Absolutely Bounded Matrices

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Introduction. An infinite matrix (α_{ij}) is said to be *bounded* (respectively, *absolutely bounded*) if the matrix (α_{ij}) (respectively, the matrix $(|\alpha_{ij}|)$) induces a bounded operator on ℓ^2.

The Hilbert matrix is an example of a bounded matrix that is not absolutely bounded. The existence of such matrices makes desirable a characterization of those matrices that are or are unitarily equivalent to the absolutely bounded ones.

A sufficient condition for a matrix to be absolutely bounded was given by Schur ([6]). This was independently generalized to a theorem about integral operators by several people ([1], [2] and [3]). The first section of this paper contains a sort of interpolation theorem for integral operators on function spaces. This result is shown to include the generalized Schur test as a special case.

As for conditions for a bounded matrix to be unitarily equivalent to an absolutely bounded one, no non-trivial necessary conditions seem to be known. It may even be the case that every bounded matrix has a unitary transform that is absolutely bounded. The second section of this paper characterizes those bounded matrices (α_{ij}) which fulfill the stronger requirement that every unitary transform of (α_{ij}) is absolutely bounded. Theorem II asserts that such matrices are precisely the ones of the form $\lambda \cdot I + K$, where λ is a scalar, I is the identity matrix, and K is a Hilbert-Schmidt matrix.

§1. An interpolation theorem. Throughout this section, (X, μ) will denote a fixed separable, σ-finite measure space. Let $\mathcal{M}(X)$ be the vector space of equivalence classes of measurable functions on X (where, of course, functions are considered equivalent precisely when they agree a.e.). In the sequel, the symbol $\mathcal{E}$ will always denote a linear subspace of $\mathcal{M}(X)$.

Definition 1.1. $\mathcal{E}$ is said to be

(i) an *order-ideal* of $\mathcal{M}(X)$ if $f \in \mathcal{E}$ whenever $|f| \leq |g|$ a.e. for some g in $\mathcal{E}$;

(ii) *exhausting* if there exists an increasing sequence $\{E_n\}_{n=1}^{\infty}$ of measurable sets of finite measure such that $\chi_{E_n} \in \mathcal{E}$ for all n, and $\mu\left(X - \bigcup_n E_n\right) = 0$.

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(For instance, each $L^p(X)$ (for $1 \leq p \leq \infty$) is an exhausting order-ideal of $\mathcal{M}(X)$.)

Definition 1.2. A measurable function k on $X \times X$ is said to *induce an integral operator* A on $\mathcal{E}$ if, for each f in $\mathcal{E}$,

(a) $\int |k(x, y)f(y)|d\mu(y) < \infty$ a.e.; and

(b) $Af(x) = \int k(x, y)f(y)d\mu(y) \in \mathcal{E}$.

As is well known, if $\mathcal{E}$ is equipped with a norm $\|\cdot\|$, the collection $\mathcal{B}(\mathcal{E})$ of bounded operators on $(\mathcal{E}, \|\cdot\|)$ constitutes a normed algebra with respect to the operator norm: $\|A\| = \sup\{\|Af\|: f \in \mathcal{E}, \|f\| \leq 1\}$. A norm $\|\cdot\|$ on $\mathcal{E}$ will be said to be an *ideal norm* if $\|f\| \leq \|g\|$ whenever f and g are in $\mathcal{E}$ and such that $|f| \leq |g|$ a.e. (The usual norm $\|\cdot\|_p$ on $L^p(X)$ is clearly an example of an ideal norm, for $1 \leq p \leq \infty$.) Observe that if $\|\cdot\|$ is an ideal norm defined on an order ideal $\mathcal{E}$, then $f \in \mathcal{E}$ if and only if $|f| \in \mathcal{E}$, and $\|f\| = \| |f| \|$ for all f in $\mathcal{E}$.

Finally, if k is a measurable function on $X \times X$, the adjoint function k^* (on $X \times X$) is defined by $k^*(x, y) = \overline{k(y, x)}$.

Theorem I. *If $\mathcal{E}$ is an order-ideal of $\mathcal{M}(X)$, equipped with an ideal norm $\|\cdot\|$, and if k is a non-negative measurable function on $X \times X$ such that k and k^* induce bounded integral operators A and B respectively on $(\mathcal{E}, \|\cdot\|)$, then k induces a bounded integral operator T on $L^2(X)$ such that*

$$\|T\|_2 \leq (\|A\| \, \|B\|)^{\frac{1}{2}} .$$

(Here, and in the proof, the symbol $\|\cdot\|$ will be used for the norms on $\mathcal{E}$ and $\mathcal{B}(\mathcal{E})$, while $\|\cdot\|_2$ will denote the norms on $L^2(X)$ and $\mathcal{B}(L^2(X))$.)

Proof. Let h be any measurable function on $X \times X$ such that $|h| \leq k$ a.e. Then, for any f in $\mathcal{E}$,

$$\int |h(x, y)f(y)|d\mu(y) \leq \int |k(x, y)f(y)|d\mu(y)$$
$$< \infty \text{ a.e. (cf. (a) of Definition 1.2).}$$

Further,

$$\left| \int h(x, y)f(y)d\mu(y) \right| \leq \int k(x, y)|f(y)|d\mu(y)$$
$$= A|f|(x) \text{ a.e.,}$$

and so, $\int h(x, y)f(y)d\mu(y) \in \mathcal{E}$. (Recall that $\mathcal{E}$ is an order-ideal.) Further, since $\|\cdot\|$ is an ideal norm,

$$\left\|\int h(x, y)f(y)d\mu(y)\right\| \leq \||A|f|\,\| \leq \|A\|\,\|\,|f|\,\| = \|A\|\,\|f\|.$$

Hence, h induces a bounded integral operator on $\mathscr{E}$, whose norm is no greater than $\|A\|$.

Case (i): $k = k^*$. Let $\{E_n\}_{n=1}^{\infty}$ be an increasing sequence of subsets of X which demonstrates that $\mathscr{E}$ is exhausting; *i.e.*, $\mu(E_n) < \infty$, $\chi_{E_n} \in \mathscr{E}$ and $\mu\left(X - \bigcup_n E_n\right) = 0$. Choose a sequence $\{h_n\}_{n=1}^{\infty}$ of non-negative simple functions such that $h_n \uparrow k$ a.e., and such that $h_n = h_n^*$ for all n. Setting $k_n = \chi_{E_n \times E_n} h_n$, it is seen that $0 \leq k_n \uparrow k$ a.e., and that $k_n = k_n^*$ for all n.

Fix n temporarily. Since k_n is a simple function supported on a set of finite measure, it follows that $k_n \in L^2(X \times X)$. This, and the fact that $k_n = k_n^*$ imply that k_n induces a Hermitian Hilbert-Schmidt operator T_n on $L^2(X)$. Further, since $|k_n| \leq k$, the kernel k_n also induces a bounded integral operator A_n on $\mathscr{E}$ such that $\|A_n\| \leq \|A\|$.

The next step in the proof is to show that $\|T_n\|_2 \leq \|A_n\|$. If $k_n \equiv 0$, there is nothing to prove, since $\|T_n\|_2 = 0 = \|A_n\|$ in this case. So, suppose $k_n \not\equiv 0$. Then $\|T_n\|_2 > 0$. Since T_n is a compact Hermitian operator on $L^2(X)$, it follows that $\|T_n\|_2 = \max\{|\lambda|: \lambda \text{ is an eigenvalue of } T_n\}$. Thus, there exists an eigenvalue λ of T_n such that $|\lambda| = \|T_n\|_2 > 0$. Let f_0 be an eigenvector of T_n corresponding to λ. The aim of what follows is to show that $f_0 \in \mathscr{E}$, which would imply that

$$\|T_n\|_2 = |\lambda| = \frac{\|A_n f_0\|}{\|f_0\|} \leq \|A_n\|,$$

as desired.

Observe, to start with, that

$$\lambda f_0(x) = \int k_n(x, y) f_0(y) d\mu(y) \text{ a.e.}$$

In particular, $f_0(x) = 0$ for almost every x in E_n'. Thus $f_0 = f_0\chi_{E_n}$. Further, since k_n is a simple function, there is a constant $M > 0$ such that $|k_n| \leq M$ a.e. Thus,

$$\begin{aligned}|\lambda f_0(x)| &= \left|\int k_n(x, y) f_0(y) d\mu(y)\right| \\ &\leq \int_{E_n} |k_n(x, y) f_0(y)| d\mu(y) \\ &\leq M \int_{E_n} |f_0(y)| d\mu(y) \\ &\leq M\, \|f_0\|_2\, \mu(E_n)^{\frac{1}{2}}.\end{aligned}$$

Since f_0 is supported on E_n, this implies that

$$|f_0| \leq \left[\frac{M||f_0||_2 \mu(E_n)^{\frac{1}{2}}}{|\lambda|} \right] \chi_{E_n},$$

and consequently, that $f_0 \in \mathscr{E}$. (Recall that $\chi_{E_n} \in \mathscr{E}$ and that $\mathscr{E}$ is an order-ideal.) Hence, by an earlier remark, $||T_n||_2 \leq ||A_n||$. Since $||A_n|| \leq ||A||$, and since n was arbitrary, this means that

$$\sup_n ||T_n||_2 \leq ||A||.$$

It is to be shown that k induces a bounded integral operator T on $L_2(X)$ such that $||T||_2 \leq ||A||$. So, pick any f in $L^2(X)$. Let

$$g_n(x) = \int k_n(x, y)|f(y)|d\mu(y)$$

and

$$g(x) = \int k(x, y)|f(y)|d\mu(y).$$

Observe that $g_n = T_n|f|$ and conclude that

$$(1.1) \qquad ||g_n||_2 \leq ||T_n||_2 \, ||f||_2 \leq ||A|| \, ||f||_2.$$

Since $0 \leq k_n(x, y) \nearrow k(x, y)$ for almost every (x, y) in $X \times X$, Fubini's theorem implies that except for x's in a set of measure zero,

$$0 \leq k_n(x, y)|f(y)| \nearrow k(x, y)|f(y)|$$

for almost every y in X. The monotone convergence theorem implies that $0 \leq g_n(x) \nearrow g(x)$ for the "good" x's, and hence a.e. A second application of the monotone convergence theorem shows that

$$||g||_2^2 = \lim_{n \to \infty} ||g_n||_2^2 \leq ||A||^2 \, ||f||_2^2 \text{ (by (1.1))}.$$

In particular $|g| < \infty$ a.e. Thus, if $f \in L^2(X)$,

$$\int |k(x, y)f(y)|d\mu(y) < \infty \text{ a.e.},$$

and

$$\int \left| \int k(x, y)f(y)d\mu(y) \right|^2 d\mu(x) \leq ||A||^2 ||f||_2^2.$$

In other words, k induces a bounded integral operator on $L^2(X)$ whose norm is no greater than $||A||$, and the proof of Case (i) is complete.

Case (ii): k arbitrary. Let $k_1 = \dfrac{k + k^*}{2}$, $k_2 = \dfrac{k - k^*}{2i}$. Since k and k^* both induce bounded integral operators on $\mathscr{E}$, the same is clearly true of k_1 and k_2. Since however, $k_i^* = k_i$ (for $i = 1, 2$), it follows (from Case (i)) that both k_1 and k_2 induce bounded integral operators on $L^2(X)$; the same is necessarily true of $k = k_1 + ik_2$.

Let T be the bounded integral operator on $L^2(X)$ induced by k. A moment's thought and some easy verification shows that the kernel $\tilde{k}$ defined by

$$\tilde{k}(x, y) = \int k^*(x, z)k(z, y)d\mu(z)$$

induces the bounded integral operators T^*T and BA on $L^2(X)$ and $\mathscr{E}$ respectively. Since however, $\tilde{k} = \tilde{k}^*$, another appeal to Case (i) shows that

$$\|T^*T\|_2 \leq \|BA\|.$$

Hence,

$$\|T\|_2 = \|T^*T\|_2^{\frac{1}{2}} \leq \|BA\|^{\frac{1}{2}} \leq (\|A\|\,\|B\|)^{\frac{1}{2}},$$

and the proof is complete.

Corollary. (Schur test). *Let k be a non-negative measurable function on $X \times X$. Suppose there exist positive constants α and β, and a strictly positive measurable function p on X such that*

$$\int k(x, y)p(y)d\mu(y) \leq \alpha p(x) \text{ a.e.}$$

and

$$\int k(x, y)p(x)d\mu(x) \leq \beta p(y) \text{ a.e.}$$

Then k induces a bounded integral operator on $L^2(X)$ with norm no greater than $(\alpha\beta)^{\frac{1}{2}}$.

Proof. Let $\mathscr{E} = \left\{f \in \mathscr{M}(X): \dfrac{f}{p} \in L^\infty(X)\right\}$; define $\|f\| = \left\|\dfrac{f}{p}\right\|_\infty$ for f in $\mathscr{E}$.

It is easily seen that $\mathscr{E}$ is an order-ideal and that $\|\cdot\|$ is an ideal norm on $\mathscr{E}$. Since X is σ-finite, there exists an increasing sequence $\{F_n\}_{n=1}^\infty$ of sets of finite measure, whose union is X.

Let $E_n = F_n \cap \left\{p > \dfrac{1}{n}\right\}$. Then $\bigcup_n E_n = X$ and $\mu(E_n) \leq \mu(F_n) < \infty$.

Further, $\left|\dfrac{\chi_{E_n}}{p}\right| \leq n$, and so $\chi_{E_n} \in \mathscr{E}$. Thus $\mathscr{E}$ is exhausting.

Finally, observe that if $f \in \mathscr{E}$,

$$\begin{aligned}\int |k(x, y)f(y)|d\mu(y) &= \int k(x, y)p(y)\left|\frac{f(y)}{p(y)}\right| d\mu(y)\\ &\leq \|f\| \int k(x, y)p(y)d\mu(y)\\ &\leq \|f\|\alpha p(x)\\ &< \infty \text{ a.e.}\end{aligned}$$

Hence, if $Af(x) = \int k(x, y)f(y)d\mu(y)$, it follows that $|Af| \leq \alpha\|f\|p$ a.e., and hence that $\left\|\frac{Af}{p}\right\|_\infty \leq \alpha\|f\|$. Thus, $f \in \mathscr{E} \Rightarrow Af \in \mathscr{E}$ and $\|Af\| \leq \alpha\|f\|$. So k induces a bounded integral operator A on $\mathscr{E}$ such that $\|A\| \leq \alpha$.

Similarly, the inequality

$$\int k(x, y)p(x)d\mu(x) \leq \beta p(y) \text{ a.e.}$$

can be shown to imply that k^* induces a bounded integral operator B on $\mathscr{E}$ such that $\|B\| \leq \beta$.

An application of Theorem I completes the proof. ∎

Note. The original Schur test was the particular case of Corollary 1 with $X = \{1, 2, 3, \cdots\}$, μ = counting measure on X, and $p(n) \equiv 1$.

§2. Universally absolutely bounded matrices.

Definition 2.1. A matrix A is said to be *universally absolutely bounded* if U^*AU is absolutely bounded, for every unitary matrix U.

Let $\mathscr{U}$ denote the class of universally absolutely bounded matrices. The symbols $\mathscr{B}$ and $\mathscr{A}$ will denote the classes of bounded and absolutely bounded matrices, respectively. If $A = (\alpha_{ij})$, the induced absolute matrix $(|\alpha_{ij}|)$ will be referred to as $|A|$.

Lemma 2.2. *Let ω_n be a primitive n-th root of unity. Then the $n \times n$ matrix U_n, whose (i, j)-th entry is $\frac{1}{\sqrt{n}}\,\omega_n^{ij}$, is unitary. Further, if A is the $n \times n$ diagonal matrix*

$$\begin{bmatrix} \mu_1 & 0 & \cdots & 0 \\ \vdots & \mu_2 & \cdots & 0 \\ & \vdots & \ddots & \vdots \\ 0 & 0 & & \mu_n \end{bmatrix},$$

then

$$\|\,|U_nA|\,\| = \left(\sum_{i=1}^{n} |\mu_i|^2\right)^{\frac{1}{2}}.$$

Proof. It is easily verified that U_n is unitary. A simple computation shows that

$$|U_nA| = \frac{1}{\sqrt{n}}\begin{bmatrix} |\mu_1| & |\mu_2| & \cdots & |\mu_n| \\ |\mu_1| & |\mu_2| & \cdots & |\mu_n| \\ \vdots & \vdots & & \vdots \\ |\mu_1| & |\mu_2| & \cdots & |\mu_n| \end{bmatrix}.$$

Hence,

$$\| |U_nA| \| = \sup \left\{ \frac{1}{\sqrt{n}} \left[\sum_{i=1}^{n} \left| \sum_{j=1}^{n} |\mu_j| x_j \right|^2 \right]^{\frac{1}{2}} : \sum_{j=1}^{n} |x_j|^2 \leq 1 \right\}$$

$$= \sup \left\{ \left| \sum_{j=1}^{n} |\mu_j| x_j \right|^{\frac{1}{2}} : \sum_{j=1}^{n} |x_j|^2 \leq 1 \right\}$$

$$= \left(\sum_{j=1}^{n} |\mu_j|^2 \right)^{\frac{1}{2}} . \qquad \blacksquare$$

Theorem II. *A matrix is universally absolutely bounded if and only if it is the sum of a Hilbert-Schmidt matrix and a scalar multiple of the identity matrix.*

Proof. Recall that a matrix (α_{ij}) is a Hilbert-Schmidt matrix if $\sum_{i,j=1}^{\infty} |\alpha_{ij}|^2 < \infty$. Such matrices are clearly absolutely bounded. Since every unitary transform of a Hilbert-Schmidt matrix is again one such matrix, Hilbert-Schmidt matrices are universally absolutely bounded. Since $\mathcal{U}$ is clearly a linear subspace of $\mathcal{B}$, and since $I \in \mathcal{U}$, it follows that every matrix of the form "scalar + Hilbert-Schmidt" is universally absolutely bounded.

Conversely, suppose $A \in \mathcal{U}$. It is clear that $A^* \in \mathcal{U}$ (where, of course, A^* is the conjugate transpose of A), and hence, that $A_R = \frac{A + A^*}{2}$ and $A_I = \frac{A - A^*}{2i}$ are both in $\mathcal{U}$. If A_R and A_I were both of the form "scalar + Hilbert-Schmidt", the same would also be true of $A = A_R + iA_I$. In other words, it suffices to prove the theorem under the further assumption that A is Hermitian.

If A is Hermitian, the Weyl-von Neumann theorem (*cf.* [7]) allows a decomposition: $A = A_1 + K_1$, where A_1 is a diagonalizable Hermitian matrix and K_1 is a Hilbert-Schmidt matrix. Since $K_1 \in \mathcal{U}$—this follows from the already established half of this theorem—A_1 is also universally absolutely bounded. There is clearly no loss of generality in assuming that A_1 is a diagonal matrix. (Since it is to be proved that A_1 is of the form "scalar + Hilbert-Schmidt", and since some unitary transform of A_1 is diagonal, this reduction is possible.) Suppose $A_1 = \operatorname{diag}(\lambda_1, \lambda_2, \lambda_3, \cdots)$. Since $\{\lambda_n\}_{n=1}^{\infty}$ is a bounded set, it has a cluster point λ. Pick a subsequence $\{\lambda_{n_k}\}_{k=1}^{\infty}$ such that $\sum_k |\lambda_{n_k} - \lambda|^2 < \infty$. Let $K_2 = \operatorname{diag}(\mu_1, \mu_2, \mu_3, \cdots)$, where

$$\mu_n = \begin{cases} \lambda_{n_k} - \lambda, & \text{if} \quad n = n_k \\ 0, & \text{if} \quad n \notin \{n_k\}_{k=1}^{\infty}. \end{cases}$$

The choice of the λ_{n_k}'s implies that K_2 is a Hilbert-Schmidt matrix. Define

$$\xi_n = \begin{cases} \lambda_n - \lambda, & \text{if} \quad n \notin \{n_k\}_{k=1}^{\infty} \\ 0, & \text{if} \quad n = n_k \text{ for some } k, \end{cases}$$

and let $A_2 = \text{diag}(\xi_1, \xi_2, \xi_3, \cdots)$. Then, $A = \lambda I + (K_1 + K_2) + A_2$.

Since A, λI, K_1 and K_2 are in $\mathcal{U}$, it is seen that $A_2 \in \mathcal{U}$. The proof will be completed by showing that A_2 is a Hilbert-Schmidt matrix. Observe that A_2 is of the form $\begin{bmatrix} B & 0 \\ 0 & 0 \end{bmatrix}$, where the blocks are infinite and B is a diagonal matrix. Relabel the eigenvalues of B and suppose that $B = \text{diag}(\eta_1, \eta_2, \eta_3, \cdots)$, In view of the foregoing remarks, it suffices to show that $\sum_k |\eta_k|^2 < \infty$.

Let P be any bounded matrix such at $\|P\| \leq 1$. It follows from the existence of unitary dilations of contractions (*cf.*, for instance [5], Problem 177) that for suitably chosen Q, R and S in $\mathcal{B}$, the operator matrix $U = \begin{bmatrix} P & Q \\ R & S \end{bmatrix}$ is unitary. Observe that

$$U^*A_2U = \begin{bmatrix} P & R^* \\ Q^* & S^* \end{bmatrix} \begin{bmatrix} B & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} P & Q \\ R & S \end{bmatrix}$$
$$= \begin{bmatrix} P^*BP & P^*BQ \\ Q^*BP & Q^*BQ \end{bmatrix};$$

hence,

$$|U^*A_2U| = \begin{bmatrix} |P^*BP| & |P^*BQ| \\ |Q^*BP| & |Q^*BQ| \end{bmatrix}.$$

Since $A_2 \in \mathcal{U}$, the matrix $|U^*A_2U|$ is bounded. In particular, the matrix $|P^*BP|$ is bounded. Thus, it has been shown that $P^*BP \in \mathcal{A}$ whenever P is a contraction. Since every bounded matrix is a scalar multiple of a contraction, it follows that $P^*BP \in \mathcal{A}$ for every P in $\mathcal{B}$.

If $T \in \mathcal{B}$ and $\lambda \in \mathbb{C}$, the preceding paragraph implies that $(T + \lambda)^*B(T + \lambda)$, T^*BT and $|\lambda|^2B$ are all absolutely bounded. Hence,

$$\lambda T^*B + \bar{\lambda}BT = (T + \lambda)^*B(T + \lambda) - T^*BT - |\lambda|^2B \in \mathcal{A}.$$

For $\lambda = 1$ and $\lambda = i$, this reads:

$$T^*B + BT \in \mathcal{A};$$

$$i(T^*B - BT) \in \mathcal{A}.$$

From this, it is seen that T^*B, $BT \in \mathcal{A}$, for every bounded matrix T.

Suppose now that $\sum_k |\eta_k|^2 = +\infty$. It is clearly possible then to construct a sequence $\{m_k\}_{k=0}^{\infty}$ of integers such that $0 = m_0 < m_1 < m_2 < \cdots$, and

$$\sum_{j=m_{k-1}+1}^{m_k} |\eta_j|^2 \geq k^2, \quad \text{for} \quad k = 1, 2, \cdots. \tag{2.1}$$

Let

$$B_k = \begin{bmatrix} \eta_{m_{k-1}+1} & 0 & \cdots & 0 \\ 0 & \eta_{m_{k-1}+2} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \eta_{m_k} \end{bmatrix}, \text{ for } k = 1, 2, \cdots,$$

so that $B = \bigoplus_{k=1}^{\infty} B_k$. Let $T = \bigoplus_{k=1}^{\infty} U^*_{(m_k - m_{k-1})}$ (with U_m as in Lemma 2.2). Then T is clearly bounded (in fact, unitary). Observe that

$$|T^*B| = \bigoplus_{k=1}^{\infty} |U_{(m_k - m_{k-1})} B_k|. \tag{2.2}$$

It follows from (2.1) and Lemma 2.2 that $\| \, |U_{(m_k - m_{k-1})} B_k| \, \| \geq k$, for each k. This, together with (2.2), shows that $|T^*B|$ is not bounded, contradicting the already established fact that $S^*B \in \mathscr{A}$ for every S in $\mathscr{B}$. This contradiction implies that $(\eta_k)_{k=1}^{\infty} \in \ell^2$; thus B (and hence $A_2 = B \oplus 0$) is a Hilbert-Schmidt matrix. Hence, $A = \lambda + (K_1 + K_2 + A_2)$ is a decomposition of A as the sum of a scalar and a Hilbert Schmidt-matrix.

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