tfrac{1}{4} H^2 R^3 .\end{aligned} \tag{125}$$

Let the sphere be now deformed in such a way that the equation of the bounding surface is

$$r(\mu) = R + \epsilon P_l(\mu), \tag{126}$$

where $\epsilon \ll R$, $\mu = \cos\theta$ (θ being the polar angle), and $P_l(\mu)$ denotes, as usual, the Legendre polynomial of order l. We shall call such a deformation of the sphere a "P_l-deformation." We shall investigate the stability of the sphere by examining whether or not it is stable to a P_l-deformation.

14. *The change in the magnetic energy of the sphere due to a P_l-deformation.*—As we have already pointed out in § 8, an arbitrary deformation of an incompressible body can be thought of as the result of applying to each point of the body a displacement $\boldsymbol{\xi}$. And if, as in § 8 (eqs. [59] and [60]), we express $\boldsymbol{\xi}$ as the gradient of a scalar function, ψ, the solution of Laplace's equation satisfied by ψ appropriate to a P_l-deformation of a sphere is

$$\psi = A r^l P_l(\mu), \tag{127}$$

where A is a constant. The corresponding expressions for the components of $\boldsymbol{\xi}$ are

$$\xi_r = \frac{\partial\psi}{\partial r} = A l r^{l-1} P_l(\mu) \tag{128}$$

and

$$\xi_\theta = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = -A r^{l-1}(1-\mu^2)^{1/2} P_l'(\mu), \tag{128a}$$

where a prime is used to denote differentiation with respect to μ. According to equation (126), at $r = R$, $\xi_r = \epsilon P_l(\mu)$; this determines A, and we have

$$\xi_r = \epsilon \left(\frac{r}{R}\right)^{l-1} P_l(\mu) \quad \text{and} \quad \xi_\theta = -\frac{\epsilon}{l}\left(\frac{r}{R}\right)^{l-1} (1-\mu^2)^{1/2} P_l'(\mu) \,. \tag{129}$$

Now the deformation of a body will alter the prevailing magnetic field; and, since in a medium of infinite electrical conductivity a change in the existing magnetic field can be effected only by bodily pushing aside the lines of force, it follows that

$$\delta \boldsymbol{H} = \text{curl}\,(\boldsymbol{\xi} \times \boldsymbol{H}) \,. \tag{130}$$

[The truth of this last relation can be established in the following way: Suppose that the displacement $\boldsymbol{\xi}$ takes place as a slow continuous movement so that if $\boldsymbol{u}$ denotes the velocity, $\boldsymbol{u} = \partial \boldsymbol{\xi}/\partial t$ (i.e., if quantities of the second order of smallness are neglected). On the other hand, when the electrical conductivity is infinite,

$$\delta \boldsymbol{E} = -\boldsymbol{u} \times \boldsymbol{H} \,,$$

where $\delta \boldsymbol{E}$ is the electrical field resulting from the changing magnetic field $\delta \boldsymbol{H}$ in accordance with Maxwell's equation,

$$\text{curl}\ \delta \boldsymbol{E} = -\frac{\partial}{\partial t}\,\delta \boldsymbol{H} \,.$$

Combining the last two equations, we have

$$\text{curl}\left(\frac{\partial \boldsymbol{\xi}}{\partial t} \times \boldsymbol{H}\right) = \frac{\partial}{\partial t}(\delta \boldsymbol{H}) \,.$$

The relation (130) is simply the integrated form of this equation.]

When the fluid is incompressible (i.e., when div $\boldsymbol{\xi} = 0$ in addition to div $\boldsymbol{H} = 0$), equation (130) can be written alternatively in the form

$$\delta \boldsymbol{H} = (\boldsymbol{H}\cdot \text{grad})\boldsymbol{\xi} - (\boldsymbol{\xi}\cdot\text{grad})\boldsymbol{H} \,. \tag{131}$$

And when the initial field is homogeneous, equation (131) simplifies still further to

$$\delta \boldsymbol{H} = (\boldsymbol{H}\cdot\text{grad})\boldsymbol{\xi} \,. \tag{132}$$

In spherical polar co-ordinates the foregoing equation is equivalent to

$$\delta H_r = \left(H_r \frac{\partial}{\partial r} + \frac{H_\theta}{r}\frac{\partial}{\partial \theta}\right)\xi_r - \frac{H_\theta \xi_\theta}{r} \tag{133}$$

and

$$\delta H_\theta = \left(H_r \frac{\partial}{\partial r} + \frac{H_\theta}{r}\frac{\partial}{\partial \theta}\right)\xi_\theta + \frac{H_\theta \xi_r}{r} \,. \tag{133a}$$

These equations in conjunction with equations (123) and (129) give

$$\delta H_r^{(i)} = \epsilon H\,(l-1)\frac{r^{l-2}}{R^{l-1}} P_{l-1}(\mu) \tag{134}$$

and

$$\delta H_\theta^{(i)} = -\epsilon H \frac{r^{l-2}}{R^{l-1}}(1-\mu^2)^{1/2} P_{l-1}'(\mu) \,. \tag{134a}$$

The corresponding change in the internal magnetic energy density is given by

$$\delta\left(\frac{|\boldsymbol{H}|^2}{8\pi}\right)=\frac{1}{4\pi}\boldsymbol{H}^{(i)}\cdot\delta\boldsymbol{H}^{(i)}$$
$$=\epsilon\frac{H^2}{4\pi}\frac{r^{l-2}}{R^{l-1}}\{(l-1)\mu P_{l-1}(\mu)+(1-\mu^2)P'_{l-1}(\mu)\}\,. \tag{135}$$

On further simplification this reduces to

$$\delta\left(\frac{|\boldsymbol{H}|^2}{8\pi}\right)=\epsilon(l-1)\frac{H^2}{4\pi}\frac{r^{l-2}}{R^{l-1}}P_{l-2}(\mu)\,. \tag{136}$$

Hence, when averaged over all directions, this is zero except when $l = 2$, in which case

$$\delta\left(\frac{|\boldsymbol{H}|^2}{8\pi}\right)=\frac{\epsilon}{4\pi}\frac{H^2}{R}\qquad(l=2)\,; \tag{137}$$

the corresponding change in the internal magnetic energy, $\Delta\mathfrak{M}^{(i)}$, is given by

$$\Delta\mathfrak{M}^{(i)}=\tfrac{1}{3}\epsilon H^2R^2\,. \tag{138}$$

15. *The change in the external magnetic energy of the sphere due to a P_l-deformation.*—Writing the magnetic field outside the deformed sphere in the form

$$H_r^{(e)}=H\left(\frac{R}{r}\right)^3\mu+\delta H_r^{(e)} \tag{139}$$

and

$$H_\theta^{(e)}=\tfrac{1}{2}H\left(\frac{R}{r}\right)^3(1-\mu^2)^{1/2}+\delta H_\theta^{(e)}\,, \tag{139a}$$

we shall suppose that $\delta H_r^{(e)}$ and $\delta H_\theta^{(e)}$ are derivable from a magnetic potential $\delta\phi^{(e)}$. Since the magnetic potential satisfies Laplace's equation, the solution for $\delta\phi^{(e)}$ must be expressible as a linear combination of the fundamental solutions $P_j(\mu)/r^{j+1}$, which vanish at infinity.

We shall find it convenient to write the solution for $\delta\phi^{(e)}$ in the form

$$\delta\phi^{(e)}=-\epsilon H\left\{\frac{l-1}{l}\left(\frac{R}{r}\right)^l P_{l-1}(\mu)+\Sigma A_j\left(\frac{R}{r}\right)^{j+1}P_j(\mu)\right\}, \tag{140}$$

where the A_j's are coefficients to be determined. The expressions for $\delta H_r^{(e)}$ and $\delta H_\theta^{(e)}$ derived from this potential are

$$\delta H_r^{(e)}=\epsilon H\left\{(l-1)\frac{R^l}{r^{l+1}}P_{l-1}(\mu)+\Sigma A_j(j+1)\frac{R^{j+1}}{r^{j+2}}P_j(\mu)\right\} \tag{141}$$

and

$$\delta H_\theta^{(e)}=\epsilon H\left\{\frac{l-1}{l}\frac{R^l}{r^{l+1}}P_{l-1}^1(\mu)+\Sigma A_j\frac{R^{j+1}}{r^{j+2}}P_j^1(\mu)\right\}. \tag{141a}$$

The coefficients A_j in equations (141) and (141a) can be determined from the condition that the component of the magnetic field normal to a bounding surface must be continuous. To the first order in ϵ this condition requires that

$$\{H_r^{(e)}\}_{R+\epsilon P_l}+\{H_\theta^{(e)}\}_R\frac{\epsilon}{R}(1-\mu^2)^{1/2}\frac{\partial P_l}{\partial\mu}$$
$$=\{H_r^{(i)}\}_{R+\epsilon P_l}+\{H_\theta^{(i)}\}_R\frac{\epsilon}{R}(1-\mu^2)^{1/2}\frac{\partial P_l}{\partial\mu}, \tag{142}$$

where $-(\epsilon/R)(1-\mu^2)^{1/2}\partial P_l/\partial\mu$ is the angle (to the first order in ϵ) which the deformed boundary makes with the θ-direction; the terms in H_θ in the foregoing equation arise from this latter circumstance. Now, according to equations (124), (139), and (140),

$$\{H_r^{(e)}\}_{R+\epsilon P_l}+\{H_\theta^{(e)}\}_R\frac{\epsilon}{R}(1-\mu^2)^{1/2}\frac{\partial P_l}{\partial\mu}=H\mu\left(1-3\frac{\epsilon}{R}P_l\right)$$
$$+\tfrac{1}{2}H\frac{\epsilon}{R}(1-\mu^2)\frac{\partial P_l}{\partial\mu}+\frac{\epsilon}{R}H\{(l-1)P_{l-1}+\Sigma A_j(j+1)P_j\}\,, \tag{143}$$

while, according to equations (123) and (134),

$$\{H_r^{(i)}\}_{R+\epsilon P_l}+\{H_\theta^{(i)}\}_R\frac{\epsilon}{R}(1-\mu^2)^{1/2}\frac{\partial P_l}{\partial\mu}=H\mu$$
$$+\frac{\epsilon}{R}H(l-1)P_{l-1}-\frac{\epsilon}{R}H(1-\mu^2)\frac{\partial P_l}{\partial\mu}; \tag{143a}$$

and the equality of the expressions on the right-hand sides of equations (143) and (143a) requires

$$\Sigma A_j(j+1)P_j=3\mu P_l-\tfrac{3}{2}(1-\mu^2)\frac{\partial P_l}{\partial\mu}$$
$$=\frac{3}{2(2l+1)}\{(l+1)(l+2)P_{l+1}-l(l-1)P_{l-1}\}\,. \tag{144}$$

Hence

$$A_{l-1}=-\frac{3(l-1)}{2(2l+1)}\,,\qquad A_{l+1}=\frac{3(l+1)}{2(2l+1)}\,, \tag{145}$$

and

$$A_j=0\qquad\text{for}\qquad j\neq l-1\qquad\text{or}\qquad l+1\,. \tag{145a}$$

Inserting these values of A in equations (141) and (141a), we obtain

$$\delta H_r^{(e)}=\epsilon H\left\{\frac{(l-1)(l+2)}{2(2l+1)}\frac{R^l}{r^{l+1}}P_{l-1}(\mu)+\frac{3(l+1)(l+2)}{2(2l+1)}\frac{R^{l+2}}{r^{l+3}}P_{l+1}(\mu)\right\} \tag{146}$$

and

$$\delta H_\theta^{(e)}=\epsilon H\left\{\frac{(l-1)(l+2)}{2l(2l+1)}\frac{R^l}{r^{l+1}}P_{l-1}^1(\mu)+\frac{3(l+1)}{2(2l+1)}\frac{R^{l+2}}{r^{l+3}}P_{l+1}^1(\mu)\right\}. \tag{146a}$$

Returning to equations (139) and (139a), we can write the change in the external magnetic energy, $\Delta\mathfrak{M}^{(e)}$, to the first order in ϵ, in the form

$$\Delta\mathfrak{M}^{(e)}=\frac{H^2}{8\pi}\iiint_{R+\epsilon P_l\geq r\geq R}\left(\frac{R}{r}\right)^6\{\mu^2+\tfrac{1}{4}(1-\mu^2)\}r^2dr\,d\mu\,d\varphi$$
$$+\frac{H}{8\pi}\iiint_{r>R}\left(\frac{R}{r}\right)^3\{2P_1(\mu)\,\delta H_r^{(e)}+P_1^1(\mu)\,\delta H_\theta^{(e)}\}r^2dr\,d\mu\,d\varphi\,. \tag{147}$$

After some minor reductions we find

$$\Delta\mathfrak{M}^{(e)} = \tfrac{1}{4}\epsilon H^2 R^2 \int_{-1}^{+1} \{\tfrac{1}{2}P_2(\mu) + \tfrac{1}{2}\} P_l(\mu)\, d\mu$$

$$+ \tfrac{1}{2}\epsilon H^2 \int_R^\infty dr\, r^2 \int_{-1}^{+1} d\mu \left(\frac{R}{r}\right)^3 P_1(\mu) \left\{ \frac{(l-1)\,(l+2)}{2\,(2l+1)} \frac{R^l}{r^{l+1}} P_{l-1}(\mu) + \frac{3\,(l+1)\,(l+2)}{2\,(2l+1)} \frac{R^{l+2}}{r^{l+3}} P_{l+1}(\mu) \right\}$$

$$+ \tfrac{1}{4}\epsilon H^2 \int_R^\infty dr\, r^2 \int_{-1}^{+1} d\mu \left(\frac{R}{r}\right)^3 P_1^1(\mu) \left\{ \frac{(l-1)\,(l+2)}{2l\,(2l+1)} \frac{R^l}{r^{l+1}} P_{l-1}^1(\mu) + \frac{3\,(l+1)}{2\,(2l+1)} \frac{R^{l+2}}{r^{l+3}} P_{l+1}^1(\mu) \right\}. \tag{148}$$

From this equation it is evident that $\Delta\mathfrak{M}^{(e)}$ vanishes (to the first order in ϵ) for all deformations except a P_2-deformation. And for a P_2-deformation we have

$$\Delta\mathfrak{M}^{(e)} = \tfrac{1}{8}\epsilon H^2 R^2 \int_{-1}^{+1} [P_2(\mu)]^2 d\mu + \tfrac{1}{5}\epsilon H^2 R^5 \int_R^\infty \int_{-1}^{+1} \frac{dr}{r^4} \{\tfrac{1}{2}P_2(\mu) + \tfrac{1}{2}\}\, d\mu \tag{149}$$

or

$$\Delta\mathfrak{M}^{(e)} = \tfrac{7}{60}\,\epsilon H^2 R^2 \qquad (l=2)\,. \tag{150}$$

Finally, combining equations (138) and (150), we obtain

$$\Delta\mathfrak{M} = \Delta\mathfrak{M}^{(i)} + \Delta\mathfrak{M}^{(e)} = \tfrac{9}{20}\,\epsilon H^2 R^2\,, \tag{151}$$

for the total change in the magnetic energy due to a P_2-deformation; it vanishes to this order for all higher deformations.

We have, therefore, shown that *the change in the magnetic energy is of the second order in ϵ for all deformations of the sphere except a P_2-deformation; and for a P_2-deformation it is of the first order in ϵ and is given by (151)*. Moreover, for a P_2-deformation $\Delta\mathfrak{M} > 0$ when the deformation is in the sense of making the sphere into a prolate spheroid; and $\Delta\mathfrak{M} < 0$ when the deformation is in the sense of making the sphere into an oblate spheroid.

16. *The change in the gravitational potential energy and the instability of the sphere to a P_2-deformation.*—The change in the potential energy, $\Delta\Omega$, due to a P_l-deformation can also be computed. The result is well known for a P_2-deformation. For a general P_l-deformation we can evaluate $\Delta\Omega$ by following the procedure used in § 8. We shall not give here the details of the calculations, which lead to the result

$$\Delta\Omega = \frac{3\,(l-1)}{(2l+1)^2} \left(\frac{\epsilon}{R}\right)^2 \frac{GM^2}{R}\,. \tag{152}$$

The change in the potential energy is therefore always positive and is of the second order in ϵ. This is in contrast to $\Delta\mathfrak{M}$, which, as we have seen, is of the first order in ϵ for a P_2-deformation and is negative for a deformation which tends to make it oblate. We can therefore conclude that the sphere is unstable and that it will tend to collapse toward an oblate spheroidal shape. To estimate the extent to which this collapse may proceed, let us consider $\Delta\Omega + \Delta\mathfrak{M}$ for a P_2-deformation. We have (cf. eqs. [151] and [152])

$$\Delta\Omega + \Delta\mathfrak{M} = \frac{3}{25}\frac{GM^2}{R^3}\epsilon^2 + \tfrac{9}{20} H^2 R^2 \epsilon \qquad (l=2)\,. \tag{153}$$

As a function of ϵ, $\Delta\Omega + \Delta\mathfrak{M}$ has a minimum which it takes when

$$\frac{6}{25}\frac{GM^2}{R^3}\epsilon + \frac{9}{20}H^2R^2 = 0 , \tag{154}$$

or

$$\frac{\epsilon}{R} = -\frac{15}{8}\frac{H^2R^4}{GM^2} . \tag{155}$$

If H_* denotes the value of the constant magnetic field inside the sphere for which $\mathfrak{M}$ (given by eq. [125]) is equal to the numerical value of the gravitational potential energy Ω $(= -3GM^2/5R)$, then

$$\tfrac{1}{4}H_*^2R^3 = \frac{3}{5}\frac{GM^2}{R} . \tag{156}$$

In terms of H_* defined in this manner, we can rewrite equation (155) in the form

$$\frac{\epsilon}{R} = -4.5\left(\frac{H}{H_*}\right)^2 . \tag{157}$$

We may interpret this relation by saying that when a star has a magnetic field approaching the limit set by the virial theorem (cf. Sec. I), then it tends to become highly oblate; in this respect the magnetic field has the same effect as a rotation.

V. THE GRAVITATIONAL INSTABILITY OF AN INFINITE HOMOGENEOUS MEDIUM IN THE PRESENCE OF A MAGNETIC FIELD

17. *The statement of the problem.*—It is well known that, by considering the propagation of a wave in an infinite homogeneous medium and allowing for the gravitational effects of the density fluctuations, Jeans[6] showed that the velocity of wave propagation is given by

$$V_J = \sqrt{(c^2 - 4\pi G\rho/k^2)} , \tag{158}$$

where $c = \sqrt{(\gamma p/\rho)}$ denotes the convectional velocity of sound and k is the wave number. Accordingly, when

$$k < c(4\pi\rho G)^{-1/2} , \tag{159}$$

the velocity of wave propagation becomes imaginary; and under these circumstances the amplitude of the wave will increase exponentially with time. The inequality (159) is therefore the condition for gravitational instability; this is Jeans's result. In Section V we shall show that Jeans's condition (159) is unaffected by the presence of a magnetic field. The physical reason for this is evident for a deformation in which the density waves are perpendicular to the lines of force because the motion of the particles in this case will be parallel to the lines of force and therefore will not be impeded by the magnetic field. But also a density wave forming an angle with the lines of force may be obtained by particle motions parallel to the lines of force, as shown in Figure 2.

18. *The three modes of wave propagation in the presence of a magnetic field and the condition for gravitational instability.*—Consider an extended homogeneous gaseous medium of infinite electrical conductivity, and suppose that there is present a uniform magnetic

[6] *Astronomy and Cosmogony* (Cambridge: At the University Press, 1929), pp. 345–347.

field of intensity $\boldsymbol{H}$. Then the fluctuations in density ($\delta\rho$), pressure (δp), magnetic field ($\boldsymbol{h}$), and gravitational potential (δV) are governed by the equations

$$\rho\frac{\partial \boldsymbol{u}}{\partial t}=\frac{1}{4\pi}(\text{curl } \boldsymbol{h}\times\boldsymbol{H})-\text{grad } \delta p+\rho \text{ grad } \delta V\,,$$

$$\frac{\partial \boldsymbol{h}}{\partial t}=\text{curl }(\boldsymbol{u}\times\boldsymbol{H})\,, \tag{160}$$

$$\frac{\partial}{\partial t}\delta\rho=-\rho \text{ div } \boldsymbol{u}\,,$$

and

$$\nabla^2\delta V=-4\pi G\delta\rho\,.$$

If the changes in pressure and density are assumed to take place adiabatically, then

$$\delta p=c^2\delta\rho\,. \tag{161}$$

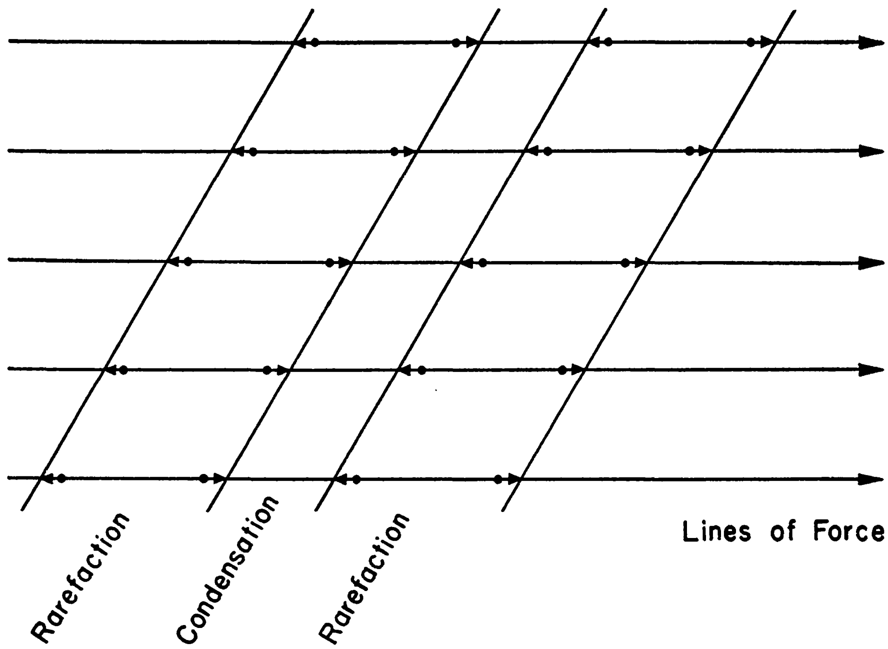

FIG. 2.—Illustrating why the presence of a magnetic field does not affect Jeans's condition for the gravitational instability of an infinite homogeneous medium.

We shall seek the solutions of equations (160) and (161) which correspond to the propagation of waves in the z-direction. Then $\partial/\partial z$ is the only nonvanishing component of the gradient. And if we further suppose that the orientation of the co-ordinate axes is so chosen that

$$\boldsymbol{H}=(0, H_y, H_z)\,, \tag{162}$$

it readily follows that $h_z=0$; and we find that equations (160) and (161) break up into the two noncombining systems:

$$\rho\frac{\partial u_x}{\partial t}=\frac{H_z}{4\pi}\frac{\partial h_x}{\partial z}\,,\qquad \frac{\partial h_x}{\partial t}=H_z\frac{\partial u_x}{\partial z}\,; \tag{163}$$

and

$$\rho \frac{\partial u_y}{\partial t} - \frac{H_z}{4\pi} \frac{\partial h_y}{\partial z} = 0 ,$$

$$\rho \frac{\partial u_z}{\partial t} + \frac{H_y}{4\pi} \frac{\partial h_y}{\partial z} + c^2 \frac{\partial}{\partial z} \delta \rho - \rho \frac{\partial}{\partial z} \delta V = 0 ,$$

$$\frac{\partial h_y}{\partial t} + H_y \frac{\partial u_z}{\partial z} - H_z \frac{\partial u_y}{\partial z} = 0 , \tag{164}$$

$$\frac{\partial}{\partial t} \delta \rho + \rho \frac{\partial u_z}{\partial z} = 0 ,$$

$$\frac{\partial^2}{\partial z^2} \delta V + 4\pi G \delta \rho = 0 .$$

Equations (163) can be combined to give

$$\frac{\partial^2 h_x}{\partial t^2} = \frac{H_z^2}{4\pi\rho} \frac{\partial^2 h_x}{\partial z^2} \quad \text{and} \quad \frac{\partial^2 u_x}{\partial t^2} = \frac{H_z^2}{4\pi\rho} \frac{\partial^2 u_x}{\partial t^2} . \tag{165}$$

These equations are the same as those leading to the ordinary hydromagnetic waves of Alfvén propagated with the velocity

$$V_A = \frac{H_z}{\sqrt{(4\pi\rho)}} . \tag{166}$$

This mode of wave propagation is therefore unaffected by gravitation and compressibility.

Turning next to solutions of equations (164), which also represent the propagation of waves in the z-direction, we can write

$$\frac{\partial}{\partial t} = i\omega \quad \text{and} \quad \frac{\partial}{\partial z} = -ik , \tag{167}$$

where ω denotes the frequency and k the wave number. Making the substitutions (167) in equations (164), we obtain a system of linear homogeneous equations which can be written in matrix notation in the following form:

$$\left\| \begin{array}{ccccc} \rho\omega & k\dfrac{H_z}{4\pi} & 0 & 0 & 0 \\ 0 & -k\dfrac{H_y}{4\pi} & \rho\omega & -kc^2 & k\rho \\ kH_z & \omega & -kH_y & 0 & 0 \\ 0 & 0 & -k\rho & \omega & 0 \\ 0 & 0 & 0 & 4\pi G & -k^2 \end{array} \right\| \left| \begin{array}{c} u_y \\ h_y \\ u_z \\ \delta\rho \\ \delta V \end{array} \right| = 0 . \tag{168}$$

The condition that equation (168) has a nontrivial solution is that the determinant of the matrix on the left-hand side should vanish. Expanding the determinant, we find that it can be reduced to the form

$$\left(\frac{\omega}{k}\right)^4 - \left\{\frac{H^2}{4\pi\rho} + \left(c^2 - \frac{4\pi G\rho}{k^2}\right)\right\}\left(\frac{\omega}{k}\right)^2 + \frac{H_z^2}{4\pi\rho}\left(c^2 - \frac{4\pi G\rho}{k^2}\right) = 0\,. \tag{169}$$

In terms of the velocity of wave propagation, $V = \omega/k$, we can rewrite equation (169) in the form

$$V^4 - (V_A^2 \sec^2\vartheta + V_J^2)V^2 + V_A^2 V_J^2 = 0\,, \tag{170}$$

where ϑ denotes the angle between the directions of $\boldsymbol{H}$ and of wave propagation and V_J and V_A have the same meanings as in equations (158) and (166).

It is seen that equation (170) allows two modes of wave propagation. If V_1 and V_2 denote the velocities of propagation of these two modes, we conclude from equation (170) that

$$V_1 V_2 = V_A V_J$$

and

$$V_1^2 + V_2^2 = V_A^2 \sec^2\vartheta + V_J^2\,. \tag{171}$$

Accordingly, *if V_J is imaginary, then either V_1 or V_2 must be imaginary*. In other words, one of the two modes of wave propagation will be unstable if Jeans's condition (159) is satisfied. The condition for gravitational instability is therefore unaffected by the presence of the magnetic field. However, as to which of the two modes will become unstable will depend on the strength of the magnetic field. Thus for $H \to 0$, the two modes given by equation (170) approach, respectively, Jeans's mode and Alfvén's mode. And if we suppose that

$$V_1 \to V_J \quad \text{and} \quad V_2 \to V_A \quad \text{as} \quad H \to 0\,, \tag{172}$$

then it follows from equation (170) that so long as $V_J^2 > 0$,

$$V_1 \to V_A \sec\vartheta \quad \text{and} \quad V_2 \to V_J \cos\vartheta \quad \text{as} \quad H \to \infty\,. \tag{173}$$

Hence, for $H \to \infty$, the mode which will become unstable when Jeans's condition is satisfied will be the mode which for $H \to 0$ is Alfvén's mode; and the mode which for $H \to 0$ is Jeans's mode becomes a hydromagnetic wave for $H \to \infty$ and is unaffected by gravitation. This "crossing-over" of the two modes with increasing strength of the magnetic field is in agreement with what is known[7] from the theory of wave propagation in a compressible medium in the absence of gravitation.

[7] Cf. H. van de Hulst, *Symposium: Problems of Cosmical Aerodynamics* (Dayton, Ohio: Central Air Documents Office, 1951), chap. vi; also N. Herlofson, *Nature*, **165**, 1020, 1950.