ON THE RADIATIVE EQUILIBRIUM OF A STELLAR ATMOSPHERE. V

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ABSTRACT

In this paper the methods developed in earlier papers are extended to solving the problem of radiative transfer in curved atmospheres, i.e., to solving the equation of transfer

$$\mu \frac{\partial I}{\partial r} + \frac{1-\mu^2}{r^2}\frac{\partial I}{\partial \mu} = -\kappa\rho I + \tfrac{1}{2}\kappa\rho \int_{-1}^{+1} I d\mu ,$$

where $\kappa\rho$ is a function only of r. After outlining a general method for replacing this partial integrodifferential equation by an equivalent system of $2n$ linear equations in the nth approximation, the most convenient forms of the equations for the first two approximations are found. The equations of the second approximation for the astrophysically important case $\kappa\rho \propto r^{-n}$ $(n > 1)$ are explicitly solved and found to involve quadratures over Bessel functions of purely imaginary arguments. For the case $n = 2$ the solutions have been found in their numerical forms. Finally, the physically interesting case of diffusion through a homogeneous sphere is also considered.

1. *Introduction.*—In the earlier papers of this series[1] an attempt has been made to present the solutions to the various plane problems of radiative transfer in the theory of stellar atmospheres in forms which would enable one to derive results to any desired degree of accuracy without much difficulty. In this paper we propose to extend this discussion to include the case of the radiative equilibrium of "extended atmospheres" in which it is not permissible to ignore the curvature of the outer layers. First approximations to the solution of this latter problem in curved atmospheres have been given by N. A. Kosirev[2] and the writer.[3] But all attempts to improve on the "first approximations" given by these writers have so far proved unsuccessful.[4] However, in view of the fact that the theory of extended atmospheres is finding increasing applications to a variety of practical problems,[5] it would appear worth while to re-examine the basic problem with a view toward developing systematic methods of approximation for obtaining solutions of higher accuracy. This is the object of this paper.

2. *The reduction of the equation of transfer to an equivalent system of* 2n *linear equations in the* n*th approximation.*—Let r denote the distance measured outward from the center of symmetry of the atmosphere and ϑ the angle measured from the positive direction of the radius vector. The equation of transfer which we have to deal with is

$$\cos\vartheta \frac{\partial I}{\partial r} - \frac{\sin\vartheta}{r}\frac{\partial I}{\partial \vartheta} = -\kappa\rho I + \tfrac{1}{2}\kappa\rho \int_0^{\pi} I(r, \vartheta)\sin\vartheta d\vartheta , \tag{1}$$

where I, κ, and ρ have their usual meanings. Writing μ for $\cos\vartheta$, we can re-write equation (1) in the form

$$\mu \frac{\partial I}{\partial r} + \frac{1-\mu^2}{r}\frac{\partial I}{\partial \mu} = -\kappa\rho I + \tfrac{1}{2}\kappa\rho \int_{-1}^{+1} I(r, \mu) d\mu . \tag{2}$$

[1] *Ap. J.*, **99**, 180; **100**, 76, 117, and 355, 1944. These papers will be referred to as "I," "II," "III," and "IV," respectively.

[2] *M.N.*, **94**, 430, 1934.

[3] S. Chandrasekhar, *M.N.*, **94**, 444, 1934; see also *Proc. Cambridge Phil. Soc.*, **31**, 390, 1935, and *Russian Astr. J.*, **11**, 550, 1934.

[4] Cf. L. Gratton, *Soc. astr. italiana*, **10**, 309, 1937.

[5] See F. L. Whipple and C. Payne-Gaposchkin, *Harvard Circ.*, No. 413, 1936, and C. Payne-Gaposchkin and Sergei Gaposchkin, *Ap. J.*, **101**, 56, 1945.

According to the ideas developed in papers II and III, we replace the integral which occurs on the right-hand side of this equation by a sum according to Gauss's formula for numerical quadratures. This permits the replacement of the integrodifferential equation (2) by a system of linear equations which in the nth approximation is

$$\mu_i \frac{dI_i}{dr} + \frac{1-\mu_i^2}{r}\left(\frac{\partial I}{\partial \mu}\right)_{\mu=\mu_i} = -\kappa\rho I_i + \tfrac{1}{2}\kappa\rho\Sigma a_j I_j \quad (i = \pm 1, \ldots, \pm n), \quad (3)$$

where we have used I_j to denote $I(r, \mu_j)$. Further, as in papers II, III, and IV, the μ_j's are the zeros of the Legendre polynomial $P_{2n}(\mu)$ and the a_j's are the appropriate Gaussian weights. It is at once seen that our present system of equations (cf. eq. [3]) differs in an essential way from those which occurred in our earlier studies on the plane problems; for equation (3) now involves $(\partial I/\partial\mu)_{\mu=\mu_i}$, and before we can proceed any further we must know the values which we are to assign to $\partial I/\partial\mu$ at the points of the Gaussian division μ_i in our present scheme of approximation. At first it might be supposed that the assignment of values to $\partial I/\partial \mu$ at $\mu = \mu_i$, $i = \pm 1, \ldots, \pm n$, is largely arbitrary, particularly when n is small. However, on consideration it appears that this assignment can be done in a satisfactory manner in only one way and, indeed, according to the following device:

Define the polynomials $Q_m(\mu)$ according to the formula

$$P_m(\mu) = -\frac{dQ_m}{d\mu} \qquad (m = 1, \ldots, 2n), \quad (4)$$

and adjust the constant of integration in Q_m by requiring that

$$Q_m = 0 \quad \text{for} \quad |\mu| = 1 . \quad (5)$$

This can always be accomplished, since when m is odd, Q_m is even, and when m is even, the indefinite integral of $P_m(\mu)$ already contains $(1-\mu^2)$ as a factor.[6] The first few of the polynomials $Q_m(\mu)$ are given in Table 1.

TABLE 1

m	$P_m(\mu)$	$Q_m(\mu)$	$\mathcal{Q}_m(\mu)$
1.	μ	$\frac{1}{2}(1-\mu^2)$	$\frac{1}{2}$
2.	$\frac{1}{2}(3\mu^2-1)$	$\frac{1}{2}\mu(1-\mu^2)$	$\frac{1}{2}\mu$
3.	$\frac{1}{2}(5\mu^3-3\mu)$	$\frac{1}{8}(5\mu^2-1)(1-\mu^2)$	$\frac{1}{8}(5\mu^2-1)$
4.	$\frac{1}{8}(35\mu^4-30\mu^2+3)$	$\frac{1}{8}\mu(7\mu^2-3)(1-\mu^2)$	$\frac{1}{8}\mu(7\mu^2-3)$
5.	$\frac{1}{8}(63\mu^5-70\mu^3+15\mu)$	$\frac{1}{16}(21\mu^4-14\mu^2+1)(1-\mu^2)$	$\frac{1}{16}(21\mu^4-14\mu^2+1)$

Now by an integration by parts we arrive at the identity

$$\int_{-1}^{+1} Q_m(\mu)\frac{\partial I}{\partial\mu}\,d\mu = -\int_{-1}^{+1} I\,\frac{dQ_m}{d\mu}\,d\mu = \int_{-1}^{+1} IP_m(\mu)\,d\mu . \quad (6)$$

Expressing the first and the last integrals in equation (6) as sums according to Gauss's formula, we have in the nth approximation

$$\Sigma a_i Q_m(\mu_i)\left(\frac{\partial I}{\partial\mu}\right)_{\mu=\mu_i} = \Sigma a_i I_i P_m(\mu_i) \quad (m = 1, \ldots, 2n). \quad (7)$$

Equation (7) provides us with exactly the right number of equations to express

[6] Actually, $Q_m(\mu) = [(P_{m-1} - P_{m+1})/(2\mu+1)] + \text{constant}$.

$(\partial I/\partial\mu)_{\mu=\mu_i}$, $i = \pm 1, \ldots, \pm n$, as linear combinations of I_i.[7] Accordingly, equations (3) and (7), together, provide the required reduction of the equation of transfer (2) to an equivalent system of linear equations for the I_i's in the nth approximation.

For purposes of practical solution it appears most convenient to combine equations (3) and (7) in the following manner:

Since we have arranged $Q_m(\mu)$ to be divisible by $1 - \mu^2$, we can clearly write

$$Q_m(\mu) = \mathcal{Q}_m(\mu)(1-\mu^2). \tag{8}$$

The first few of the polynomials $\mathcal{Q}_m(\mu)$, defined according to equation (8), are also listed in Table 1.

Now multiply equation (3) by $a_i\mathcal{Q}_m(\mu_i)$ and sum over all i's. We obtain

$$\left.\begin{aligned}\frac{d}{dr}\Sigma a_i\mu_i\mathcal{Q}_m(\mu_i)I_i+\frac{1}{r}\Sigma a_i(1-\mu_i^2)\mathcal{Q}_m(\mu_i)\left(\frac{\partial I}{\partial\mu}\right)_{\mu=\mu_i} &= -\kappa\rho\Sigma a_i\mathcal{Q}_m(\mu_i)I_i\\ +\tfrac{1}{2}\kappa\rho(\Sigma a_iI_i)[\Sigma a_i\mathcal{Q}_m(\mu_i)] \qquad\qquad & (m=1,\ldots,2n).\end{aligned}\right\} \tag{9}$$

But, according to equations (7) and (8),

$$\left.\begin{aligned}\Sigma a_i(1-\mu_i^2)\mathcal{Q}_m(\mu_i)\left(\frac{\partial I}{\partial\mu}\right)_{\mu=\mu_i} &= \Sigma a_iQ_m(\mu_i)\left(\frac{\partial I}{\partial\mu}\right)_{\mu=\mu_i}\\ &= \Sigma a_iP_m(\mu_i)I_i;\end{aligned}\right\} \tag{10}$$

and equation (9) reduces to

$$\left.\begin{aligned}\frac{d}{dr}\Sigma a_i\mu_i\mathcal{Q}_m(\mu_i)I_i+\frac{1}{r}\Sigma a_iP_m(\mu_i)I_i &= -\kappa\rho\Sigma a_i\mathcal{Q}_m(\mu_i)I_i\\ +\tfrac{1}{2}\kappa\rho(\Sigma a_iI_i)[\Sigma a_i\mathcal{Q}_m(\mu_i)] \qquad\qquad & (m=1,\ldots,2n),\end{aligned}\right\} \tag{11}$$

which is the required system of linear equations in the nth approximation.

Equation (11) for the case $m = 1$ admits of immediate integration. For, when $m = 1$,

$$P_1(\mu)=\mu \quad\text{and}\quad \mathcal{Q}_1(\mu)=\tfrac{1}{2}, \tag{12}$$

and equation (11) yields

$$\frac{1}{2}\frac{d}{dr}\Sigma a_i\mu_iI_i+\frac{1}{r}\Sigma a_i\mu_iI_i=0. \tag{13}$$

Accordingly,

$$\Sigma a_i\mu_iI_i=\frac{1}{2}\frac{F_0}{r^2}, \tag{14}$$

where F_0 is a constant of integration. Equation (14) is the expression in our present approximation of the flux integral

$$F=2\int_{-1}^{+1}I\mu\,d\mu=\frac{F_0}{r^2}, \tag{15}$$

which the equation of transfer (2) admits directly.

Again, since $\mathcal{Q}_m(\mu)$ is odd when m is even, equation (11) reduces for even values of m to the form

$$\left.\begin{aligned}\frac{d}{dr}\Sigma a_i\mu_i\mathcal{Q}_m(\mu_i)I_i+\frac{1}{r}\Sigma a_iP_m(\mu_i)I_i &= -\kappa\rho\Sigma a_i\mathcal{Q}_m(\mu_i)I_i\\ & (m=2,4,\ldots,2n).\end{aligned}\right\} \tag{16}$$

[7] Essentially what eq. (7) allows is to determine in a "best possible way" the derivatives of a function in terms of its values at the points of the Gaussian division. This problem has apparently not been considered before.

For $m = 2n$ the foregoing equation further simplifies to

$$\frac{d}{dr}\Sigma a_i\mu_i \mathcal{Q}_{2n}(\mu_i) I_i = -\kappa\rho\Sigma a_i \mathcal{Q}_{2n}(\mu_i) I_i , \tag{17}$$

since the μ_i's are by definition the zeros of the polynomial $P_{2n}(\mu)$.

Finally, for further reference we may note here the explicit forms of equation (11) for the first few values of m. Using the definition of the polynomials P_m and $\mathcal{Q}_m$ given in Table 1, we find

$$\frac{d}{dr}\Sigma a_i\mu_i I_i + \frac{2}{r}\Sigma a_i\mu_i I_i = 0 , \tag{18}$$

$$\frac{d}{dr}\Sigma a_i\mu_i^2 I_i + \frac{1}{r}\Sigma a_i(3\mu_i^2 - 1) I_i = -\kappa\rho\Sigma a_i\mu_i I_i , \tag{19}$$

$$\frac{d}{dr}\Sigma a_i\mu_i(5\mu_i^2 - 1) I_i + \frac{4}{r}\Sigma a_i\mu_i(5\mu_i^2 - 3) I_i = -\tfrac{5}{3}\kappa\rho\Sigma a_i(3\mu_i^2 - 1) I_i , \tag{20}$$

$$\frac{d}{dr}\Sigma a_i\mu_i^2(7\mu_i^2 - 3) I_i + \frac{1}{r}\Sigma a_i(35\mu_i^4 - 30\mu_i^2 + 3) I_i = -\kappa\rho\Sigma a_i\mu_i(7\mu_i^2 - 3) I_i , \tag{21}$$

et cetera.

3. *The first approximation.*—In the first approximation we consider equation (11) for $m = 1$ and 2 only, with (cf. II, eq. [33])

$$a_1 = a_{-1} = 1 \quad \text{and} \quad \mu_1 = -\mu_{-1} = \frac{1}{\sqrt{3}} . \tag{22}$$

We have (cf. eqs. [14] and [19])

$$I_1 - I_{-1} = \frac{\sqrt{3}}{2}\frac{F_0}{r^2} \tag{23}$$

and

$$\frac{1}{3}\frac{d}{dr}(I_1 + I_{-1}) = -\kappa\rho\frac{1}{\sqrt{3}}(I_1 - I_{-1}) . \tag{24}$$

Combining equations (23) and (24), we have

$$\frac{d}{dr}(I_1 + I_{-1}) = -\frac{3}{2}\kappa\rho\frac{F_0}{r^2} . \tag{25}$$

Hence,

$$I_1 + I_{-1} = \frac{3}{2}F_0\int_r^R \frac{\kappa\rho dr}{r^2} + \text{constant} , \tag{26}$$

where $r = R$ defines the extent of the atmosphere. The constant of integration in equation (26) is determined by the condition that $I_{-1} = 0$ at $r = R$. In this manner we obtain for the source function J the solution

$$J = \tfrac{1}{2}(I_1 + I_{-1}) = \frac{\sqrt{3}}{4}\frac{F_0}{R^2} + \frac{3}{4}F_0\int_r^R \frac{\kappa\rho dr}{r^2} , \tag{27}$$

which is to be compared with the solution given earlier by Kosirev[8] and by Chandrasekhar.[9]

For an atmosphere which extends to infinity we should require that both I_1 and I_{-1} tend to zero as $r \to \infty$. In this case the solution for J reduces to

$$J = \frac{3}{4}F_0\int_r^\infty \frac{\kappa\rho dr}{r^2} , \tag{28}$$

[8] *M.N.*, **94**, 430, 1934. See particularly eq. (8) in this paper.

[9] *M.N.*, **94**, 444, 1934. See eqs. (49)–(51).

or, alternatively,

$$J=\frac{3}{4}F_0\int_0^{\tau}\frac{d\tau}{r^2}, \tag{29}$$

where τ denotes the radial optical thickness. It is in the form (29) that the solution to the problem of the extended atmospheres has been used in practice.

4. *The equations for the second approximation.*—In the second approximation we choose for the μ_i's the zeros of $P_4(\mu)$ and for the a_i's the appropriate Gaussian weights (II, eq. [38]). The equations which we have now to deal with are (cf. eqs. [14] and [19]–[21])

$$\Sigma a_i\mu_i I_i=\frac{1}{2}\frac{F_0}{r^2}, \tag{30}$$

$$\frac{d}{dr}\Sigma a_i\mu_i^2 I_i+\frac{1}{r}\Sigma a_i(3\mu_i^2-1)I_i=-\kappa\rho\Sigma a_i\mu_i I_i, \tag{31}$$

$$\frac{d}{dr}\Sigma a_i\mu_i(5\mu_i^2-1)I_i+\frac{4}{r}\Sigma a_i\mu_i(5\mu_i^2-3)I_i=-\tfrac{5}{3}\kappa\rho\Sigma a_i(3\mu_i^2-1)I_i, \tag{32}$$

and

$$\frac{d}{dr}\Sigma a_i\mu_i^2(7\mu_i^2-3)I_i=-\kappa\rho\Sigma a_i\mu_i(7\mu_i^2-3)I_i. \tag{33}$$

The foregoing equations can be written more compactly in terms of the quantities J, H, K, L, and M, defined as follows:

$$\left.\begin{aligned}&\tfrac{1}{2}\Sigma a_i I_i=J\,; &&\tfrac{1}{2}\Sigma a_i\mu_i^3 I_i=L\,,\\ &\tfrac{1}{2}\Sigma a_i\mu_i I_i=H\,; &&\tfrac{1}{2}\Sigma a_i\mu_i^4 I_i=M\,.\\ &\tfrac{1}{2}\Sigma a_i\mu_i^2 I_i=K\,; &&\end{aligned}\right\} \tag{34}$$

We have

$$H=\frac{1}{4}\frac{F_0}{r^2}, \tag{34'}$$

$$\frac{dK}{dr}+\frac{1}{r}(3K-J)=-\kappa\rho H, \tag{35}$$

$$\frac{d}{dr}(5L-H)+\frac{4}{r}(5L-3H)=-\tfrac{5}{3}\kappa\rho(3K-J), \tag{36}$$

and

$$\frac{d}{dr}(7M-3K)=-\kappa\rho(7L-3H). \tag{37}$$

Equations (35)–(37) provide us with three equations for the four unknowns J, K, L, and M. However, in our present approximation we can express M linearly in terms of J and K; for, since the μ_i's ($i=\pm 1, \pm 2$) are now defined as the zeros of $P_4(\mu)$, we have identically

$$\Sigma a_i P_4(\mu_i)I_i\equiv 0, \tag{38}$$

or, substituting for $P_4(\mu)$, we have

$$\Sigma a_i(35\mu_i^4-30\mu_i^2+3)I_i\equiv 0. \tag{39}$$

In other words,

$$35M-30K+3J=0. \tag{40}$$

Accordingly,

$$7M-3K=3K-\tfrac{3}{5}J\,, \tag{41}$$

and equation (37) becomes

$$\frac{d}{dr}\left(3K-\tfrac{3}{5}J\right)=-\kappa\rho\left(7L-3H\right). \tag{42}$$

Equations (34)–(36) and (42), together with the relevant boundary conditions on the I_i's, make the problem determinate.

We now transform equations (35), (36), and (42) to more convenient forms. Letting X and Y stand for

$$X=3K-J \quad \text{and} \quad Y=5L-3H\,, \tag{43}$$

we can re-write equation (35) as

$$\frac{dK}{dr}=-\frac{X}{r}-\frac{1}{4}\kappa\rho\frac{F_0}{r^2}, \tag{44}$$

where we have substituted for H according to equation (34′). Equation (44) can be formally integrated to give

$$K=-\int^r X\frac{dr}{r}-\frac{1}{4}F_0\int^r\frac{\kappa\rho dr}{r^2}. \tag{45}$$

Considering next equation (36), we obtain, after some minor reductions,

$$\frac{dY}{dr}+\frac{4}{r}Y=-\tfrac{5}{3}\kappa\rho X+\frac{F_0}{r^3}. \tag{46}$$

Again, since

$$3K-\tfrac{3}{5}J=\tfrac{6}{5}K+\tfrac{3}{5}\left(3K-J\right)=\tfrac{6}{5}K+\tfrac{3}{5}X\,, \tag{47}$$

equation (42) is clearly equivalent to

$$2\frac{dK}{dr}+\frac{dX}{dr}=-\tfrac{5}{3}\kappa\rho\left(7L-3H\right). \tag{48}$$

We can eliminate dK/dr from this equation by using equation (44). We obtain, in this manner,

$$\frac{dX}{dr}-\frac{2}{r}X=-\tfrac{7}{3}\kappa\rho Y. \tag{49}$$

Equations (46) and (49) provide us with a pair of simultaneous equations for X and Y. These equations can also be written in the forms

$$r^2\frac{d}{dr}\left(\frac{X}{r^2}\right)=-\tfrac{7}{3}\kappa\rho Y \tag{50}$$

and

$$\frac{1}{r^4}\frac{d}{dr}\left(r^4 Y\right)=-\tfrac{5}{3}\kappa\rho X+\frac{F_0}{r^3}. \tag{51}$$

From these equations we can readily eliminate X or Y to obtain a single second-order differential equation for either of them. We shall not perform this elimination at this stage, since to solve any of these equations we need a prior assumption concerning the dependence of $\kappa\rho$ on r.

5. *The solution in the second approximation for the case* $\kappa\rho\propto r^{-n}$, *when* $n>1$.—In terms of the radial optical thickness defined by

$$d\tau=-\kappa\rho dr\,, \tag{52}$$

the equations of the second approximation derived in the preceding section (eqs. [45], [46], and [49]) become

$$K=\int^{\tau} X \frac{d\tau}{\kappa\rho r}+\frac{1}{4}F_0\int^{\tau}\frac{d\tau}{r^2}, \tag{53}$$

$$\frac{dX}{d\tau}+\frac{2}{\kappa\rho r}X=\tfrac{7}{3}Y, \tag{54}$$

and

$$\frac{dY}{d\tau}-\frac{4}{\kappa\rho r}Y=\tfrac{5}{3}X-\frac{F_0}{\kappa\rho r^3}. \tag{55}$$

We shall now show how the foregoing equations can be solved in the case where $\kappa\rho$ is assumed to vary as some inverse power of r. Suppose, then, that

$$\kappa\rho=cr^{-n}, \tag{56}$$

where c is a constant. For a dependence of $\kappa\rho$ on r of this form and $n>1$ we can define the optical depth τ measured from $r=\infty$ inward; for in that case the integral

$$\tau=\int_r^{\infty}\kappa\rho dr, \tag{57}$$

converges, and, in fact, we have

$$\tau=\frac{c}{n-1}\frac{1}{r^{n-1}} \qquad (n>1). \tag{58}$$

For $n\leqslant 1$ the integral (57) diverges; but, as these cases have no astrophysical interest, we shall not consider them here.

From equations (56) and (58) we obtain the relations

$$\kappa\rho r=(n-1)\tau \tag{59}$$

and

$$\tau=\left(\frac{R}{r}\right)^{n-1}, \tag{60}$$

where R denotes the distance at which $\tau=1$. Using these relations in the equations (53), (54), and (55) and measuring the various quantities J, H, K, L, X, and Y in units of F_0/R^2 (i.e., in units of the emergent flux at $r=R$), we obtain the equations

$$K=\frac{1}{n-1}\int_0^{\tau}X\frac{d\tau}{\tau}+\frac{n-1}{4(n+1)}\tau^{(n+1)/(n-1)}, \tag{61}$$

$$\frac{dX}{d\tau}+\frac{2}{(n-1)\tau}X=\tfrac{7}{3}Y, \tag{62}$$

and

$$\frac{dY}{d\tau}-\frac{4}{(n-1)\tau}Y=\tfrac{5}{3}X-\frac{1}{n-1}\tau^{(3-n)/(n-1)}. \tag{63}$$

It will be noticed that in the integral occurring in equation (61) we have set the lower limit for τ as zero. This is in accordance with the requirement that, since the atmosphere now extends to infinity, all the quantities must vanish at $r=\infty$, i.e., at $\tau=0$.

Eliminating Y between equations (62) and (63), we find for X the differential equation

$$\frac{d^2X}{d\tau^2}-\frac{2}{(n-1)\tau}\frac{dX}{d\tau}-\frac{2(n+3)}{(n-1)^2\tau^2}X=\tfrac{35}{9}X-\frac{7}{3(n-1)}\tau^{(3-n)/(n-1)}. \tag{64}$$

With the substitutions

$$z = q\tau \qquad (q = \sqrt{35}/3 = 1.9720), \tag{65}$$

and

$$X = \tau^{(n+1)/2(n-1)} f = q^{-(n+1)/2(n-1)} z^{(n+1)/2(n-1)} f(z), \tag{66}$$

we find that equation (64) can be brought to the form

$$z^2 \frac{d^2 f}{dz^2} + z \frac{df}{dz} - (z^2 + \nu^2) f = -k z^{\mu+1}, \tag{67}$$

where we have written

$$\nu = \frac{n+5}{2(n-1)}; \qquad \mu = \frac{3-n}{2(n-1)}, \tag{68}$$

and

$$k = \frac{7}{3(n-1)} q^{-(n+1)/2(n-1)}. \tag{69}$$

The accompanying short table (Table 2), giving the values of ν and μ for a few values of n, may be noted.

TABLE 2

n	ν	μ	n	ν	μ
1.25	12.5	3.5	3.0	2.0	0
1.50	6.5	1.5	4.0	1.5	$-1/6$
$1\frac{2}{3}$	5.0	1.0	7.0	1.0	$-1/3$
2.0	3.5	0.5	∞	1/2	$-1/2$

Before proceeding to the solution of equation (67) we may observe that, if ϕ be defined as the solution of

$$z^2 \frac{d^2 \phi}{dz^2} + z \frac{d\phi}{dz} - (z^2 + \nu^2) \phi = -z^{\mu+1}, \tag{70}$$

we have, according to equations (66) and (69),

$$X = q^{-(n+1)/(n-1)} \frac{7}{3(n-1)} z^{(n+1)/2(n-1)} \phi(z). \tag{71}$$

Substituting this solution for ϕ in equation (61), we obtain

$$K = q^{-(n+1)/(n-1)} \left[\frac{7}{3(n-1)^2} \int_0^z z^{(3-n)/2(n-1)} \phi(z)\, dz + \frac{n-1}{4(n+1)} z^{(n+1)/(n-1)} \right]. \tag{72}$$

Moreover, since $3K - J = X$, we have for the source function J the solution

$$\left. \begin{aligned} J = q^{-(n+1)/(n-1)} \Big[& \frac{7}{(n-1)^2} \int_0^z z^{(3-n)/2(n-1)} \phi(z)\, dz \\ & - \frac{7}{3(n-1)} z^{(n+1)/2(n-1)} \phi(z) + \frac{3(n-1)}{4(n+1)} z^{(n+1)/(n-1)} \Big]. \end{aligned} \right\} \tag{73}$$

It now remains to solve equation (70).

First, it will be observed that the homogeneous part of equation (70) is simply Bessel's equation for a purely imaginary argument. The general solution of the homogeneous equation is, accordingly, known and can be expressed as a linear combination of the fundamental solutions $I_\nu(z)$ and $K_\nu(z)$.[10] The solution of the nonhomogeneous equation, therefore, can be found most conveniently by the method of the variation of the parameters.[11] Thus, writing

$$\phi = A(z) I_\nu(z) + B(z) K_\nu(z), \tag{74}$$

we determine the functions $A(z)$ and $B(z)$ by the equations

$$A'(z) I_\nu(z) + B'(z) K_\nu(z) = 0 \tag{75}$$

and

$$A'(z) I'_\nu(z) + B'(z) K'_\nu(z) = -z^{\mu-1}. \tag{76}$$

where we have used primes to denote differentiation with respect to the argument. Using the relation (Watson, p. 80, eq. [19])

$$I_\nu(z) K'_\nu(z) - I'_\nu(z) K_\nu(z) = -\frac{1}{z}, \tag{77}$$

we readily find from equation (75) and (76) that

$$A'(z) = -z^\mu K_\nu(z) \quad \text{and} \quad B'(z) = z^\mu I_\nu(z). \tag{78}$$

Hence,

$$A(z) = \int_z^{c_1} z^\mu K_\nu(z)\,dz \quad \text{and} \quad B(z) = \int_{c_2}^z z^\mu I_\nu(z)\,dz, \tag{79}$$

where c_1 and c_2 are constants unspecified for the present. The general solution of equation (70) can, accordingly, be expressed in the form

$$\phi = I_\nu(z) \int_z^{c_1} z^\mu K_\nu(z)\,dz + K_\nu(z) \int_{c_2}^z z^\mu I_\nu(z)\,dz. \tag{80}$$

For the problem on hand the arbitrary limits c_1 and c_2 in the general solution (80) are determined by the following considerations:

First, since none of the quantities must tend to infinity exponentially as $z \to \infty$, we must require that $c_1 = \infty$ in equation (80). This readily follows from the known asymptotic behaviors of $I_\nu(z)$ and $K_\nu(z)$ as $z \to \infty$ (cf. Watson, § 7.23, p. 202). Second, the vanishing of all the quantities as $z \to 0$ requires that (cf. eq. [71])

$$z^{(n+1)/2(n-1)} \phi(z) \to 0 \qquad (z \to 0). \tag{81}$$

But $K_\nu(z)$ diverges at the origin, and condition (81) can be met only by setting $c_2 = 0$. Thus the solution for ϕ appropriate for our problem is

$$\phi = I_\nu(z) \int_z^\infty z^\mu K_\nu(z)\,dz + K_\nu(z) \int_0^z z^\mu I_\nu(z)\,dz. \tag{82}$$

With this we have formally solved the equations of the second approximation for the case $\kappa\rho \propto r^{-n}$, where $n > 1$.

[10] We shall adopt, throughout, the definitions and notations of G. N. Watson in his *Treatise on the Theory of Bessel Functions*, Cambridge, England, 1922. In our further references to this work we shall simply refer to it as "Watson." In our particlar context see pp. 77–80.

[11] The treatment of Lommel's equation

$$z^2 \frac{d^2y}{dz^2} + z\frac{dy}{dz} + (z^2 - \nu^2)y = kz^{\mu+1}$$

in Watson, § 10.7, p. 345, is followed in our discussion of eq. (70).

6. *The numerical form of the solution for the case* $\kappa\rho \propto r^{-2}$.—As an example of the solution obtained in the preceding section, we shall consider the case $\kappa\rho \propto r^{-2}$. For this case the solutions for X, J, and K given in § 5 (eqs. [71]-[73]) become

$$\left.\begin{aligned} X &= q^{-3}\tfrac{7}{3}z^{3/2}\phi(z)\,, \\ K &= q^{-3}\left[\tfrac{7}{3}\int_0^z z^{1/2}\phi(z)\,dz + \tfrac{1}{12}z^3\right], \\ J &= q^{-3}\left[7\int_0^z z^{1/2}\phi(z)\,dz - \tfrac{7}{3}z^{3/2}\phi(z) + \tfrac{1}{4}z^3\right], \end{aligned}\right\} \tag{83}$$

where it might be recalled that

$$z = q\tau \qquad \text{and} \qquad q = \frac{\sqrt{35}}{3} = 1.972\,. \tag{84}$$

Moreover, when $n = 2$, $\nu = \frac{7}{2}$ and $\mu = \frac{1}{2}$ (cf. Table 2); and the solution (82) for ϕ takes the particular form

$$\phi = I_{7/2}(z)\int_z^\infty z^{1/2}K_{7/2}(z)\,dz + K_{7/2}(z)\int_0^z z^{1/2}I_{7/2}(z)\,dz\,. \tag{85}$$

The Bessel functions $I_{7/2}(z)$ and $K_{7/2}(z)$ are known explicitly, and we have (cf. Watson, p. 78)

$$I_{7/2}(z) = \left(\frac{2}{\pi z}\right)^{1/2}\left[\left(1+\frac{15}{z^2}\right)\cosh z - \left(\frac{6}{z}+\frac{15}{z^3}\right)\sinh z\right] \tag{86}$$

and

$$K_{7/2}(z) = \left(\frac{\pi}{2z}\right)^{1/2}e^{-z}\left(1+\frac{6}{z}+\frac{15}{z^2}+\frac{15}{z^3}\right). \tag{87}$$

Accordingly, we can re-write equation (85) in the form

$$\left.\begin{aligned} \phi = \frac{1}{z^{1/2}}\Bigg\{&\left[\left(1+\frac{15}{z^2}\right)\cosh z - \left(\frac{6}{z}+\frac{15}{z^3}\right)\sinh z\right]\int_z^\infty e^{-z}\left(1+\frac{6}{z}+\frac{15}{z^2}+\frac{15}{z^3}\right)dz \\ &+ e^{-z}\left(1+\frac{6}{z}+\frac{15}{z^2}+\frac{15}{z^3}\right)\int_0^z\left[\left(1+\frac{15}{z^2}\right)\cosh z - \left(\frac{6}{z}+\frac{15}{z^3}\right)\sinh z\right]dz\Bigg\}. \end{aligned}\right\} \tag{88}$$

It is seen that the first of the two integrals in equation (88) can be expressed in terms of known functions. We have

$$\int_z^\infty e^{-z}\left(1+\frac{6}{z}+\frac{15}{z^2}+\frac{15}{z^3}\right)dz = e^{-z}\left(1+\frac{15}{2z}+\frac{15}{2z^2}\right) - \tfrac{3}{2}Ei(z)\,, \tag{89}$$

where $Ei(z)$ stands for the exponential integral

$$Ei(z) = \int_1^\infty \frac{e^{-zt}}{t}\,dt\,. \tag{90}$$

The functions q^3X, q^3K, and q^3J, defined as in the foregoing paragraph, have been evaluated numerically for a range of values for z; and the results are given in Table 3.[12] For comparison we have also tabulated the function $z^3/4$, which is the solution for q^3J in the first approximation (cf. eq. [29]). It is seen that the second approximation introduces

[12] I am indebted to Miss Frances Herman for carrying out the necessary numerical work.

corrections to the extent of about 10 per cent. In view of this, it would be of interest further to examine the solution given in § 5 for other values of n. The case of $n = \frac{3}{2}$ would be of particular interest.

7. *Diffusion through a homogeneous sphere.*—The discussion of this case, while it has no special interest for astrophysics, is, however, likely to be of importance for problems of diffusion in physics.[13] But, apart from possible applications, the consideration of this

TABLE 3

z	q^3X	q^3K	q^3J	$\frac{1}{4}z^3$	z	q^3X	q^3K	q^3J	$\frac{1}{4}z^3$
0..	0	0	0	0	2.8..	2.5081	2.8719	6.1075	5.4880
0.1..	0.0002322	0.0001605	0.0002500	0.00025	2.9..	2.7048	3.1664	6.7944	6.0973
0.2..	0.0018377	0.0012825	0.0020097	0.00200	3.0..	2.9068	3.4791	7.5304	6.7500
0.3..	0.0061122	0.0043092	0.0068154	0.00675	3.1..	3.1137	3.8103	8.3172	7.4478
0.4..	0.014237	0.010158	0.016238	0.01600	3.2..	3.3254	4.1606	9.1564	8.1920
0.5..	0.027265	0.019718	0.031890	0.03125	3.3..	3.5414	4.5303	10.0494	8.9843
0.6..	0.046111	0.033844	0.055420	0.05400	3.4..	3.7617	4.9199	10.9979	9.8260
0.7..	0.071559	0.053357	0.088512	0.08575	3.5..	3.9858	5.3297	12.0033	10.7188
0.8..	0.10426	0.079047	0.13288	0.12800	3.6..	4.2136	5.7603	13.0671	11.6640
0.9..	0.14475	0.11167	0.19026	0.18225	3.7..	4.4448	6.2119	14.1910	12.6633
1.0..	0.19345	0.15195	0.26241	0.25000	3.8..	4.6792	6.6852	15.3763	13.7180
1.1..	0.25067	0.20059	0.35109	0.33275	3.9..	4.9165	7.1804	16.6246	14.8298
1.2..	0.31661	0.25824	0.45813	0.43200	4.0..	5.1565	7.6979	17.9373	16.0000
1.3..	0.39146	0.32557	0.58523	0.54925	4.1..	5.3991	8.2383	19.3159	17.2303
1.4..	0.47522	0.40317	0.73429	0.68600	4.2..	5.6440	8.8019	20.7618	18.5220
1.5..	0.56792	0.49165	0.90703	0.84375	4.3..	5.8911	9.3892	22.2766	19.8768
1.6..	0.66950	0.59158	1.10524	1.02400	4.4..	6.1401	10.0006	23.8617	21.2960
1.7..	0.77985	0.70352	1.33072	1.22825	4.5..	6.3909	10.6365	25.5184	22.7813
1.8..	0.89876	0.82800	1.58525	1.45800	4.6..	6.6434	11.2973	27.2484	24.3340
1.9..	1.0261	0.96556	1.8706	1.7148	4.7..	6.8975	11.9835	29.0529	25.9558
2.0..	1.1616	1.1167	2.1884	2.0000	4.8..	7.1528	12.6954	30.9334	27.6480
2.1..	1.3051	1.2819	2.5405	2.3153	4.9..	7.4095	13.4336	32.8914	29.4123
2.2..	1.4563	1.4616	2.9286	2.6620	5.0..	7.6672	14.1985	34.9283	31.2500
2.3..	1.6148	1.6564	3.3545	3.0418	5.1..	7.9260	14.9905	37.0454	33.1628
2.4..	1.7805	1.8667	3.8197	3.4560	5.2..	8.1856	15.8100	39.2443	35.1520
2.5..	1.9529	2.0930	4.3259	3.9063	5.3..	8.4461	16.6574	41.5263	37.2193
2.6..	2.1319	2.3356	4.8749	4.3940	5.4..	8.7072	17.5333	43.8928	39.3660
2.7..	2.3171	2.5951	5.4682	4.9208	5.5..	8.9690	18.4381	46.3453	41.5938

case is of definite interest, inasmuch as it provides the simplest illustration of the use of the equations of the second approximation obtained in § 4.

For a homogeneous sphere we naturally assume that

$$\kappa \rho = \text{constant} = \kappa_0 \,(\text{say}) . \tag{91}$$

As κ_0 is of dimensions $(\text{length})^{-1}$ it is convenient to measure length in units of $1/\kappa_0$ and intensity in units of $F_0 \kappa_0^2$ (i.e., in units of the emergent flux at $r = 1/\kappa_0$). In these units, equations (45), (46), and (49) now reduce to the forms

$$K = -\int \frac{X}{r} dr + \frac{1}{4r} , \tag{92}$$

$$\frac{dX}{dr} - \frac{2}{r} X = -\tfrac{7}{3} Y , \tag{93}$$

and

$$\frac{dY}{dr} + \frac{4}{r} Y = -\tfrac{5}{3} X + \frac{1}{r^3} . \tag{94}$$

[13] Cf. W. Bothe, *Zs. f. Phys.*, **119**, 493, 1942.

Eliminating Y between equations (93) and (94), we obtain for X the differential equation

$$r^2 \frac{d^2 X}{d r^2} + 2 r \frac{d X}{d r} - 6 X = \tfrac{35}{9} X r^2 - \frac{7}{3 r} . \tag{95}$$

Making the substitutions

$$z = \frac{\sqrt{35}}{3} r \tag{96}$$

and

$$X = z^{-1/2} y , \tag{97}$$

we find that equation (95) is transformed to

$$z^2 \frac{d^2 y}{d z^2} + z \frac{d y}{d z} - \left(z^2 + \tfrac{25}{4} \right) y = - \frac{7 \sqrt{35}}{9} z^{-1/2} . \tag{98}$$

We verify that

$$y = \frac{7 \sqrt{35}}{9} z^{-5/2} \tag{99}$$

represents a particular integral of equation (98); and, as the homogeneous part of this equation is Bessel's equation of order $\frac{5}{2}$ for a purely imaginary argument, we can write the general solution of equation (98) in the form

$$y = \frac{7 \sqrt{35}}{9} z^{-5/2} + A I_{5/2} (z) + B K_{5/2} (z) , \tag{100}$$

where A and B are two constants to be determined by the boundary conditions appropriate to the problem on hand. At this stage we can determine the constant B. For, as $z \to 0$,

$$K_{5/2} (z) \to 3 \left(\frac{\pi}{2} \right)^{1/2} z^{-5/2} \qquad (z \to 0) ; \tag{101}$$

and this is seen to be too high an order for the singularity at the origin. But by choosing

$$B = - \left(\frac{2}{\pi} \right)^{1/2} \frac{7 \sqrt{35}}{27} , \tag{102}$$

we can lower the order of the singularity at $z = 0$ by one to $z^{-3/2}$; for, with this choice of B the term in $z^{-5/2}$ in $K_{5/2}(z)$ cancels the particular integral (99). Accordingly, we write for ϕ the solution

$$y = \frac{7 \sqrt{35}}{9} z^{-5/2} - \left(\frac{2}{\pi} \right)^{1/2} \frac{7 \sqrt{35}}{27} K_{5/2} (z) + A I_{5/2} (z) , \tag{103}$$

where A is a constant to be determined later by conditions at the boundary of the sphere.

Corresponding to the solution (103) for y, we have

$$X = \frac{7 \sqrt{35}}{9} z^{-3} - \left(\frac{2}{\pi} \right)^{1/2} \frac{7 \sqrt{35}}{27} z^{-1/2} K_{5/2} (z) + A z^{-1/2} I_{5/2} (z) . \tag{104}$$

Equation (93) now enables us to determine Y; for, writing this equation in the form

$$Y = - \frac{3}{7} r^2 \frac{d}{d r} \left(\frac{X}{r^2} \right) = - \frac{\sqrt{35}}{7} z^2 \frac{d}{d z} \left(\frac{X}{z^2} \right) \tag{105}$$

and substituting for X according to equation (104), we find that

$$\left.\begin{aligned} Y &= 5L-3H \\ &= \tfrac{175}{9}z^{-4}-\left(\frac{2}{\pi}\right)^{1/2}\tfrac{35}{27}z^{-1/2}K_{7/2}(z)-\frac{\sqrt{35}}{27}A\,z^{-1/2}I_{7/2}(z)\,. \end{aligned}\right\} \qquad (106)$$

In obtaining the foregoing result we have made use of the recurrence relations satisfied by the Bessel functions (cf. Watson, p. 79, eq. [67]).

Again substituting for X in equation (92) according to equation (104), we similarly find that

$$\left.\begin{aligned} K=\frac{7\sqrt{35}}{27}z^{-3}-\left(\frac{2}{\pi}\right)^{1/2}\frac{7\sqrt{35}}{27}z^{-3/2}K_{3/2}(z)-A\,z^{-3/2}I_{3/2}(z) \\ +\frac{\sqrt{35}}{12z}+C\,, \end{aligned}\right\} \qquad (107)$$

where C is another constant of integration. Finally, remembering that $J = 3K - X$, we obtain for the source function J the solution

$$\left.\begin{aligned} J=\left(\frac{2}{\pi}\right)^{1/2}\frac{7\sqrt{35}}{27}z^{-3/2}[zK_{5/2}(z)-3K_{3/2}(z)]-A\,z^{-3/2}[zI_{5/2}(z) \\ +3I_{3/2}(z)]+\frac{\sqrt{35}}{4z}+3C\,. \end{aligned}\right\} \qquad (108)$$

With this we have explicitly solved all the equations of the second approximation for the case under consideration. It only remains to determine the two constants of integrations A and C. These can be determined by the conditions at the boundary of the sphere, namely, that here both I_{-1} and I_{-2} must vanish. But we shall not continue here with the details of the elementary calculations necessary for the determination of these constants. We may, however, note that for the case of an *infinite* homogeneous sphere the constants A and C must vanish and that the solution for J reduces to

$$J=\left(\frac{2}{\pi}\right)^{1/2}\frac{7\sqrt{35}}{27}z^{-3/2}(zK_{5/2}-3K_{3/2})+\frac{\sqrt{35}}{4z}\,; \qquad (109)$$

or, substituting for $K_{5/2}$ and $K_{3/2}$ their known explicit forms and reverting to the original variable r, we find

$$J=\frac{3}{4r}(1+\tfrac{28}{27}e^{-1.972r})\,, \qquad (110)$$

which is to be contrasted with what would be obtained on the first approximation, namely,

$$J=\frac{3}{4r} \qquad \text{(first approximation)}\,. \qquad (111)$$

It is seen that in going to the second approximation we introduce a "correction" term which amounts to 14.5 per cent at $r = 1$ and which further decreases rapidly for increasing r.

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