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Cherenkov radiation in spatially dispersive media

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The Cherenkov fields of a proton, and a neutron, moving with a relativistic velocity in a spatially dispersive medium are studied in the rest frame of the particle. The model of the medium used is typical of the behaviour of a dielectric near an exciton transition, and includes as a special case a screening medium like an isotropic plasma. The Fourier integral for the field of a proton is shown to split up into three integrals, each of which is identical to that in an ordinary medium but for a weight factor dependent on the frequency of the Fourier component. Each of these integrals is associated with one mode of Cherenkov emission, with its own threshold. The motion of the charge gives rise to three coaxial diffuse circular field cones with an azimuthally symmetric intensity distribution. The output of photons in each mode is evaluated. The field and output of a relativistic neutron are also evaluated for different orientations of the magnetic moment of the neutron relative to the direction of motion. It is shown that there are only two cones in this case, consistent with the fact that magnetic sources cannot excite the longitudinal plasma mode in a medium which is spatially dispersive only in its electrical properties.

1. INTRODUCTION

The phenomenological electrodynamics of spatially dispersive media has been the object of increasing research interest. This stems not only from the theoretical significance that attaches to the problem of solving Maxwell's equations with a non-local polarization (Agarwal *et al.* 1974; Pekar 1957), but also from the variety of effects that have their origin in the phenomenon of spatial dispersion. Some of these effects, for example, optical activity (Landau & Lifshitz 1960) and Debye screening (Pines & Bohm 1952), are very well known, while some others have been found only recently, for example, double refraction in cubic crystals (Gross & Kaplyanskii 1960), Doppler spatial dispersion in a plasma (Neufeld 1961), anomalous reflexion near an exciton band in a solid (Hopfield & Thomas 1963), and, in general, the optical effects of polaritons (Burstein & Martini 1974).

Irrespective of the mechanism that is responsible for the non-local character of polarization in space–time, it manifests itself as a dependence of the permeabilities on the wavevector $\boldsymbol{k}$ in addition to the frequency ω. This results in an increase in the degree of the dispersion equation, which gives rise to the possibility of several waves propagating in the medium in the same direction but with different velocities. In addition to this, the solutions of the equation $\hat{\epsilon}(\boldsymbol{k}, \omega) = 0$ are no longer restricted to a zero group-velocity, so that longitudinal waves become an acceptable mode of propagation in the medium.

Some of the consequences of incorporating a non-local polarization in classical electrodynamics were reviewed by Rukhadze & Silin (1961). The optics of crystals with allowance for spatial dispersion was investigated by Agranovich & Ginzburg (1966). The controversial boundary-value problem of reflexion and refraction of light from a spatially dispersive slab was solved entirely within the framework of Maxwell's electrodynamics by Maradudin & Mills (1973) and Agarwal *et al.* (1974). More recently, the transient optical response of a spatially dispersive medium has been investigated by Frankel & Birman (1977).

In this paper we work out the Cherenkov effect in an unbounded spatially dispersive medium, adopting the simplifying features of the special Lorentz frame moving with the particle. We calculate the radiation fields and output of a charge (typified by a proton) and a magnetic dipole (typified by a neutron). The techniques used in this paper follow from the earlier work on a non-dispersive medium by Majumdar & Pal (1970). The model of the medium chosen describes well a dielectric in the exciton régime (Agranovich & Ginzburg 1966).

We set up the Fourier integral for the potential of a charge in § 2 and find that it splits into three similar integrals, each of which corresponds to one possible mode of Cherenkov emission. In § 3 we evaluate the radiative and Coulomb fields and from these obtain the Cherenkov output in the various modes. In § 4, we exploit the duality symmetry of the augmented Maxwell's equations and derive the fields and output of a relativistic magnetic monopole and a magnetic dipole of arbitrary orientation.

2. Fourier synthesis of the Cherenkov field

We wish to evaluate the Cherenkov and Coulomb fields of a charge e moving with a uniform velocity βc along the x_1^0 axis in an isotropic spatially dispersive medium. As the Cherenkov effect and allied problems of electromagnetic radiation in material media involve a relative motion of the sources and media of varying complexity, the rest frame of the medium (Σ^0) and that of the source (Σ) suggest themselves as specially suitable for a solution of these problems. Whereas the source function is simply described in Σ, the constitutive relations take simple forms in Σ^0. For a particle moving uniformly in an unbounded medium, an evaluation in Σ is simpler than one in Σ^0 even though an isotropic, non-magneto-electric, spatially non-dispersive medium in Σ^0 appears to behave as a uni-axial, magneto-electric, spatially dispersive medium in Σ (Sastry 1978). This complication is however offset by the field becoming static in the rest frame of the particle, and hence it can be derived from two scalar potentials ϕ_e and ϕ_m (Majumdar & Pal 1970). If one uses the technique of evaluating the output from the work done on the particle by the retarding force set up by the polarization of the medium (Landau & Lifshitz

1960), one needs to evaluate only one of these potentials, and that only at the site of the particle.

Let the homogeneous, isotropic, spatially dispersive medium be characterized by the following rest-frame permeabilities:

$$\hat{\epsilon}(\boldsymbol{k}^0, \omega^0) = \epsilon(\omega^0)\left[1+\frac{\eta^2(\omega^0)}{(\boldsymbol{k}^0)^2-\psi^2(\omega^0)}\right]; \quad \hat{\mu}(\boldsymbol{k}^0, \omega^0) = \mu(\omega^0), \tag{2.1}$$

where ϵ, μ, η and ψ are in general functions of the frequency ω^0. Equation (2.1) gives the general form of the permeabilities: with η^2 and ψ^2 chosen as appropriate functions of the exciton mass and lifetime of the excited state, it describes a dielectric medium in the exciton régime; with $\psi = 0$ and $\eta = r_{\mathrm{D}}^{-1}$, it describes a screening medium, for example, an isotropic plasma, r_{D} denoting the Debye length of the medium; while, with $\eta = 0$, we recover an ordinary spatially non-dispersive medium.

The constitutive matrix in Σ can be evaluated by a Lorentz transformation of the fourth-rank four-tensor T_{ijkl} connecting the induction and field tensors H_{ij} and F_{kl} (Majumdar & Pal 1970). Inserting the matrix into Maxwell's equations, one obtains

$$\phi^{\mathrm{e}}(\boldsymbol{r}) = -\frac{e}{8\pi^3}\int \frac{\gamma^2(\beta^2\hat{\epsilon}\hat{\mu}-1)}{\hat{\epsilon}}\,\frac{\exp{(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r})}}{k_3^2+k_2^2-\gamma^2(\beta^2\hat{\epsilon}\hat{\mu}-1)\,k_1^2}\,\mathrm{d}^3k, \tag{2.2}$$

where $\gamma^2 = (1-\beta^2)^{-1}$.

Since the medium is assumed isotropic, the only preferred direction in the problem is the x_1-axis. Therefore there is symmetry in k_2 and k_3 as borne out by (2.2). The special Lorentz transformation to the rest frame of the charge implies

$$k_1^0 = \gamma k_1, \quad k_2^0 = k_2, \quad k_3^0 = k_3, \quad \omega^0 = \gamma\beta c k_1. \tag{2.3}$$

Equation (2.3) shows that k_1 is proportional to the frequency ω^0 in the rest frame of the medium. Hence the k_1-integral in (2.2) can be evaluated only if we assume the explicit functional dependence of ϵ, μ, η and ψ on ω^0. The integrals over k_2 and k_3 can be evaluated in any order since there is perfect symmetry in x_2 and x_3. We shall start by evaluating first the k_3-integral and then the k_2-integral, and expect to obtain a final result symmetric in x_2 and x_3. Substituting (2.1) and (2.3) into (2.2), we obtain

$$\phi^{\mathrm{e}}(\boldsymbol{r}) = -\frac{e}{8\pi^3}\int \frac{\mu\gamma^2}{n^2}\,\frac{N^{\mathrm{e}}}{D^{\mathrm{e}}}\exp{(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r})}\,\mathrm{d}^3k, \tag{2.4}$$

where $n^2 = \epsilon\mu$; D^{e} is a product of three factors:

$$D^{\mathrm{e}} = \prod_{j=1}^{3}(k_3^2-A_j^2), \tag{2.5}$$

where
$$A_j^2 = \xi_j^2k_1^2-k_2^2, \quad \xi_j^2 = \gamma^2(\beta^2n_j^2-1) \tag{2.6}$$

and
$$n_1^2 = \frac{1}{2}\left(n^2+\frac{\psi^2}{k_0^2}\right)+\left[\frac{\eta^2n^2}{k_0^2}+\frac{1}{4}\left(n^2-\frac{\psi^2}{k_0^2}\right)^2\right]^{\frac{1}{2}}, \tag{2.7a}$$

$$n_2^2 = \frac{1}{2}\left(n^2+\frac{\psi^2}{k_0^2}\right)-\left[\frac{\eta^2n^2}{k_0^2}+\frac{1}{4}\left(n^2-\frac{\psi^2}{k_0^2}\right)^2\right]^{\frac{1}{2}}, \tag{2.7b}$$

$$n_3^2 = (\psi^2-\eta^2)/k_0^2, \tag{2.7c}$$

with $k_0^2 = \omega^{0\,2}/c^2$. The denominator of the Fourier integral in the rest frame of the charge is just the dispersion equation (in disguise) in the rest frame of the medium. From this it follows that the triple product in (2.5) has its genesis in the existence of three possible modes of propagation in the spatially dispersive medium. By working out the appropriate dispersion equations in Σ^0, it can be checked that the n_j of (2.7) are precisely the three refractive indices of the medium, two for transverse modes ($j = 1, 2$) and one for the longitudinal plasma mode ($j = 3$).

The numerator N^{e} of (2.4) has the form

$$N^{\mathrm{e}}(k_3^2) = [(k_3^2 - A_3^2)(\beta^2 n^2 - 1) + \eta^2](k_3^2 - A_3^2 - \eta^2). \tag{2.8}$$

Splitting the Fourier integrals (2.4) into partial fractions, we obtain

$$\phi^{\mathrm{e}}(\boldsymbol{r}) = \sum_{j=1}^{3} \phi_j^{\mathrm{e}}(\boldsymbol{r}), \tag{2.9}$$

where
$$\phi_j^{\mathrm{e}}(\boldsymbol{r}) = -\frac{e}{8\pi^3}\int \frac{\mu\gamma^2}{n^2}\left[\frac{N^{\mathrm{e}}(k_3^2 = A_j^2)}{(A_k^2 - A_j^2)(A_i^2 - A_j^2)}\right]\frac{\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r})}{k_3^2 - A_j^2}\,\mathrm{d}^3k, \tag{2.10}$$

the i, j, k being cyclic permutations of 1, 2, 3. The total field (2.9) is thus an uncoupled mixture of the three partial fields (2.10).

3. Evaluation of the fields and the Cherenkov output

Cherenkov processes occur in those bands of frequencies where the imaginary parts of the refractive indices are negligibly small. Radiation in these bands is observable even at large distances from the line of motion of the charge. On the other hand, it is rapidly absorbed when the imaginary parts of n_j become appreciably large (Sastry & Parida 1978). The Cherenkov effect in these bands merely serves as one more contribution to the stopping power of the medium in addition to excitation, ionization and ohmic losses (Bohr 1948; Fermi 1940; Price 1955).

To evaluate the scalar potential ϕ^{e}, we start by making the k_3-integration in (2.10), using the residue theorem in the complex k_3-plane. The structure of the field is decided by the configuration of the poles of the k_3-integrand, with radiative fields arising from the poles lying close to the real axis. Apart from the factor in the square brackets, which is independent of k_3, the k_3-integrand of (2.10) is identical to that in an ordinary (spatially non-dispersive) medium, and real poles occur only if the particle velocity satisfies the condition

$$\beta^2 n_j^2 > 1, \quad j = 1, 2, 3. \tag{3.1}$$

The poles determine the three threshold velocities for the three modes of Cherenkov emission.

For velocities of the particle exceeding the Cherenkov threshold of the jth mode, the poles of the integrand lie on the real k_3-axis as long as $k_2^2 < \xi_j^2 k_1^2$, and shift to the imaginary axis for $k_2^2 > \xi_j^2 k_1^2$. Viewed from the rest frame Σ^0 of the medium, the field of the moving charge can be resolved into an angular spectrum of plane waves that in general consists of both homogeneous and evanescent components. The occurrence of these two types of plane waves corresponds respectively with the presence of the real and imaginary poles in the k_3-integrand in the rest frame Σ of the particle. When poles occur on the real axis, the contour along it is indented in accord with the usual requirements of causality on the sign of the infinitesimal imaginary part of the refractive index n_j. As the factor in the square brackets of (2.10) is also independent of k_2, we find that the k_2-integral too is identical with that in an ordinary medium but for a factor which is a function only of k_1. We thus obtain (Sastry & Majumdar 1974)

$$\phi^{\mathrm{e}}_{j,\,\mathrm{Cherenkov}} = -\frac{\mathrm{i}e}{8\pi}\int \frac{\mu\gamma^2}{n^2} F^{\mathrm{e}}_j(k_1)\exp(\mathrm{i}k_1x_1) \times [\Theta(k_1)\,H_0^{(1)}(\xi_j\,|k_1|\,\rho) - \Theta(-k_1)\,H_0^{(2)}(\xi_j\,|k_1|\,\rho)]\,\mathrm{d}k_1, \qquad (3.2)$$

where Θ is the Heaviside step function, $H^{(1)}_0$, $H^{(2)}_0$ are the Hankel functions $J_0 \pm \mathrm{i}N_0$, $\rho = (x_2^2+x_3^2)^{\frac{1}{2}}$ is the distance of the field point from the line of motion, and

$$F^{\mathrm{e}}_1 = \frac{[(n_1^2-n_3^2)(\beta^2n^2-1)+\eta^2/k_0^2]\,(n_1^2-n_3^2-\eta^2/k_0^2)}{(n_2^2-n_1^2)\,(n_3^2-n_1^2)}, \qquad (3.3a)$$

$$F^{\mathrm{e}}_2 = \frac{[(n_2^2-n_3^2)(\beta^2n^2-1)+\eta^2/k_0^2]\,(n_2^2-n_3^2-\eta^2/k_0^2)}{(n_3^2-n_2^2)\,(n_1^2-n_2^2)}, \qquad (3.3b)$$

$$F^{\mathrm{e}}_3 = \frac{\eta^4}{k_0^4(n_1^2-n_3^2)\,(n_3^2-n_2^2)}. \qquad (3.3c)$$

Since k_1 is proportional to the frequency ω^0, the result of the final integration in (3.2) depends on the detailed dispersive behaviour of $n_j(\omega^0)$. If the medium is non-dispersive, we can easily check that the Fourier transforms of the Hankel functions in (3.2) give rise to the sharp infinite Mach cones in an ordinary medium. However, since the medium under study is intrinsically dispersive, the cones become diffuse and the Cherenkov pulses acquire a finite width. For the special choice of a step in the refractive index, the final integration in (3.2) can be made; it yields a linear combination of the Fresnel integrals C and S, having an oscillatory build-up and decay near the site of the non-dispersive field cone. If the particle velocity exceeds the Cherenkov thresholds of all the three modes, three distinct coaxial diffuse Cherenkov rings of light will be observed at three different angles from the line of motion of the charge. Since the F^{e}_j, the intensity factors of these pulses, are all independent of the azimuthal angle around the axis of the cones, we conclude that the cones are circular in cross section with a uniform azimuthal intensity-distribution.

If the particle velocity falls below the Cherenkov threshold of the jth mode, the poles of the k_3-integrand in (2.10) lie on the imaginary axis for all values of k_2 and k_1, and we now have

$$\phi^{\mathrm{e}}_{j,\,\mathrm{Coulomb}} = -\frac{e}{4\pi^2}\int \frac{\mu\gamma^2}{n^2} F^{\mathrm{e}}_j(k_1)\exp(\mathrm{i}k_1x_1)\,K_0(\xi'_j\,|k_1|\,\rho)\,\mathrm{d}k_1, \qquad (3.4)$$

where the F_j^{e} are the boosting factors that occurred in (3.2), K_0 is the Bessel function of the third kind, and $\xi_j'^2 = -\xi_j^2$. The Fourier transform in (3.4) would give in a non-dispersive medium the Coulomb potential with foreshortened ellipsoidal equipotential surfaces. In particular, for $\psi = 0$ and $\beta = 0$, we obtain the well known Yukawa potential in a screening medium:

$$\phi_{\mathrm{Scr}}^{\mathrm{e}} = (4\pi\epsilon)^{-1} \exp(-\eta r)/r, \tag{3.5}$$

where η^{-1} is the Debye length of the plasma.

For an arbitrary particle velocity, we can thus write

$$\phi^{\mathrm{e}} = \sum_{j=1}^{3} \phi_j^{\mathrm{e}}, \tag{3.6}$$

where
$$\phi_j^{\mathrm{e}} = \Theta(\xi_j^2)\, \phi_{j,\,\mathrm{Cherenkov}}^{\mathrm{e}} + \Theta(\xi_j'^2)\, \phi_{j,\,\mathrm{Coulomb}}^{\mathrm{e}}. \tag{3.7}$$

The energy output per unit path length is equal to the work done on the particle by the retarding force produced by the polarization in the medium (Landau & Lifshitz 1960), and can be obtained as

$$\frac{\mathrm{d}W^{\mathrm{e}}}{\mathrm{d}l} = -(eE_1^0)_{r_0=\beta c t_0} = -(eE_1)_{r=0} = e\left(\frac{\partial \phi^{\mathrm{e}}}{\partial x_1}\right)_{r=0}. \tag{3.8}$$

Substituting (3.6) in (3.8), we then obtain

$$\frac{\mathrm{d}^2 N_{\mathrm{p}}^{\mathrm{e}}(\omega^0)}{\mathrm{d}l\,\mathrm{d}\omega^0} = \sum_{j=1}^{3} \frac{\mathrm{d}^2 N_{\mathrm{p}j}^{\mathrm{e}}(\omega^0)}{\mathrm{d}l\,\mathrm{d}\omega^0}, \tag{3.9}$$

with
$$\frac{\mathrm{d}^2 N_{\mathrm{p}j}^{\mathrm{e}}(\omega^0)}{\mathrm{d}l\,\mathrm{d}\omega^0} = \Theta(\xi_j^2)\, \frac{e^2}{4\pi c^2 \hbar}\, \frac{\mu F_j^{\mathrm{e}}(\omega^0)}{\beta^2 n^2}, \tag{3.10}$$

where $N_{\mathrm{p}j}^{\mathrm{e}}(\omega^0)$ is the number of photons of frequency ω^0 emitted in the jth mode (Jelley 1958). In the special case when the particle velocity exceeds the Cherenkov thresholds of all the three modes, we obtain

$$\frac{\mathrm{d}^2 N_{\mathrm{p}}^{\mathrm{e}}(\omega^0)}{\mathrm{d}l\,\mathrm{d}\omega^0} = \frac{e^2}{4\pi c^2 \hbar}\, \frac{\mu}{\beta^2 n^2} \sum_{j=1}^{3} F_j^{\mathrm{e}}(\omega^0). \tag{3.11}$$

It can be checked from (3.3) that

$$\sum_{j=1}^{3} F_j^{\mathrm{e}}(\omega^0) = \beta^2 n^2 - 1, \tag{3.12}$$

so that in this case, the output takes the simple form

$$\frac{\mathrm{d}^2 N_{\mathrm{p}}^{\mathrm{e}}(\omega^0)}{\mathrm{d}l\,\mathrm{d}\omega^0} = \frac{e^2}{4\pi c^2 \hbar}\, \mu \left(1 - \frac{1}{\beta^2 n^2}\right), \tag{3.13}$$

a formula which is reminiscent of the well known Frank–Tamm result for the Cherenkov output in an ordinary dielectric. We shall see in the Appendix that such a result also holds in a very general spatially dispersive medium.

In spectral regions where the imaginary parts of n_j are appreciably large the poles of the k_3-integrand in (2.10) are not confined to the real and imaginary k_3-axes, but are scattered over the complex plane. Energy loss now occurs both below and above the Cherenkov threshold (Sastry & Parida 1978), which is now given by the condition (Hoenders & Pattanayak 1976)

$$\beta^2(n_{j\mathrm{r}}^2 - n_{j\mathrm{i}}^2) > 0, \quad j = 1, 2, 3, \tag{3.14}$$

where $n_{j\mathrm{r}}$ and $n_{j\mathrm{i}}$ are the real and imaginary parts of n_j.

4. Cherenkov output of a neutron

When augmented by the formal addition of magnetic monopoles and their currents, Maxwell's equations become perfectly symmetric. This symmetry allows us to write down at once the magnetic scalar potential ϕ^{m} of a magnetic monopole m from the electric scalar potential ϕ^{e} of an electric charge e, by making the following substitutions (Majumdar & Pal 1970; Papas 1965):

$$\phi^{\mathrm{e}} \to \phi^{\mathrm{m}}, \quad e \to m, \quad \hat{\epsilon} \to \hat{\mu}, \quad \hat{\mu} \to \hat{\epsilon}. \tag{4.1}$$

However, the medium under study is asymmetric in its electrical and magnetic properties, since it is spatially dispersive only in $\hat{\epsilon}$ and not in $\hat{\mu}$. For this reason, the magnetic scalar potential ϕ^{m} of a magnetic monopole has a slightly different structure from that of ϕ^{e} in (2.4), which is in fact much simpler, taking the form

$$\phi^{\mathrm{m}}(\boldsymbol{r}) = -\frac{m}{8\pi^3}\int \frac{\epsilon\gamma^2}{n^2}\frac{N^{\mathrm{m}}}{D^{\mathrm{m}}} \exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r})\,\mathrm{d}^3k, \tag{4.2}$$

where
$$D^{\mathrm{m}} = \prod_{j=1}^{2} (k_3^2 - A_j^2), \tag{4.3}$$

and
$$N^{\mathrm{m}} = (k_3^2 - A_3^2)(\beta^2 n^2 - 1) + \eta^2. \tag{4.4}$$

We see that the denominator of the Fourier integral now contains a product of only two factors corresponding to the two transverse modes ($j = 1, 2$). Thus a magnetic monopole does not excite the longitudinal electric mode in our electrically spatially dispersive medium. The threshold velocities are again given by (3.1) with $j = 1, 2$. The evaluation of (4.2) proceeds along exactly the same lines as that of (2.5), and the final result is again formally identical with (3.6) but with the replacements $e \to m$, $\hat{\mu} \rightleftarrows \hat{\epsilon}$. However, the $F_j^{\mathrm{m}}(j = 1, 2)$ are now given by

$$F_1^{\mathrm{m}} = \frac{k_0^2(n_1^2 - n_3^2)(\beta^2 n^2 - 1) + \eta^2}{k_0^2(n_1^2 - n_2^2)}, \tag{4.5a}$$

$$F_2^{\mathrm{m}} = \frac{k_0^2(n_2^2 - n_3^2)(\beta^2 n^2 - 1) + \eta^2}{k_0^2(n_2^2 - n_1^2)}. \tag{4.5b}$$

There are now only two diffuse Cherenkov cones, and the fields and output of photons are again given by (3.2), (3.4), (3.6), (3.7), (3.9) and (3.10) with the substitutions $e \to m$, $\mu \rightleftarrows \epsilon$. When the particle velocity exceeds the Cherenkov thresholds for both the transverse modes, we again obtain a simple and interesting result like (3.13). However, the magnetic analogue of (3.5) will not contain any exponential term. This is quite in order, as the static field of a magnetic monopole cannot be screened off by a plasma made up of electric charges.

The dipole moment of a neutron can formally be viewed as being due to a juxtaposition of two monopole moments (Frank 1943; Balazs 1956), and the output of energy can be written as (Sastry & Bhattacharya 1977)

$$\mathrm{d}W^{\mathrm{M}}/\mathrm{d}l = -M_\mu M_\nu \partial_\mu \partial_\nu \partial_1 (\phi^{\mathrm{m}}/m)_{r=0}, \tag{4.6}$$

where the $M_\mu (\mu = 1, 2, 3)$ are the components of the dipole moment along the coordinate axes. It can easily be checked that the cross terms vanish in the output (4.6) with ϕ^{m} given by (4.2), so that the total output is merely a sum of the three diagonal terms $(W^{\mathrm{M}})_\mu$. Because of the rotational symmetry of the medium around the line of motion, $(W^{\mathrm{M}})_2 = (W^{\mathrm{M}})_3 \equiv (W^{\mathrm{M}})_\perp$ and $(W^{\mathrm{M}})_1 \equiv (W^{\mathrm{M}})_\parallel$ are the only two independent outputs. Substituting (4.2) in (4.6) and integrating as before, we obtain

$$\frac{\mathrm{d}^2 N^{\mathrm{M}}_{\mathrm{p}}(\omega^0)}{\mathrm{d}l\,\mathrm{d}\omega^0} = \sum_{j=1}^{2} \frac{\mathrm{d}^2 N^{\mathrm{M}}_{\mathrm{p}j}(\omega^0)}{\mathrm{d}l\,\mathrm{d}\omega^0}, \tag{4.7}$$

where

$$\left(\frac{\mathrm{d}^2 N^{\mathrm{M}}_{\mathrm{p}j}(\omega^0)}{\mathrm{d}l\,\mathrm{d}\omega^0}\right)_\parallel = \Theta(\xi_j^2)\,\frac{M_\parallel^2}{4\pi\beta^4\gamma^2c^4\hbar}\,\frac{\epsilon}{n^2}\,F_j^{\mathrm{m}}\omega^{0\,2} \tag{4.8}$$

and

$$\left(\frac{\mathrm{d}^2 N^{\mathrm{M}}_{\mathrm{p}j}(\omega^0)}{\mathrm{d}l\,\mathrm{d}\omega^0}\right)_\perp = \Theta(\xi_j^2)\,\frac{M_\perp^2}{8\pi\beta^4c^4\hbar}\,\frac{\epsilon}{n^2}\,(\beta^2 n_j^2 - 1)\,F_j^{\mathrm{m}}\omega^{0\,2}. \tag{4.9}$$

5. Conclusions

In this paper we have evaluated the field and the Cherenkov output of a relativistic proton, and a neutron, moving in a spatially dispersive medium. The model of the medium chosen describes a dielectric in the exciton régime. We have found it convenient to exploit the simplifying features of the rest frame of the particle. It is found that a proton emits in general three Cherenkov rings, but a neutron can emit only two.

The discussion in this paper is confined to *isotropic* spatially dispersive media. In general, however, exciton transitions are frequently associated with crystalline media (Agranovich & Ginzburg 1966). To compare the theoretical formulae with possible experimental results on realistic samples, it is profitable to extend the calculations of this paper to anisotropic spatially dispersive media. We note in this connection that calculations of the Cherenkov field and output have so far proved intractable in general for *biaxial* crystals, even in the absence of spatial dispersion. The situation in respect of *uniaxial* crystals is more promising. The theory of the Cherenkov effect in spatially non-dispersive uniaxial crystals is complicated but well understood, and has found ample experimental confirmation in the photographs of the Cherenkov rings in calcite (Zrelov 1964). We hope that the theory of the simpler case developed here of isotropic spatially dispersive media will serve as the basis for an extension to include uniaxial anisotropy. A proton moving in such a crystal can be expected to emit six Cherenkov rings, and a neutron four.

Appendix

When the particle velocity exceeds the Cherenkov thresholds of all the modes, and all the modes lie in typical Cherenkov bands, the total output of photons is given by (3.13). This result is not only simple, compact and *formally* identical to the Frank–Tamm result, but is independent of the spatially dispersive parameters η and ψ. This implies that when the single Cherenkov cone emitted by a relativistic charge is split up into a number of cones owing to spatial dispersion near an exciton band, the total photon output of all these cones is at best equal to that of the single Cherenkov cone in the absence of the exciton band.

We wish to show that such a result also holds for a much more general model of the medium. The dispersion formula (2.1) is a first-order approximation in $\boldsymbol{k}^{0\,2}$ (Agranovich & Ginzburg 1966). We now generalize (2.1) in such a manner that both the permeability and the permittivity are expressed to an arbitrary degree in $\boldsymbol{k}^{0\,2}$ by the formulae

$$\hat{\epsilon}(\boldsymbol{k}^0,\omega^0) = \epsilon(\omega^0)\left[1+\sum_{j=1}^{m}\frac{\eta^2_{\mathrm{e}j}(\omega^0)}{(\boldsymbol{k}^0)^{2j}-\psi^2_{\mathrm{e}j}(\omega^0)}\right] \tag{A 1a}$$

and

$$\hat{\mu}(\boldsymbol{k}^0,\omega^0) = \mu(\omega^0)\left[1+\sum_{j=1}^{n}\frac{\eta^2_{\mathrm{m}j}(\omega^0)}{(\boldsymbol{k}^0)^{2j}-\psi^2_{\mathrm{m}j}(\omega^0)}\right]. \tag{A 1b}$$

It can easily be deduced from Maxwell's equations that in the medium described by (A 1) there can be $\frac{1}{2}m(m+1)+\frac{1}{2}n(n+1)+1$ transverse modes, $\frac{1}{2}m(m+1)$ longitudinal electric modes, and $\frac{1}{2}n(n+1)$ longitudinal magnetic modes. A charged particle travelling through such a medium can excite at most $p = [m(m+1) + \frac{1}{2}n(n+1)+1]$ modes, but not the $\frac{1}{2}n(n+1)$ longitudinal magnetic modes. Accordingly it can emit at most p distinct Cherenkov cones, while a magnetic monopole can emit only $q = [\frac{1}{2}m(m+1)+n(n+1)+1]$ cones.

The energy loss per unit path-length of a charged particle can then be written as

$$\frac{\mathrm{d}W^e}{\mathrm{d}l} = \frac{\mathrm{i}e^2}{8\pi^3}\int\frac{\gamma^2(\beta^2\hat{\epsilon}\hat{\mu}-1)\,k_1\,\mathrm{d}k_1\,\mathrm{d}k_2\,\mathrm{d}k_3}{\hat{\epsilon}[k_3^2+k_2^2-\gamma^2(\beta^2\hat{\epsilon}\hat{\mu}-1)\,k_1^2]}. \tag{A 2}$$

Since we have

$$k_1^0 = \gamma k_1,\quad k_2^0 = k_2,\quad k_3^0 = k_3,\quad \omega^0 = \gamma\beta c k_1, \tag{A 3}$$

inserting (A 1) and (A 3) into (A 2) gives

$$\frac{\mathrm{d}W^e}{\mathrm{d}l} = \frac{\mathrm{i}e^2}{8\pi^3}\int\frac{\mu\gamma^2}{n^2}\,Gk_1\,\mathrm{d}k_1\,\mathrm{d}k_2\,\mathrm{d}k_3, \tag{A 4}$$

where $n = (\epsilon\mu)^{\frac{1}{2}}$ is the background refractive index, and

$$G = \sum_{j=0}^{p-1} c_j k_3^{2j} \Big/ \prod_{i=1}^{p} (k_3^2 - A_i^2), \tag{A 5}$$

where $A_i^2 = \gamma^2(\beta^2 n_i^2 - 1)\, k_1^2 - k_2^2$, and n_i is the refractive index of the ith mode. The coefficients c_j in the polynomial in (A 5) are in general functions of k_1, k_2 and the spatially dispersive parameters η and ψ.

Splitting G into partial fractions gives

$$G = \sum_{r=1}^{p} \frac{F_r}{k_3^2 - A_r^2}. \tag{A 6}$$

We note the interesting result that the coefficients F_r become independent not only of k_3 but also of k_2. Hence, we can again evaluate the k_3-integral in (A 4) by contour integration in the Cherenkov bands. Since the F_r are independent of k_2, the k_2-integration also becomes trivially identical to that in an ordinary medium. This means that the total Cherenkov output splits up into an uncoupled mixture of p Cherenkov outputs (Jelley 1958):

$$\frac{\mathrm{d}^2 N}{\mathrm{d}l\,\mathrm{d}\omega^0} = \frac{e^2}{4\pi c^2 \hbar} \frac{\mu}{\beta^2 n^2} \sum_{r=1}^{p} F_r \Theta(\beta^2 n_r^2 - 1). \tag{A 7}$$

If the particle velocity exceeds the Cherenkov thresholds of all the p modes, we have for the total output,

$$\left(\frac{\mathrm{d}^2 N}{\mathrm{d}l\,\mathrm{d}\omega^0}\right)_{\mathrm{tot}} = \frac{e^2}{4\pi c^2 \hbar} \frac{\mu}{\beta^2 n^2} \sum_{r=1}^{p} F_r. \tag{A 8}$$

Equating (A 5) and (A 6) and comparing the coefficients of $k_3^{2(p-1)}$ on either side, we obtain

$$\sum_{r=1}^{p} F_r = c_{p-1}. \tag{A 9}$$

Substituting (A 1)–(A 4) into (A 5), we obtain

$$c_{p-1} = \beta^2 n^2 - 1. \tag{A 10}$$

(We do not present the detailed calculations here as they are unwieldy, although the final result is very simple.) We thus have

$$\left(\frac{\mathrm{d}^2 N}{\mathrm{d}l\,\mathrm{d}\omega^0}\right)_{\mathrm{tot}} = \frac{e^2}{4\pi c^2 \hbar} \mu \left(1 - \frac{1}{\beta^2 n^2}\right), \tag{A 11}$$

which is identical to (3.13). This proves the assertion stated at the beginning of the Appendix. By using the substitutions of (4.1), we can check that an identical result holds for the total photon output of a magnetic monopole.

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