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ON THE STABILITY OF AXISYMMETRIC SYSTEMS TO AXISYMMETRIC PERTURBATIONS IN GENERAL RELATIVITY. IV. ALLOWANCE FOR GRAVITATIONAL RADIATION IN AN ODD-PARITY MODE

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ABSTRACT

In the present paper, the variational principle derived in Paper II is clarified; and it is shown how it may be used to treat the damping of the axisymmetric oscillations of a uniformly rotating star, by the emission of gravitational radiation in an odd-parity mode. It is further shown that the expression, for the imaginary part of the frequency as a surface integral (at infinity), which follows from the variational principle, is consistent with the requirements of the conservation of energy.
Subject headings: gravitation — relativity — rotation

I. INTRODUCTION

In the second paper of this series (Chandrasekhar and Friedman 1972*a*, *b*, *c*; these papers will be referred to hereafter as Papers I, II, and III, respectively) a general variational principle was derived for determining the characteristic frequencies of axisymmetric oscillations of uniformly rotating stars in general relativity. In the applications of the principle, attention was restricted to the case of slow rotation and to obtaining a criterion for the onset of instability via a neutral mode of quasi-stationary deformation, i.e., to situations in which the emission of gravitational radiation plays no role. In this paper, in considering the more general situations in which gravitational radiation does play a role, we shall restrict ourselves to the case in which gravitational radiation is emitted in a mode of odd parity, i.e., a mode which leads to vanishing radiation in the limit of slow rotation. We shall not be concerned in this paper with the modes of radiation which in the limit of slow rotation tend to the even-parity modes of nonradial oscillations (with "m" = 0) of spherical stars; we postpone their consideration to a later paper.

II. THE OUTLINE OF THE PROCEDURE

The basic equations of the problem have been assembled in Paper II (§ II). They govern the evolution of axisymmetric perturbations of a fluid system initially axisymmetric, uniformly rotating with an angular velocity Ω, and described by a metric of the form

$$ds^2 = -e^{2\nu}(dt)^2 + e^{2\psi}(d\phi - \omega dt)^2 + e^{2\mu_2}(dx^2)^2 + e^{2\mu_3}(dx^3)^2 . \qquad (1)^1$$

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[1] For the sake of brevity we shall not define all the symbols: it would be invidious to attempt it. And moreover, the present paper cannot, in any case, be followed without a detailed familiarity with Papers I and II: the present paper takes off where Paper II leaves (in § IV).

The perturbed configuration is described by equations that are appropriate to the metric

$$ds^2 = -e^{2\nu}(dt)^2 + e^{2\psi}(d\phi - \omega dt - q_{2,0}dx^2 - q_{3,0}dx^3)^2 + e^{2\mu_2}(dx^2)^2 + e^{2\mu_3}(dx^3)^2 , \quad (2)$$

linearized about equations appropriate to the metric (1).

The equations that describe the evolution of the perturbation are of two kinds: the *initial-value equations*, those that ensure the conservation of baryon number, entropy, and angular momentum (II, eqs. [3], [4], and [5]) and those that represent the integrated forms of the linearized versions of the (0, α)- and the (1, α)-components of the fully time-dependent field-equations (II, eqs. [9], [10], [13], and [14]); and the *dynamical equations*, which include the pulsation equation (II, eq. [18]), the integrability condition (II, eq. [15]) of the (1, α)-components of the field equations, and the linearized versions of two remaining field-equations (which we may select from I, eqs. [166], [168], and [169]).

In our present context, the most important dynamical equation is the integrability condition (II, eq. [15])

$$\begin{aligned}(e^{-3\psi+\nu+\mu_3-\mu_2}Q_{,2})_{,2} + (e^{-3\psi+\nu+\mu_2-\mu_3}Q_{,3})_{,3} + \sigma^2 e^{-3\psi-\nu+\mu_2+\mu_3}Q \\ = -[\omega_{,2}(3\delta\psi - \delta\nu + \delta\mu_3 - \delta\mu_2)]_{,3} + [\omega_{,3}(3\delta\psi - \delta\nu + \delta\mu_2 - \delta\mu_3)]_{,2} \\ + 16\pi\{[(\epsilon + p)u^0 u_1 e^{-2\psi+2\nu+2\mu_2}\xi^2]_{,3} - [(\epsilon + p)u^0 u_1 e^{-2\psi+2\nu+2\mu_3}\xi^3]_{,2}\} \\ \equiv \mathfrak{S} \quad (\text{say}) ,\end{aligned} \quad (3)$$

where

$$Q = e^{3\psi+\nu-\mu_2-\mu_3}(q_{2,3} - q_{3,2}) . \quad (4)$$

Equation (3) has the form of a wave equation; and the source terms $\mathfrak{S}$ on the right-hand side are clearly derived from the currents provided by the rotation: they vanish when the rotation ceases. It may, therefore, be expected that it is equation (3) that determines the emission of gravitational radiation in an odd-parity mode of "magnetic-type" (cf. Ipser 1971); it is the mode in which we shall be primarily interested in this paper. Our procedure, then, will be the following. First, we shall derive the asymptotic behavior of the solutions of equation (3) which have the forms of outgoing (and ingoing) waves. Next, we shall relate the behavior, at large distances, of the other field-variables $\delta\psi$, $\delta\nu$, $\delta\mu$, and $\delta\tau$ to the solution for Q. And finally, we shall return to the variational principle derived in Paper II (eqs. [25], [31], and [32]) and elucidate how it can be used to determine and characterize both the real standing modes and the quasi-normal outgoing complex modes (in the sense of Thorne and Campolattaro 1967).

III. THE ASYMPTOTIC BEHAVIOR OF THE SOLUTION FOR Q

In considering equation (3), it is convenient to introduce instead of Q the variable

$$\Phi = e^{-2\psi}Q = e^{\psi+\nu-\mu_2-\mu_3}(q_{2,3} - q_{3,2}) . \quad (5)$$

We find that Φ satisfies the equation

$$\begin{aligned}(e^{-2\mu_2}\Phi_{,2}\sqrt{-g})_{,2} + (e^{-2\mu_3}\Phi_{,3}\sqrt{-g})_{,3} + \sigma^2 e^{-2\nu}\Phi\sqrt{-g} \\ + 2e^{\psi+\nu}\{e^{\mu_3-\mu_2}[\psi_{,22} + \psi_{,2}(-\psi + \nu - \mu_2 + \mu_3)_{,2}] \\ + e^{\mu_2-\mu_3}[\psi_{,33} + \psi_{,3}(-\psi + \nu + \mu_2 - \mu_3)_{,3}]\}\Phi = e^{2\psi}\mathfrak{S} .\end{aligned} \quad (6)$$

The terms in braces on the left-hand side of this equation can be simplified by making use of I, equations (70) and (78), governing equilibrium. We find

$$(e^{-2\mu_2}\Phi_{,2}\sqrt{-g})_{,2} + (e^{-2\mu_3}\Phi_{,3}\sqrt{-g})_{,3} + \sigma^2 e^{-2\nu}\Phi\sqrt{-g}$$
$$- 2e^{\psi+\nu}\left[2e^{\mu_3-\mu_2}(\psi_{,2})^2 + 2e^{\mu_2-\mu_3}(\psi_{,3})^2 + \tfrac{1}{2}e^{2\psi-2\nu}X\right.$$
$$\left. + 4\pi e^{\mu_2+\mu_3}\frac{\epsilon(1+V^2) - p(1-3V^2)}{1-V^2}\right]\Phi = e^{2\psi}\mathfrak{S}\,. \quad (7)$$

In this form it is manifest that Φ satisfies a d'Alembertian equation with a potential and a source.

The asymptotic behavior of the solutions of equation (7) is most conveniently considered in the system of spherical-polar coordinates introduced in Paper I (eqs. [82]). Since we are here considering the time-dependent problem (albeit in the linearized form) it is necessary to generalize I, equations (82) in the manner,

$$e^{\psi} = r\sin\theta e^{\eta+\zeta}\,, \qquad e^{\mu_2} = e^{\eta-\zeta-\delta\tau}\,, \qquad \text{and} \qquad e^{\mu_3} = re^{\eta-\zeta+\delta\tau}\,, \quad (8)$$

to allow for the fact that the coordinate condition $e^{\mu_3-\mu_2} = r$ imposed under stationary conditions cannot obtain when the conditions cease to be stationary. (In the perturbed state, η, ζ, and ν will differ from their equilibrium values by $\delta\eta$, $\delta\zeta$, and $\delta\nu$.)

In terms of the new variables

$$Q = r^2\sin^2\theta e^{2(\eta-\zeta)}\Phi\,, \qquad \Phi = e^{3\zeta-\eta+\nu}\sin\theta\,(q_{2,3} - q_{3,2})\,,$$

and

$$q_{2,3} - q_{3,2} = \frac{Q}{r^2\sin^3\theta}\,e^{-5\zeta-\eta-\nu}\,; \quad (9)$$

and the equation for Φ becomes

$$\frac{1}{r^2}(e^{\eta+\zeta+\nu}r^2\Phi_{,2})_{,2} + \frac{1}{r^2\sin\theta}(e^{\eta+\zeta+\nu}\sin\theta\,\Phi_{,3})_{,3} + \sigma^2 e^{3\eta-\zeta-\nu}\Phi$$
$$- 2e^{\eta+\zeta+\nu}\left[2|\text{grad}\,\psi|^2 + \tfrac{1}{2}r^2\sin^2\theta\,e^{2\eta+2\zeta-2\nu}|\text{grad}\,\omega|^2\right.$$
$$\left. + 4\pi e^{2\eta-2\zeta}\frac{\epsilon(1+V^2) - p(1-3V^2)}{1-V^2}\right]\Phi = \sin\theta\,e^{2\eta+2\zeta}\mathfrak{S}\,. \quad (10)$$

In the vacuum, outside the fluid sources, the equation is

$$\frac{1}{r^2}(e^{\eta+\zeta+\nu}r^2\Phi_{,2})_{,2} + \frac{1}{r^2\sin\theta}(e^{\eta+\zeta+\nu}\sin\theta\,\Phi_{,3})_{,3} + \sigma^2 e^{3\eta-\zeta-\nu}\Phi$$
$$- 2e^{\eta+\zeta+\nu}[2|\text{grad}\,\psi|^2 + \tfrac{1}{2}r^2\sin^2\theta\,e^{2\eta+2\zeta-2\nu}|\text{grad}\,\omega|^2]\Phi$$
$$= \sin\theta\,e^{2\eta+2\zeta}\Big\{-[\omega_{,2}(3\delta\eta + 3\delta\zeta - \delta\nu + 2\delta\tau)]_{,3}$$
$$+ [\omega_{,3}(3\delta\eta + 3\delta\zeta - \delta\nu - 2\delta\tau)]_{,2}\Big\}\,. \quad (11)$$

By the known asymptotic behaviors of the stationary potentials (Paper I, § VII, eqs. [101], [102], and [109], or Paper II, eqs. [55]–[58]),

$$\eta + \zeta + \nu = \frac{1}{r^2}(A - \tfrac{1}{4}M^2) + O(r^{-3}),$$

$$3\eta - \zeta - \nu = \frac{4M}{r} + \frac{1}{r^2}[A(1 + 2\cos 2\theta) - \tfrac{3}{4}M^2] + O(r^{-3}),$$

$$|\text{grad}\,\psi|^2 = \frac{1}{r^2 \sin^2\theta} + O(r^{-3}), \qquad |\text{grad}\,\omega|^2 = O(r^{-8}),$$

$$\omega_{,2} = O(r^{-4}), \qquad \text{and} \qquad \omega_{,3} = O(r^{-5}). \tag{12}$$

Using these results in equation (11), we obtain

$$\frac{1}{r^2}\left\{r^2\left[1 + \frac{1}{r^2}(A - \tfrac{1}{4}M^2) + O(r^{-3})\right]\Phi_{,r}\right\}_{,r} + \frac{1}{r^2\sin\theta}\{[1 + O(r^{-2})]\Phi_{,\theta}\sin\theta\}_{,\theta}$$
$$+ \sigma^2\left[1 + \frac{4M}{r} + \frac{1}{r^2}(A + 2A\cos 2\theta + \tfrac{29}{4}M^2) + O(r^{-3})\right]\Phi$$
$$- \frac{4\Phi}{r^2\sin^2\theta} + O\left(\frac{\Phi}{r^3}\right) = 0, \tag{13}$$

or, alternatively,

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) - \frac{4}{r^2\sin^2\theta}\Phi$$
$$+ \sigma^2\left[1 + \frac{4M}{r} + \frac{1}{r^2}(\tfrac{15}{2}M^2 + 2A\cos 2\theta)\right]\Phi + O\left(\frac{\Phi}{r^3}\right) = 0. \tag{14}$$

We shall seek a solution of equation (14) whose asymptotic behavior at large distances is that of outgoing or ingoing waves in a Coulomb potential and expressible as a series of the form

$$\Phi = \sum_{n=1}^{\infty} \frac{\Phi^{(n)}(\theta)}{r^n} e^{\mp i\sigma(r + 2M\log r)}. \tag{15}$$

Inserting this expansion in equation (14) and making use of the formulae given in § IV below (eqs. [29] and [30]) applicable to such expansions, we obtain the relation

$$\mp 2i\sigma\Phi^{(2)} = \pm 2Mi\sigma(1 \pm 2Mi\sigma)\Phi^{(1)} + \sigma^2(\tfrac{15}{2}M^2 + 2A\cos 2\theta)\Phi^{(1)} + \mathfrak{D}\Phi^{(1)}, \tag{16}$$

where $\mathfrak{D}$ stands for the operator

$$\mathfrak{D} = \frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\right) - \frac{4}{\sin^2\theta}. \tag{17}$$

As we shall see presently, regularity conditions on the solutions at the poles require that the angular dependence of Q be divisible by $\sin^4\theta$ and, further, that it be skew-symmetric under reflection with respect to the equatorial plane, $\theta = \pi/2$. These requirements on Q imply that Φ be divisible by $\sin^2\theta$ (cf. eq. [9]) and similarly skew-symmetric about the equatorial plane. Now it can be shown[2] that $\mathfrak{D}$ *operating on* $\sin^2\theta f(\cos\theta)$, *where f is some even (or odd) polynomial in* $\cos\theta$, *yields a similar trigonometric function*, i.e., *an even (or odd) polynomial in* $\cos\theta$ *multiplied by* $\sin^2\theta$. Consequently, if we set

$$\Phi^{(1)}(\theta) = \sin^2\theta\cos\theta f(\cos^2\theta)\,, \tag{18}$$

where f, as the notation implies, is an even polynomial in $\cos\theta$, then $\mathfrak{D}\Phi^{(1)}(\theta)$ is again a trigonometric function of the same form. From this last result it follows from equation (16) that $\Phi^{(2)}(\theta)$ is again a function of the same form as $\Phi^{(1)}(\theta)$; and the angular dependence of Φ (to an order consistent with eq. [14]) will meet the requirements of the problem. We conclude, then, that equation (14) allows solutions whose asymptotic behavior at large distances is given by

$$\Phi = \frac{Q_0}{r}\,e^{\pm i\sigma(r+2M\log r)}\sin^2\theta\cos\theta f(\cos^2\theta)\,, \tag{19}$$

[2] While the result stated can be verified directly, the following demonstration may be of some general interest.

With the substitution

$$\Phi = \operatorname{cosec}^2\theta\, P(\theta)\,, \tag{i}$$

the equation

$$\mathfrak{D}\Phi + k\Phi = 0 \tag{ii}$$

becomes

$$\sin^3\theta\frac{d}{d\theta}\left(\sin^{-3}\theta\frac{dP}{d\theta}\right) + (k-2)P = 0\,. \tag{iii}$$

This equation allows solutions regular at $\theta = 0$ and $\theta = \pi$ only when

$$k = (m-1)(m-2) \quad\text{and}\quad k - 2 = m(m-3)\,, \tag{iv}$$

where m is an integer. Equation (iii) then defines the Gegenbauer polynomials $P_m(\cos\theta| -3)$ of index -3 (cf. Sommerfeld 1949). It will suffice to consider these polynomials for $m \geq 4$; and for $m \geq 4$, the substitution

$$P_m(\cos\theta| -3) = \sin^4\theta f_m(\cos\theta)(=\sin^2\theta\,\Phi_m(\theta))\,, \tag{v}$$

reduces equation (iii), for the case $k = (m-1)(m-2)$, to

$$(1-\mu^2)\frac{d^2 f_m}{d\mu^2} - 6\mu\frac{df_m}{d\mu} + (m+1)(m-4)f_m = 0 \qquad (\mu = \cos\theta)\,. \tag{vi}$$

This last equation defines a polynomial of degree $(m-4)$ in μ; thus,

$$f_4 = \text{constant}\,, \qquad f_5 = \mu\,, \qquad f_6 = \mu^2 - \tfrac{1}{7}\,, \qquad\text{and}\qquad f_7 = \mu^3 - \tfrac{1}{3}\mu\,. \tag{vii}$$

Clearly, any polynomial $f(\cos\theta)$ can be expressed as a linear combination of the polynomials $f_m(\cos\theta)$:

$$f(\cos\theta) = \sum A_m f_m(\cos\theta) \tag{viii}$$

By equations (i), (ii), (iv), and (v),

$$\begin{aligned}\mathfrak{D}\sin^2\theta f(\cos\theta) &= \sum A_m\mathfrak{D}\sin^2\theta f_m(\cos\theta)\\ &= \sum A_m\mathfrak{D}\Phi_m(\cos\theta)\\ &= -\sum(m-1)(m-2)A_m\Phi_m(\cos\theta)\\ &= -\sin^2\theta\sum(m-1)(m-2)A_m f_m(\cos\theta)\,;\end{aligned} \tag{ix}$$

and the result stated follows.

where Q_0 is a constant. The corresponding asymptotic behaviors of the solutions for Q and $(q_{2,3} - q_{3,2})$ are (cf. eqs. [9])

$$Q \to Q_0 r e^{\pm i\sigma(r+2M\log r)} \sin^4\theta \cos\theta\, f(\cos^2\theta) \tag{20}$$

and

$$(q_{2,3} - q_{3,2}) \to \frac{Q_0}{r} e^{\pm i\sigma(r+2M\log r)} \sin\theta \cos\theta\, f(\cos^2\theta)\,. \tag{21}$$

IV. THE ASYMPTOTIC BEHAVIORS OF THE REMAINING FIELD-VARIABLES

With the asymptotic behavior of the solution for Q determined in § III, we can now complete the corresponding solutions for the remaining field variables. For this purpose we consider the initial-value equations II, (9) and (10), and the linearized versions of two of the field equations which we shall presently specify.

In terms of the new variables (eqs. [8]), II, equations (9) and (10), take the forms

$$\delta\eta_{,2} - \nu_{,2}\delta\eta + \left[(\eta+\zeta)_{,2} + \frac{1}{r}\right]\delta\zeta = e^{2(\eta-\zeta)}\left(4\pi\frac{\epsilon+p}{1-V^2}\xi^2 - \frac{Q\omega_{,3}}{4\sqrt{-g}}\right) - \tfrac{1}{2}\delta\tau_{,2} - \tfrac{1}{2}\left[(3\eta-\zeta-\nu)_{,2} + \frac{3}{r}\right]\delta\tau \tag{22}$$

and

$$\delta\eta_{,3} - \nu_{,3}\delta\eta + [(\eta+\zeta)_{,3} + \cot\theta]\delta\zeta = r^2 e^{2(\eta-\zeta)}\left(4\pi\frac{\epsilon+p}{1-V^2}\xi^3 + \frac{Q\omega_{,2}}{4\sqrt{-g}}\right) + \tfrac{1}{2}\delta\tau_{,3} + \tfrac{1}{2}[(3\eta-\zeta-\nu)_{,3} + \cot\theta]\delta\tau\,. \tag{23}$$

In view of the known asymptotic behaviors of the stationary potentials (II, eqs. [55]–[58]), we can now write

$$\delta\eta_{,2} + \frac{1}{r}\delta\zeta = -\tfrac{1}{2}\delta\tau_{,2} - \frac{3}{2r}\delta\tau + O\left(\frac{\delta\eta,\,\delta\zeta,\,\delta\tau}{r^2}\right) + O(r^{-6}) \tag{24}$$

and

$$\delta\eta_{,3} + \cot\theta\,\delta\zeta = \tfrac{1}{2}\delta\tau_{,3} + \tfrac{1}{2}\cot\theta\,\delta\tau - \tfrac{3}{2}Q_0 J\frac{e^{-i\sigma(r+2M\log r)}}{r^3}\sin^3\theta\cos\theta\, f(\cos^2\theta) + O\left(\frac{\delta\eta}{r^3}, \frac{\delta\zeta}{r^3}, \frac{\delta\tau}{r^2}\right), \tag{25}$$

where we have used the outgoing-wave solution for Q and written J instead of J_1 used in Paper I, equation (109).

We now supplement equations (24) and (25) by I, equations (168) and (169); these equations give

$$\delta(\psi+\nu)_{,22} + \frac{3}{r}\delta(\psi+\nu)_{,2} = -4\sigma^2\left(1+\frac{4M}{r}\right)\delta\eta - \frac{2}{r}\delta\tau_{,2} + O\left(\frac{\delta\nu,\,\delta\eta,\,\delta\zeta,\,\delta\tau}{r^2}\right) \tag{26}$$

and

$$\delta\psi_{,22} + \frac{3}{r}\delta\psi_{,2} + \frac{1}{r}(\delta\nu + 2\delta\tau)_{,2} = -\sigma^2\left(1+\frac{4M}{r}\right)\delta\psi + O\left(\frac{\delta\psi,\,\delta\nu,\,\delta\tau}{r^2}\right) + O(r^{-5})\,. \tag{27}$$

We shall now assume that each of the four field-variables $\delta\eta$, $\delta\zeta$, $\delta\nu$, and $\delta\tau$ can be expanded as a series in inverse powers of r in the manner

$$\delta f = \sum_{n=m}^{\infty} \frac{\delta f^{(n)}(\theta)}{r^n} e^{-i\sigma(r+2M\log r)} \qquad (f = \eta,\ \zeta,\ \nu, \text{ and } \tau)\,, \tag{28}$$

where we are presently restricting ourselves to solutions in the form of outgoing waves. The corresponding expansions for the first and the second derivatives of f with respect to r are

$$\delta f_{,r} = \sum \frac{1}{r^n}[-i\sigma\delta f^{(n)} - (n-1+2Mi\sigma)\delta f^{(n-1)}]e^{-i\sigma(r+2M\log r)} \tag{29}$$

and

$$\delta f_{,rr} = \sum \frac{1}{r^n}[-\sigma^2\delta f^{(n)} + 2i\sigma(n-1+2Mi\sigma)\delta f^{(n-1)} + (n-1+2Mi\sigma)(n-2+2Mi\sigma)\delta f^{(n-2)}]e^{-i\sigma(r+2M\log r)}\,. \tag{30}$$

For expansions of the form assumed, equations (24)–(27) give

$$-i\sigma\delta\eta^{(n)} - (n-1+2Mi\sigma)\delta\eta^{(n-1)} + \delta\zeta^{(n-1)} = \tfrac{1}{2}[i\sigma\delta\tau^{(n)} + (n-1+2Mi\sigma)\delta\tau^{(n-1)}] - \tfrac{3}{2}\delta\tau^{(n-1)} + \text{terms in } \delta f^{(n-2)}\,, \tag{31}$$

$$\delta\eta^{(n)}{}_{,\theta} + \cot\theta\,\delta\zeta^{(n)} = \tfrac{1}{2}\delta\tau^{(n)}{}_{,\theta} + \tfrac{1}{2}\cot\theta\,\delta\tau^{(n)} - \tfrac{3}{2}\delta^n{}_3 Q_0 J\sin^3\theta\cos\theta\, f(\cos^2\theta)\,, \tag{32}$$

$$-\sigma^2(\delta\psi+\delta\nu)^{(n)} + i\sigma(2n-5+4Mi\sigma)(\delta\psi+\delta\nu)^{(n-1)} = -4\sigma^2(\delta\eta^{(n)} + 4M\delta\eta^{(n-1)}) + 2i\sigma\delta\tau^{(n-1)} + \text{terms in } (\delta\psi+\delta\nu)^{(n-2)} \text{ and } \delta\tau^{(n-2)}\,, \tag{33}$$

and

$$-\sigma^2\delta\psi^{(n)} + 2i\sigma(n-1+2Mi\sigma)\delta\psi^{(n-1)} - 3i\sigma\delta\psi^{(n-1)} - i\sigma(\delta\nu^{(n-1)} + 2\delta\tau^{(n-1)}) = -\sigma^2\delta\psi^{(n)} - 4M\sigma^2\delta\psi^{(n-1)} + \text{terms in } \delta f^{(n-2)}\,. \tag{34}$$

Now let m be the *least value* of n for which not all $\delta\eta^{(m)}$, $\delta\zeta^{(m)}$, $\delta\nu^{(m)}$, and $\delta\tau^{(m)}$ vanish; i.e., for $n < m$ all these quantities vanish. Then, letting $n = m$ in equations (31)–(33) and $n = m + 1$ in equation (34), we obtain

$$\delta\eta^{(m)} = -\tfrac{1}{2}\delta\tau^{(m)}\,, \tag{35}$$

$$\delta\eta^{(m)}{}_{,\theta} + \cot\theta\,\delta\zeta^{(m)} = \tfrac{1}{2}\delta\tau^{(m)}{}_{,\theta} + \tfrac{1}{2}\cot\theta\,\delta\tau^{(m)} - \tfrac{3}{2}\delta_3{}^m Q_0 J\sin^3\theta\cos\theta\, f(\cos^2\theta)\,, \tag{36}$$

$$(\delta\psi+\delta\nu)^{(m)} = (\delta\eta+\delta\zeta+\delta\nu)^{(m)} = 4\delta\eta^{(m)}\,, \tag{37}$$

and

$$(2m-3)\delta\psi^{(m)} = (2m-3)(\delta\eta+\delta\zeta)^{(m)} = \delta\nu^{(m)} + 2\delta\tau^{(m)}\,. \tag{38}$$

In view of equation (35), equation (37) gives

$$\delta\zeta^{(m)} = \tan\theta\,\delta\tau^{(m)}{}_{,\theta} + \tfrac{1}{2}\delta\tau^{(m)} - \tfrac{3}{2}\delta^m{}_3 Q_0 J\sin^4\theta\, f(\cos^2\theta)\,; \tag{39}$$

and equations (37) and (38) can be written alternatively in the forms

$$3\delta\eta^{(m)} - \delta\zeta^{(m)} = \delta\nu^{(m)} \tag{40}$$

and

$$(2m + 1)\delta\eta^{(m)} + (2m - 3)\delta\zeta^{(m)} = \delta\nu^{(m)} . \tag{41}$$

Subtracting equation (40) from equation (41), we obtain

$$(2m - 2)[\delta\eta^{(m)} + \delta\zeta^{(m)}] = 0 . \tag{42}$$

Hence,

$$\textit{either } m = 1 \qquad \textit{or} \qquad \delta\eta^{(m)} = -\delta\zeta^{(m)} . \tag{43}$$

For the case $m = 1$, we readily verify that equations (35), (40), and (41) give

$$\delta\eta^{(1)} = -\tfrac{1}{2}\delta\tau^{(1)} , \qquad \delta\zeta^{(1)} = \tan\theta\, \delta\tau^{(1)}{}_{,\theta} + \tfrac{1}{2}\delta\tau^{(1)} ,$$

and

$$\delta\nu^{(1)} = -2\delta\tau^{(1)} - \tan\theta\, \delta\tau^{(1)}{}_{,\theta} ; \tag{44}$$

and $\delta\tau^{(1)}$ is left unspecified. It will be observed that the corresponding solutions for $\delta\eta$, $\delta\zeta$, $\delta\nu$, and $\delta\tau$ all have an r^{-1}-behavior as $r \to \infty$ and their amplitudes at infinity do not depend on Q_0.

When $m \neq 1$, equations (35), (40), and (43) combine to give

$$\delta\eta^{(m)} = -\delta\zeta^{(m)} = -\tfrac{1}{2}\delta\tau^{(m)} = \tfrac{1}{4}\delta\nu^{(m)} , \tag{45}$$

and equation (39) gives

$$\tan\theta\, \delta\tau^{(m)}{}_{,\theta} = \tfrac{3}{2}\delta^m{}_3 Q_0 J \sin^4\theta\, f(\cos^2\theta) . \tag{46}$$

Hence

$$m = 3 \qquad \text{and} \qquad \delta\tau^{(3)}{}_{,\theta} = \tfrac{3}{2}Q_0 J \sin^3\theta \cos\theta\, f(\cos^2\theta) . \tag{47}$$

It is clear that equation (47) can be integrated to give a solution of the form

$$\delta\tau^{(3)} = \tfrac{3}{8}Q_0 J \sin^4\theta\, g(\sin^2\theta) , \tag{48}$$

where g is some even polynomial in $\sin\theta$. Accordingly in this case, the solutions are dependent on the amplitude of Q at infinity. We have

$$\begin{bmatrix} \delta\eta \\ \delta\zeta \\ \delta\nu \\ \delta\tau \end{bmatrix} = \begin{bmatrix} -\frac{3}{16} \\ +\frac{3}{16} \\ -\frac{3}{4} \\ +\frac{3}{8} \end{bmatrix} \times \frac{Q_0 J}{r^3} e^{-i\sigma(r + 2M \log r)} \sin^4\theta\, g(\sin^2\theta) . \tag{49}$$

We observe that these solutions possess the required reflection-symmetry about the equatorial plane; and the solution for $\delta\zeta$ is, moreover, consistent with the requirements of regularity at the poles, namely, that, here, it behaves (at least) like $\sin^2\theta$. Not all of these requirements would have been met had we chosen an odd polynomial for f in the solution (21) for Q.

The direct proportionality of $\delta\eta$, $\delta\zeta$, $\delta\nu$, and $\delta\tau$ to ω (at large distances) is to be particularly noted.

In our further consideration, we shall restrict ourselves to the solution (49). We shall not be concerned in this paper with the alternative solution (44): it leads to finite gravitational radiation in the limit of zero rotation; and we are not presently interested in that case.

V. THE VARIATIONAL PRINCIPLE

In Paper II, a variational basis for the determination of σ^2 was derived. The basis is provided by equation II, (33), together with the surface integrals that follow from II, equations (31) and (32). But the precise way in which the variational principle is to be interpreted and used was not sufficiently clarified in Paper II. We shall attempt to do so now.

Ignoring for the present matters of convergence and uniqueness (we shall return to these questions presently), consider II, equation (34), which gives the effect on σ^2 of evaluating it in accordance with II, equation (33), for two assumed trial displacements ξ^α and $\xi^\alpha + \frac{1}{2}\delta\xi^\alpha$ and associated variations $\delta\psi$, $\delta\mu$, $\delta\tau$, and Q and $\delta\psi + \frac{1}{2}\delta^2\psi$, $\delta\mu + \frac{1}{2}\delta^2\mu$, $\delta\tau + \frac{1}{2}\delta^2\tau$, and $Q + \frac{1}{2}\delta Q$, consistent only with the initial-value equations. In II, equation (34), we now allow $\delta\xi^\alpha$ ($\alpha = 2, 3$), δQ, and $\delta^2\tau$ to be arbitrary except that they satisfy the boundary conditions required of the true variations. Since these increments must also be consistent with the initial-value equations, we must, in particular, require that they satisfy the equations

$$
\begin{aligned}
&(\delta^2\psi + \delta^2\mu)_{,2} - \nu_{,2}(\delta^2\psi + \delta^2\mu) + \psi_{,2}(\delta^2\psi - \delta^2\mu) \\
&\qquad = e^{2\mu_2}\left(8\pi\frac{\epsilon + p}{1 - V^2}\,\delta\xi^2 - \frac{\omega_{,3}\delta Q}{2\sqrt{-g}}\right) - \delta^2\tau_{,2} - (2\mu_3 + \psi - \nu)_{,2}\delta^2\tau \\
&\qquad = \delta\mathfrak{F}_2(\text{say})\,. \qquad (50)
\end{aligned}
$$

and

$$
\begin{aligned}
&(\delta^2\psi + \delta^2\mu)_{,3} - \nu_{,3}(\delta^2\psi + \delta^2\mu) + \psi_{,3}(\delta^2\psi - \delta^2\mu) \\
&\qquad = e^{2\mu_3}\left(8\pi\frac{\epsilon + p}{1 - V^2}\,\delta\xi^3 + \frac{\omega_{,2}\delta Q}{2\sqrt{-g}}\right) + \delta^2\tau_{,3} + (2\mu_2 + \psi - \nu)_{,3}\delta^2\tau \\
&\qquad = \delta\mathfrak{F}_3\ (\text{say})\,, \qquad (51)
\end{aligned}
$$

where, by hypothesis, $\delta\mathfrak{F}_2$ and $\delta\mathfrak{F}_3$ may be assumed to be known functions.

Equations (50) and (51) represent simple quasi-linear equations for $\delta^2\psi$ and $\delta^2\mu$ and can be solved by standard methods. Thus, eliminating $(\delta^2\psi - \delta^2\mu)$ from these equations, we obtain

$$\psi_{,3}[e^{-\nu}(\delta^2\psi + \delta^2\mu)]_{,2} - \psi_{,2}[e^{-\nu}(\delta^2\psi + \delta^2\mu)]_{,3} = e^{-\nu}(\psi_{,3}\delta\mathfrak{F}_2 - \psi_{,2}\delta\mathfrak{F}_3)\,. \qquad (52)$$

Letting

$$F = e^{-\nu}(\delta^2\psi + \delta^2\mu)\,, \qquad (53)$$

and expressing the general solution of equation (52) in terms of an equation not solved for F in the manner

$$\Psi \equiv \Psi(F, x^2, x^3) = 0\,, \qquad (54)$$

where Ψ is some function of F, x^2, and x^3, we have

$$\psi_{,3}\frac{\partial\Psi}{\partial x^2} - \psi_{,2}\frac{\partial\Psi}{\partial x^3} + e^{-\nu}(\psi_{,3}\delta\mathfrak{F}_2 - \psi_{,2}\delta\mathfrak{F}_3)\frac{\partial\Psi}{\partial F} = 0\,, \qquad (55)$$

in place of equation (52). The equations defining the characteristics of equation (55) are

$$\frac{dx^2}{\psi_{,3}} = -\frac{dx^3}{\psi_{,2}} = \frac{dF}{e^{-\nu}(\psi_{,3}\delta\mathfrak{F}_2 - \psi_{,2}\delta\mathfrak{F}_3)} = d\lambda \text{ (say)} . \tag{56}$$

One set of characteristics is clearly given by

$$\psi(x^2, x^3) = \text{constant} ; \tag{57}$$

and the other set is defined by the equation

$$\frac{dF}{d\lambda} = e^{-\nu}(\psi_{,3}\delta\mathfrak{F}_2 - \psi_{,2}\delta\mathfrak{F}_3) = e^{-\nu}\left(\delta\mathfrak{F}_2\frac{dx^2}{d\lambda} + \delta\mathfrak{F}_3\frac{dx^3}{d\lambda}\right) . \tag{58}$$

The solution of this equation can be expressed as

$$F = \int_C e^{-\nu}(\delta\mathfrak{F}_2 dx^2 + \delta F_3 dx^3) = \int_C e^{-\nu}\delta\boldsymbol{\mathfrak{F}}\cdot d\boldsymbol{s} , \tag{59}$$

where C is an arc along any curve ψ = constant.

The required solution for $\delta^2\psi + \delta^2\mu$ can accordingly be derived as follows. Let $\delta^2\psi + \delta^2\mu$ be specified along a curve which intersects all the characteristics ψ = constant. Then the solution for $\delta^2\psi + \delta^2\mu$, along each of the characteristics, is obtained by evaluating the line integral of $e^{-\nu}\delta\boldsymbol{\mathfrak{F}}$ along them; thus,

$$(\delta^2\psi + \delta^2\mu)_{\text{along } \Psi = \text{constant}} = e^{\nu}\int_{\Psi = \text{constant}} e^{-\nu}\delta\boldsymbol{\mathfrak{F}}\cdot d\boldsymbol{s} . \tag{60}$$

With $\delta^2\psi + \delta^2\mu$ determined in this fashion, $\delta^2\psi - \delta^2\mu$ follows from either of the equations (50) or (51) and completes the solution.

The principal result of the foregoing analysis is the demonstration that for any arbitrarily assigned values of $\delta\xi^\alpha$, δQ, and $\delta^2\tau$, the values of $\delta^2\psi$ and $\delta^2\mu$ are determined in terms of them by the initial-value equations. If we now require that $\delta\sigma^2$ given by II, equation (34), vanish *identically* for *all* arbitrarily assigned $\delta\xi^\alpha$, δQ, and $\delta^2\tau$, then it follows from this equation that four linearly independent combinations of the six dynamical equations, represented by the two components of the pulsation equation, equation (3), and the linearized versions of the [(2, 2) + (3, 3)]-, [(2, 2) − (3, 3)]-, and the (1, 1)-components of the field equations, simultaneously vanish. However, of the six dynamical equations enumerated only four can be linearly independent. Hence, *the requirement that $\delta\sigma^2$ vanish identically for all arbitrarily assigned $\delta\xi^\alpha$, δQ, and $\delta^2\tau$ implies that all the dynamical equations of the problem will be satisfied.* We clearly have an algorism for using II, equation (33), as a basis for a variational determination of σ^2.

VI. THE EMISSION OF GRAVITATIONAL RADIATION DURING AXISYMMETRIC OSCILLATIONS

Consider the expression for σ^2 given in II, equation (25), together with the surface integrals II, (31) and (32). It will be recalled that the integrals in II, equation (25), are extended over the volume included in a sphere of a sufficiently large radius R; and the integral II, (32) (which results from the various integrations by parts carried out during the reductions leading to II, eq. [25]), is extended (as noted) over the surface of the sphere of radius R. On examining the various terms in the integrands of II, equations (25), (31), and (32), we find that by virtue of the asymptotic behaviors of the