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# PENETRATION AT THE BASE OF SOLAR CONVECTION ZONE

H. M. ANTIA AND S. M. CHITRE
Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400 005, India
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## ABSTRACT

The extent of overshoot from stellar convection zones into the adjoining stable layers has been recognized to have a non-negligible influence on evolutionary tracks of stars. Recently, Stothers & Chin (1992) have carried out a detailed model-independent analysis of substantial body of observational data to conclude that the maximum permissible overshoot is 0.2 times the local pressure scale height. In the present work a realistic solar convection zone model is constructed by employing a nonlocal equation for the velocity of convective elements and by including dissipative effects in the calculations. The convection model approach and the analysis of linear eigenmodes are combined to estimate the penetration depths below the base of the convection zone and into the overlying solar atmosphere. It is demonstrated that for an arbitrary extent of overshoot into the underlying stable region, it may not be possible to find a combination of linear modes capable of reproducing the model convective flux profile over the overshoot layers. The acceptable overshoot distance below the base of the convection zone turns out to be $\leq 0.2H_p$ with a probable value of $0.1H_p$, which appears to be consistent with helioseismological data.

*Subject headings:* convection — Sun: interior

## 1. INTRODUCTION

It has been widely recognized that the extent of penetration into the adjacent stable layers beyond the classical boundary of a stellar convection zone, would have a non-negligible influence on the evolutionary history of stars. The theoretical stellar models, however, rarely incorporate the effects and extent of convective overshooting, even though in the context of laboratory, meterological, and geophysical fluids, there is ample evidence for convective penetration (cf. Massaguer 1990).

The question of convective overshoot into the bounding stable layers has been addressed by a number of investigators, albeit with widely differing conclusions. Unno (1957) considered the overshoot from the solar photosphere to conclude that the degree of penetration into the atmosphere decreases with the increasing stability of the overlying layer. In fact, for a simple two-layer model it can be shown that for linear perturbations the velocity scale-height in the penetration zone is roughly given by

$$H_p\sqrt{\frac{|\nabla - \nabla_{ad}|_{conv}}{|\nabla - \nabla_{ad}|_{over}}}, \tag{1}$$

where $H_p$ is the pressure scale height and the subscripts "conv" and "over" refer to the convection zone and overshoot layer, respectively. Adopting this expression for estimating the extent of penetration at the base of the solar convection zone, we get an overshoot of order of $0.01H_p$ into the underlying radiative zone. This result is readily understood from the fact that a convective element which is being accelerated under a small superadiabatic gradient in the convection zone experiences a strong deceleration as soon as the gradient becomes strongly subadiabatic; with the large magnitude of the subadiabatic gradient in the overshoot region compared to the superadiabatic gradient across the boundary of the convection zone, the convective elements virtually hit a solid wall and their motion is quickly contained. Saslaw & Schwarzschild (1965) performed a linear calculation in the adiabatic approximation for the interior convection zones and they found practically no penetration beyond the convective cores of upper main-sequence stars for there to be any significant evolutionary consequences.

There are two distinct approaches that have been followed to treat the problem of convective penetration:

1. Model approach: modeling of the stellar convection zone, where the mixing-length formulation may be suitably adopted to set up a nonlocal prescription.
2. Modal approach: analysis of fluctuations excited in the convective zone; in this approach, an equilibrium model is assumed and the development of some perturbation may be considered either in the linear or nonlinear regime.

It is customary to restrict the nonlinear analysis to idealized situations over a limited region, and most of these nonlinear calculations deal with cases where the magnitudes of $(\nabla - \nabla_{ad})$ on both sides of the boundary separating the unstable and stable regions are comparable. This treatment is, therefore, not strictly applicable to the stellar penetrative convection zones where such a condition does not generally obtain. The claims, based on these calculations, of substantial overshooting below the solar convection zone may consequently be regarded as somewhat unrealistic.

The models of convection zones constructed using the local mixing-length approximation tend to yield a large subadiabatic gradient below the base of the convection zone (designated to be the layer where $\nabla = \nabla_{ad}$). If magnitude of the subadiabatic gradient below this layer is reduced, then the convective element can penetrate a significant distance which may be of the same order as the local scale height. In that case, the prevailing temperature gradient and consequently the radiative flux, will turn out to be larger and in such a situation, in order to satisfy the condition of the constancy of total flux, the convective flux should become negative. This is, indeed, how a significant amount of overshooting from the interior convective cores is obtained in the model of Shaviv & Salpeter (1973), where the overshooting region is assumed to be nearly adiabatic. The temperature difference between the overshooting elements and the ambient medium, and consequently the decelerating buoyancy force is then very small and this results in penetrating elements covering a distance which can be comparable to the local pressure scale height. Using the nonlocal mixing-length theory of Shaviv & Salpeter (1973) Skaley & Stix (1991) have calculated an overshoot of up to $0.5H_p$ below the base of the solar convection zone.

It should be emphasized that if one wants to ensure the temperature gradient to remain close to the adiabatic value over a sizable overshoot distance, it is necessary that the convective flux is not only negative but its magnitude should increase with depth precisely by the amount required to make the gradient very nearly adiabatic. The occurrence of negative convective flux is then an unavoidable consequence of the downward moving convective elements being colder and hence heavier than the surrounding medium when they arrive at the $\nabla = \nabla_{ad}$ boundary. The elements continue to accelerate downwards, but during the passage through the subadiabatic region the temperature difference starts decreasing and eventually becomes negative. The elements are liable to be decelerated after this development; nevertheless, they continue their downward journey until the velocity becomes zero, but in the meanwhile the convective flux has become negative. Such a negative convective flux has appeared in the nonlinear simulations by Nordlund & Dravins (1990). However, in these calculations the magnitude of the negative convective flux seldom exceeds 10% of the total flux and also the convective flux tends to zero smoothly at the edge of the overshoot zone. This is not the case with the type of nonlocal models adopted by Shaviv & Salpeter (1973) and Langer (1986) which give rise to a sharp discontinuity in the convective flux and the temperature gradient at the overshoot boundary. Normally, a nonlocal model is expected to smoothen out sharp features inherent in local models by some kind of averaging, but in the foregoing models the motion of convective elements is characterized by a somewhat unphysical description involving infinite acceleration and sudden arrest at the edge of the overshooting region (cf. Renzini 1987).

In the linear theory it is then fair to conclude that there is some overshoot occurring beyond the unstable zone, but the extent of penetration appears to depend on the assumption about the relative degree of subadiabaticity in the adjoining stable region. The extrapolation to the nonlinear regime has been attempted by a number of investigators including Moore & Weiss (1973) and Zahn, Toomre, & Latour (1982). These studies, performed in the Boussinesq approximation express the velocity field as an expansion in a set of prescribed, horizontal planforms and have indeed found much more overshoot than would be obtained using the linear theory. Later, Massaguer et al. (1984) adopted the anelastic approximation to find again the extent of pentration depending on the degree of stability of the bounding medium and on the radio of the horizontal to vertical cell-size. The simulations for penetrative convection in a fully compressible fluid in two and three dimensions by Hurlburt, Toomre, & Massaguer (1986); Stein & Nordlund (1989); Cattaneo, Hurlburt, & Toomre (1989); Nordlund et al. (1992), show the presence of motions which are directed downward; these downdrafts are found to have a substantial overshoot into the underlying stable region. The numerical simulations, however, do not necessarily deal with realistic stellar models. Simulations are usually restricted to idealized situations where the radiative gradient may be at most twice the adiabatic gradient. In the realistic solar convection zone models, the radiative gradient could be $10^6$ times the adiabatic gradient and as a result gross extrapolation is involved in predicting the extent of overshoot from such calculations.

Zahn (1991) has stressed the importance of the Peclet number, $WL/\kappa_{rad}$ ($W$ is the mean convective velocity, $L$ is the length scale and $\kappa_{rad}$ is the radiative diffusivity) in controlling the extent of overshoot into the adjoining stable region. We should like to point out that the Peclet number and the degree of convective efficiency are equivalent physical concepts, in the sense that a large Peclet number corresponds to high convective efficiency. It may be remarked that for the types of models used in nonlinear simulations with low convective efficiency, even a straightforward linear theory will predict substantial overshoot in agreement with nonlinear calculations. Thus, until nonlinear calculations become available for situations with very high convective efficiency, it is clearly not possible to deduce the amount of overshoot below the base of the solar convection zone.

Apart from this, numerical simulations also involve approximations which may affect the final results. Thus, in direct numerical simulations, while the large-scale features are accounted for adequately, it is not altogether clear if the effect of sub-grid scales are treated in a satisfactory manner. It is well known, for example, that a full-scale simulation of the solar convection zone will require of the order of $10^{50}$ degrees of freedom which is, of course, beyond the capability of any computer. All existing numerical simulations, therefore, are either restricted to small Reynolds number which is hardly applicable to the solar case, or they make some drastic simplification like introduction of artificial viscosity to approximate the effects of small-scale motions. It is conceivable that nonlinear effects may assume importance in the overshoot layer, but at the moment we have a choice between linear calculations for a realistic convection zone model and nonlinear calculations for a highly idealized model which need not represent a realistic stellar convection zone. There is no a priori reason to believe that the latter is closer to reality than the former. In the present work we have adopted the linear approach for a realistic solar convection zone model.

Another approach adopted by Xiong (1985) and Kuhfuß (1986) is based on the use of some statistical properties of the mean flow-field, in much the same manner as the treatment of turbulent flows. Again the overshoot into the stable region is found to be substantial, although it is not clear how far the penetration length is controlled by values of the dimensionless parameters introduced in the analysis (cf. Zahn 1991). On the other hand, Canuto (1992) has recently tried an approach based on Reynold stresses to obtain equations governing various turbulent quantities. These equations can be solved to obtain the extent of overshooting.

Recently, Stothers & Chin (1992) have studied the star clusters NGC 458 and NGC 330 in the Small Magelanic Cloud to conclude that the observational data involving the maximum effective temperature of hot evolved stars and luminosity ratio of the hot and cool evolved stars are consistent with no convective core overshooting and in fact, constrain the overshoot distance to be less than $0.2H_p$. In an earlier work, Stothers (1991) has applied 14 tests for the presence of overshooting from stellar convective cores for stars in the mass range of 4–17 $M_\odot$ to find that each of these data is consistent with no overshoot, and he has put a conservative upper limit of $0.4H_p$ on the extent of overshooting from at least four of these tests.

Similarly, Monteiro, Christensen-Dalsgaard, & Thompson (1993) have attempted to estimate the extent of overshooting below the solar convection zone using the measured frequencies of solar oscillations, and fail to find any evidence of an overshoot region. Thus, there is considerable observational evidence to suggest that overshooting from stellar convection zones may not be significant.

In the present work we make an attempt to combine the convection zone model approach and the modal analysis. For this purpose, we construct the equilibrium model using an admixture of local and nonlocal mixing-length theory and then calculate the linear convective modes in such a model in order to check the consistency of the mixing-length formulation, as suggested by Narasimha & Antia (1982). Basically, the formalism is anchored on the assumption that the overshooting of individual linear convective modes would correspond to the penetration of the corresponding Fourier component of the velocity field of turbulent convection (Böhm 1963). The main thrust of this paper is to demonstrate that for an arbitrary extent of overshooting it may not be possible to find a superposition of linear convective modes which is capable of reproducing the model convective flux-profile over the unstable zone and the overshoot region. Thus, it is hoped that with the consistency requirement which, in some sense combines the model and modal approaches, it may be possible to put some constraint on the extent of pentration into the stable region.

## 2. PHYSICAL FORMULATION

A realistic stellar convection zone model can be constructed using a nonlocal equation to determine the velocity of moving elements or the convective flux. A simple nonlocal prescription for determining the convective velocity is given by Shaviv & Chitre (1968) which has been used by Antia, Chitre, & Narasimha (1984; hereafter Paper I). In this formulation the convective velocity is determined by using the nonlocal differential equation:

$$\frac{dW^2}{dr} = -\frac{\beta g T}{\rho H_p}\left(\frac{\partial \rho}{\partial T}\right)_P L(\nabla - \nabla_{\rm ad}) - \frac{DW^2}{L}, \qquad (2)$$

while the convective flux is given by the usual mixing length expression,

$$F^C = -\alpha \rho C_p W L\left(\nabla T - \nabla_{\rm ad}\frac{T}{P}\nabla P\right). \qquad (3)$$

Here $\alpha$ and $\beta$ are the usual mixing length parameters of order unity, $D = 4C_D$, $C_D$ being the aerodynamic drag coefficient, $L$ is the mixing length, $W$ is the mean velocity of convective elements, $H_p$ is the local pressure scale height,

$$\nabla = \frac{d\ln T}{d\ln P}, \quad \nabla_{\rm ad} = \left(\frac{\partial \ln T}{\partial \ln P}\right)_s,$$

and $r$ is the radial distance. It may be interesting to note that an equation similar to (2) can be obtained from equation (59) of Canuto (1992), with simplifying assumptions such as the isotropy of turbulence. Equation (2) incorporates the aerodynamic drag experienced by an ascending element which suffers resistance because of the eddies generated around it during its passage through the descending elements. Clearly, this equation can only be used to account for overshooting in the upward direction, but not the penetration below the base of the convection zone. This is evidently due to asymmetry in the differential equation for calculating the velocity, since equation (2) is set up for a convective element which is rising upward under the influence of buoyancy forces. In actual practice, there will be both ascending as well as descending elements, in general, moving with different velocities. For obtaining the equation applicable to convective elements falling under the influence of gravity, we use the simple prescription of changing the sign of $dW^2/dr$ in equation (2) to determine the velocity of downward moving elements by the nonlocal equation

$$\frac{dW^2}{dr} = \frac{\beta g T}{\rho H_p}\left(\frac{\partial \rho}{\partial T}\right)_P L(\nabla - \nabla_{\rm ad}) + \frac{DW^2}{L}, \qquad (4)$$

If we incorporate this equation, it should be possible to account for the overshoot below the base of the convection zone. In reality, we should define two distinct velocity fields and specify some prescription for determining the area covered by rising and falling elements. This will lead to a somewhat involved scheme for determining the convective flux. In this paper, we have adopted a simplified approach, as a first approximation, by merely switching from equation (2) to (4) at a point in the lower part of the convection zone where $dW^2/dr = 0$. The choice of the point at which this derivative vanishes is obviously arbitrary and will depend on the initial value of velocity from which the integration is started. Naturally, by changing the initial velocity one can adjust the extent of overshooting in the model convection zone. However, such a simplified treatment may be admissible near the bottom of the convection zone, since the temperature gradient becomes essentially adiabatic in the deeper layers of the convection zone as well as in the overshoot region. The convective flux profile consequently depends only on extent of overshooting, with the convective flux becoming negative in the penetration zone. Furthermore, below the boundary of the conventional convection zone where $\nabla = \nabla_{\rm ad}$, the magnitude of the convective flux increases rapidly with depth; typically, at a depth of $0.1H_p$ below this boundary, the convective flux $F^C \approx -0.18F^T$, where $F^T$ is the net flux. This would imply that for a substantial overshoot, the negative convective flux will also be of significant amount.

The standard stellar structure equations with the mixing-length approximation are integrated using equations (2) and (4) for the velocity. The linear stability analysis is performed with the usual hydrodynamical equations governing the conservation of mass, momentum, and energy, incorporating the turbulent thermal conductivity and turbulent viscosity, and the convective eigenmodes are computed by perturbing the equilibrium stellar convection zone model as described in Paper I. The linear stability analysis by itself clearly does not give the amplitudes of these convective modes. If these eigenmodes transport the convective flux and if the modes are assumed to be statistically independent, then it should be possible to construct a linear superposition of these modes which is capable of reproducing the required convective flux as determined by the mixing-length formulation. This is essentially a test for the self-consistency of mixing-length theory or for that matter, of any other prescription for calculating the convective flux in stellar models. Narasimha & Antia (1982) indeed demonstrated that it is possible to construct a linear superposition of unstable convective modes that can reasonably reproduce the convective flux profile inherently assumed in the local mixing-length model. Such a superposition enables one to estimate the amplitude of individual convective modes. Furthermore, the linear eigenfunctions with large values of harmonic number $l$ tend to peak near the surface, while those corresponding to small $l$ peak near the bottom of the convection zone, and it then becomes possible to identify the width of convective flux profiles of individual modes peaking at different depths with the local mixing length. This procedure can be effectively used to determine the mixing-length at a given depth, since for a convection zone model constructed using an arbitrary mixing-length prescription the width of the relevant eigenfunction will be at variance with the mixing-length. In this manner, in a self-consistent approach the choice of the mixing length is no longer an arbitrary parameter.

Let us fix the boundary of the penetration region as the furthermost place reached by the overshooting elements before their motion is arrested. The convective elements can only penetrate roughly to the depth where the negative convective flux ensures a very small subadiabatic gradient; beyond this depth, the individual elements essentially hit a solid wall (because of highly subadiabatic gradient), which destroys their downward momentum. Thus, we get into an embarrassing situation which has been aptly described by Renzini (1987) in that the extent of overshooting is small if assumed small (as in the case of the local mixing-length theory), while the overshooting can be large if assumed large. We believe this dilemma can be resolved if the consistency argument proposed by Narasimha & Antia (1982) is applied to the penetrative convection. For example, we can use an equilibrium model with substantial overshoot and check if the linear superposition of convective modes gives a significant amount of negative convective flux in the overshoot layer. In this sense, our approach to the penetration problem may be regarded as combining the stellar structure construction with the modal analysis, since we use the stellar model to calculate the equilibrium structure and then employ the modal analysis to check for its consistency.

## 3. NUMERICAL RESULT

The solar convection zone model is constructed with the usual stellar structure equations as given in Paper I along with equations (2) and (4). The integration started from a depth of 112 km below the $\tau = 1$ level is carried out in both directions. The starting values of temperature, density and mass fraction are adjusted to satisfy the outer boundary conditions

$$\tau(0) = 1 \,,$$
$$M(z_0) = M_\odot \,,$$
$$\kappa(z_0)\rho(z_0)H_\rho(z_0) = \tau(z_0) \,, \tag{5}$$

where $z_0$ is the depth (measured from the $\tau = 1$ level) up to which the integration is carried out, and $M(z)$ is the total mass contained in the sphere of radius $R_\odot - z$. Typical value of $z$. Typical value of $z_0$ is approximately $-600$ km. Further, the starting value of the velocity $W$ is adjusted to get the required extent of overshooting below the convection zone, for a particular model. Following the approach adopted in Paper I, we identify the mixing length with the full width at half maximum of the convective flux profile contributed by individual modes. It is clearly seen from the results displayed in Figure 1 that beyond some level, the width of the eigenfunctions decreases with depth. In particular the first subharmonic (C2-mode) corresponding to low $l$ has a narrow peak close to the base of the convection zone pointing to a small value of mixing length in

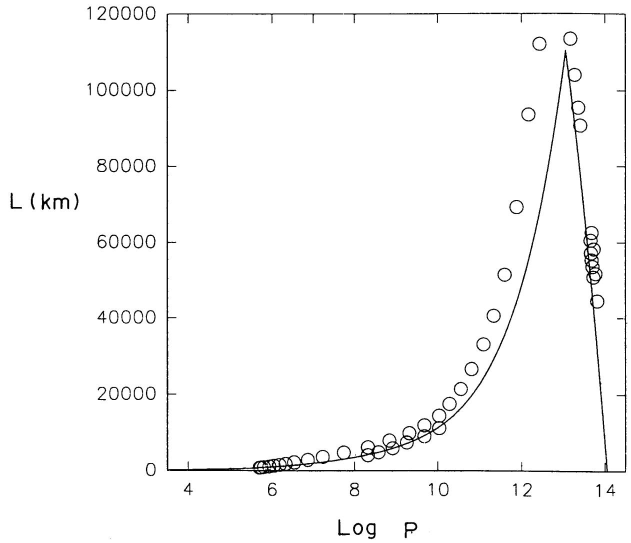


FIG. 1.—Mixing length is plotted against logarithm of the pressure for a solar envelope model with $C = 0.01$, $D = 0.1$, and $L = \min\ (z + 459, 220000 - z)$ km. The equivalent width of the luminosity profile of various convective modes is indicated by circles.

these layers. We therefore, choose the mixing length to be roughly the distance from the nearest boundary. Such a choice for the mixing length has indeed been discussed in the context of convection zone models (cf. Canuto & Mazzitelli 1991).

Another advantage of this prescription for the mixing length is that it is possible to construct convection zone models with reasonable extent of overshooting even when the drag coefficient $D$ is small. This possibility arises because the mixing length $L$ appears in the denominator of the second term on the right-hand side of equations (4) and hence as $L \to 0$, $W \to 0$ giving a reasonable overshoot distance. With the traditional form of mixing length as some multiple of the local scale height, it is very difficult to construct a convection zone model with small values of $D$, as the velocity may not approach zero even at very large depths.

With the computed realistic solar convection zone model we can study the convective eigenmodes using the linearized equations and the appropriate boundary conditions given in Paper I. From the eigenfunctions associated with the linear convective modes we can compute the flux profile due to an individual mode specified by the horizontal harmonic number $l$ and radial order $n$, using the expression (cf. Narasimha & Antia 1982)

$$F_{ln}(r) = \rho_0 v_r\left(T_0 s_1 + \frac{P_1}{\rho_0}\right). \tag{6}$$

The individual convective flux profiles may be normalized such that the maximum of $F_{ln}(r)$ over the entire convection zone equals the total solar flux at that depth. For the fundamental mode (designated C1) it is found that the flux profile has only one peak and the convective flux becomes negative at larger depths. However, it turns out that the magnitude of the negative flux is significant only for low value of $l$. The magnitude of the flux actually depends on the vale of $l$ and the corresponding growth rate $\omega$, and for small values of $l$ and $\omega$, it is possible to get a significant negative convective flux close to the convection zone boundary. Thus, it is necessary to make $\omega$ very small, which can be achieved by increasing the damping due to turbulent viscosity, i.e., by increasing the turbulent Prandtl number $\sigma_t$.

In practice, we calculate the eigenfunctions for a representative set of $l$-values, and using these profiles a least-squares fit is performed to obtain a linear superposition of these modes which attempts to reproduce the convective flux profile as computed in the mixing-length framework:

$$\sum_{l,n} a_{ln}^2 F_{ln}(r) = F_0^C(r) , \tag{7}$$

$F_0^C(r)$ is the convective flux in the equilibrium model. With the normalization mentioned above the amplitudes of these modes in the superposition is less than unity and the maximum negative flux in the superposed flux profile may not, as a result, be larger than that in any individual mode. This helps in setting an upper limit on the extent of overshooting. We explored the parameter space for the solar model as well as the turbulent Prandtl number to find that the maximum negative flux for the $l = 1$ convective eigenmode is unlikely to be larger than 80% of the solar flux. This in turn implies an upper limit of $0.5H_p$ for the overshoot region. However, in general, the numerical results indicate that the amplitude of this mode is actually much smaller and this results in a distinctly smaller overshoot.

Notice that at the edge of the overshoot region, the model flux drops abruptly to zero, while the flux profile of the individual convective modes as well as the superposed flux profile smoothly attains the value zero. We attribute this difference to the possible presence of a thin boundary layer at the base of overshoot region, which of course, has been neglected in our study; any realistic model of the overshoot zone should naturally incorporate this boundary layer in the analysis.

For estimating the extent of overshooting we must ensure that the model flux profile is in a reasonable agreement with the superposed profile in the overshoot region. It turns out that for large values of the drag coefficient $D$ in (2) and (4), the convective flux due to individual modes becomes negative long before the boundary of the convection zone where $\nabla = \nabla_{ad}$ and it becomes impossible to produce any accordance between the superposed profile and the actual convective model flux profile in the overshoot zone (cf. Fig. 2). But if we were to reduce the drag coefficient $D$ along with the turbulent pressure in the solar envelope model, we get a demonstrably better agreement. Note that the turbulent pressure is controlled by the parameter $C = P_t/(\rho W^2)$. With $D = 0.1$ and $C = 0.01$ (i.e., negligible turbulent pressure) the agreement with the model flux profile over most of the region close to the base of convection zone, in fact, turns out to be quite satisfactory. If, further, we were to reduce the turbulent Prandtl number (i.e., diminish the damping influence of turbulent viscosity) this results in even better agreement. But, with the reduction in Prandtl number, there is an increase in the growth rate and a corresponding decrease in the magnitude of the negative flux in the overshoot layer. This amounts to cutting down the extent of penetration in the solar model.

With a view to study this problem in detail, three solar convection zone models with $C = 0.01$ and $D = 0.1$ are constructed which give different extent of overshoot. We construct the following three models with different choices of the mixing length:

$$\text{Model I: } L = \text{Min}\,(z + 4.59 \times 10^5, 2.30 \times 10^8 - z)m ,$$

$$\text{Model II: } L = \text{Min}\,(z + 4.59 \times 10^5, 2.20 \times 10^8 - z)m ,$$

$$\text{Model III: } L = \text{Min}\,(z + 4.59 \times 10^5, 2.15 \times 10^8 - z)m ,$$

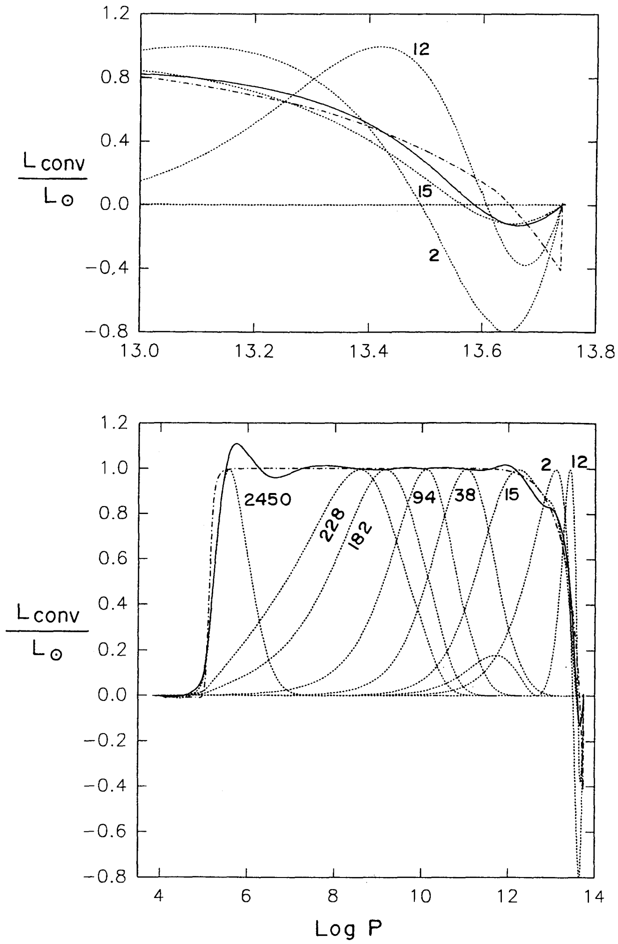


FIG. 2.—Dotted curves show the convective luminosity due to individual convective modes for various $l$-values as a function of logarithm of pressure for a solar envelope model with $C = 1$, $D = 1$, $L = \min\,(z + 459{,}22000 - z)$ km. The curve labeled 12 refers to C2-mode, while other curves pertain to C1-modes. The continuous curve shows the superposed convective luminsity profile, while the dot-dashed curve represents the model convective luminosity. The upper figure shows a blow-up of the overshoot region.

Here $z$ is the depth from the $\tau = 1$ layer close to the top of the convection zone.

The extent of overshoot in Models I, II, and III turns out to be approximately $0.30H_p$, $0.15H_p$ and $0.07H_p$, respectively. We have displayed in Figures 3 and 4 the results for Models I and III with a value of 1/3 for the Prandtl number $\sigma_t$. Note in the first two cases, the fitted convective mode profile has much smaller magnitude for the negative convective flux in the penetration zone as compared to the model flux. The fitted flux profile is nearly the same in both cases and this suggests that the extent of overshooting is probably smaller, say, about 10% of the pressure scale height. This appears to be justified from Figure 4 which shows that for Model III the fitted profile has a slightly larger magnitude for the negative flux as compared to the model flux. We have tried various values for the mixing length parameters to find that in no case is it possible to match the model convective flux profile when the extent of overshoot is larger than about $0.20H_p$.

At the top of the convective envelope, around $\tau = 1$ we get substantial overshoot into the atmosphere. In this region, most of the convective flux is contributed by convective eigenmodes

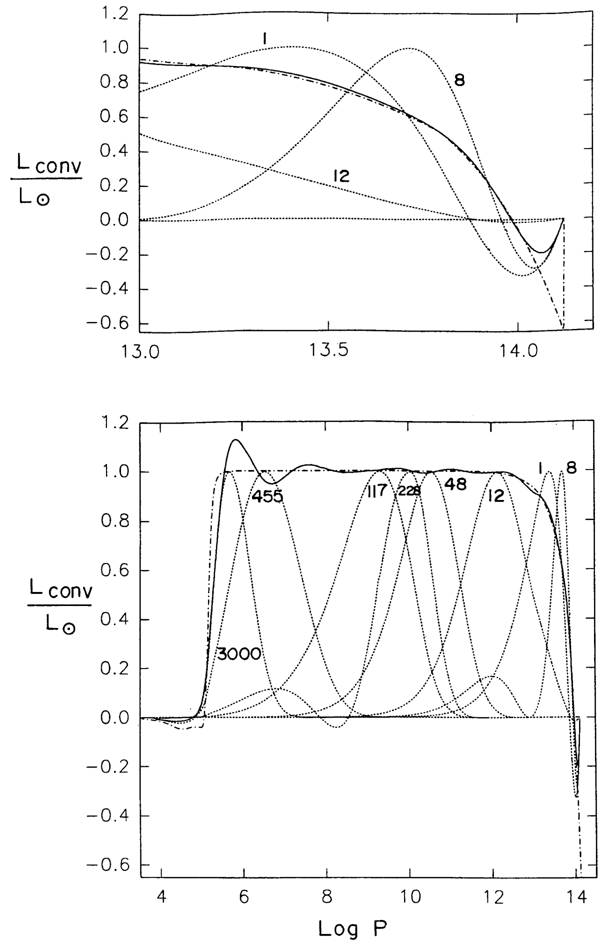


FIG. 3.—Dotted curves show the convective luminosity due to individual convective modes for various $l$-values as a funcion of logarithm of pressure for a solar envelope model with $C = 0.01$, $D = 0.1$, $L = \min\,(z + 459{,}230000 - z)$ km. The curves labeled 12 and 228 refer to C2-modes, while other curves pertain to C1-modes. The continuous curve shows the superposed convective luminosity profile, while the dot dashed curve represents the model convective luminosity. The upper figure shows a blow-up of the overshoot region.

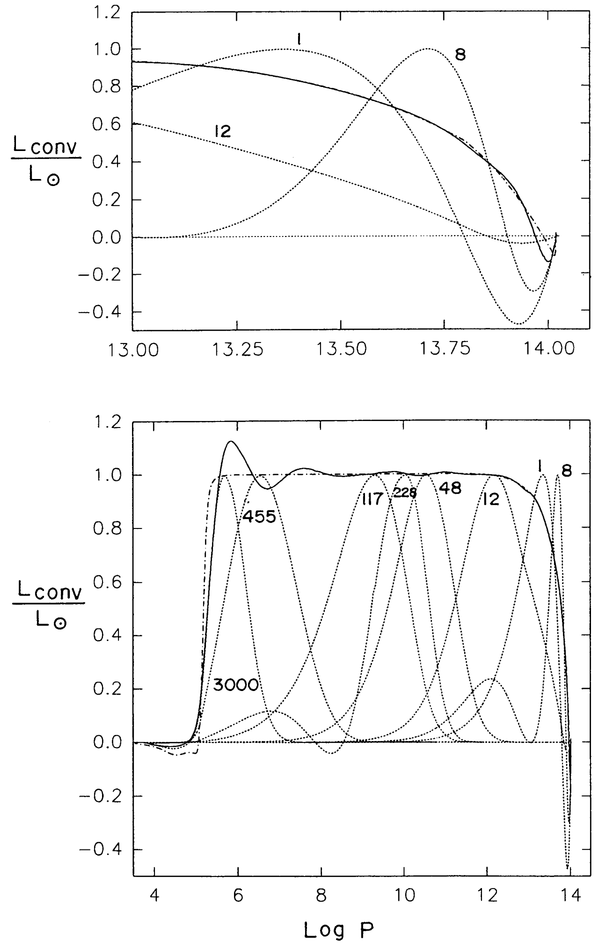


FIG. 4.—Dotted curves show the convective luminosity due to individual convective modes for various $l$-values as a function of logarithm of pressure for a solar envelope model with $C = 0.01$, $D = 0.1$, $L = \min\,(z + 459{,}215000 - z)$ km. The curves labeled 12 and 228 refer to C2-modes, while other curves pertain to C1-modes. The continuous curve shows the superposed convective luminosity profile, while the dot-dashed curve represents the model convective luminosity. The upper figure shows a blow-up of the overshoot region.

of high degree ($l > 1000$). It is clear from Figure 3 that the convective flux does indeed become negative in the overshoot region and furthermore the superposed flux smoothly tends to zero. The penetration length comes out to be of the order of 2–3$H_p$. We may stress here that our approach of matching the superposed flux profile with the modal convective flux enables us to handle the convective penetration into a strongly non-adiabatic region without any explicit assumption about the degree of adiabaticity in the neighboring stably stratified layer.

It should be stressed that we are not attempting to provide a numerical proof for the completeness of linear eigenfunctions constructed using the background temperature-density stratification in mixing-length model. Note that we have only a finite number of unstable modes and evidently, no finite set of functions can be mathematically complete. Furthermore, it is well-known (cf. Hart 1973) that in the absence of turbulent dissipation it is not possible to generate such a combination of suitably normalized eigenmodes. The main thrust of the present computation is the demonstration that even in the presence of dissipative effects, we are not able to construct such a linear superposition for arbitrary extent of overshoot. Clearly, the consistency argument adopted by us is not a mere proof of completeness. Equally, the small overshoot resulting from our computations is by no means a consequence of the use of linear eigenmodes, since adopting the same formulation we do get significant overshoot into the layers above the solar photosphere, which is of course, consistent with observations. As we have illustrated in Figures 2 and 3, even below the convection zone the linear eigenfunctions can indeed penetrate to a significant extent beyond the unstable region. However, we have rejected such models with substantial overshoot simply on the ground that no combination of the convective eigenmodes in these cases can yield the required profile of negative convective flux in the overshoot region.

## 4. DISCUSSION AND CONCLUSIONS

We have attempted to combine the model computation and the modal analysis approach to estimate the extent of convective overshoot from the convection zone into the adjoining stable region. We find that inside stars there is penetration from the convective envelopes beyond the layers defined as the convection zone boundary determined by the condition $\nabla = \nabla_{ad}$, We realize that the very definition of the extent of overshooting is somewhat ambiguous. Roughly, we might think in terms of the motion of individual convective elements and fix the edge of the overshoot region as the furthermost place reached by the elements. It is natural to define the penetration length as the distance beyond the $\nabla = \nabla_{ad}$ layer where the convective velocity becomes negligible. But, in practice, this transition layer (where $\nabla = \nabla_{ad}$) gets shifted in most nonlocal models. Thus even though the model may exhibit substantial overshooting, the overshoot region itself may not extend much beyond the layer at which the standard local mixing length theory will predict the convection zone boundary to be located. In practice, we will only be interested in this difference to estimate the penetration length. For simplicity, in this paper, we have defined the extent of overshooting as the distance between the layer where $\nabla = \nabla_{ad}$ and the layer where the convective velocity approaches zero.

The problem is further complicated by constraints; for example, the model constructed with a nonlocal version of mixing length theory that includes overshoot below the base of the solar convection zone may yield values of the radius or luminosity that are at variance with the present Sun for a given prescription of mixing length. We may then have to adjust the mixing length or some other parameter in the theory to get the correct luminosity, radius and other properties. As a result of these constraints, there may not be any significant difference between local and nonlocal models, although the nonlocal model may internally show a sizable overshoot layer.

We have another constraint provided by helioseismology. If the solar model is fitted to the observed frequencies of solar oscillations, then the depth of convection zone including any overshoot region has to be about 200,000 km, irrespective of the extent of overshooting. This can actually be used to put limits on the extent of overshooting, since a model with substantial penetration may not be able to match all the observed frequencies. To estimate the effect of overshooting at the base of convection zone, on the frequencies of solar oscillations we have constructed four solar models with different extent of penetration as follows:

Model A: No overshoot, convection zone depth 193,000 km.

Model B: Overshoot of $\approx 0.12 H_p$ with overshoot region extending to 200,000 km.

Model C: Overshoot of $\approx 0.3 H_p$ with overshoot region extending to 210,000 km.

Model D: Overshoot of $\approx 0.5 H_p$ with overshoot region extending to 220,000 km.

Figure 5 shows the difference in frequencies $\nu_D - \nu_A$ for models D and A, as a function of the frequency for different values of $l$. For large $l$, the modes are completely trapped inside the convection zone and there is no appreciable difference in the frequencies. The difference is maximum for intermediate $l$ ($\approx 50$), while for low $l$, again the difference is small. For each value of $l$ the difference becomes significant at frequencies which penetrate beyond the normal convection zone. The maximum difference in this case is approximately 10 $\mu$Hz which should be easily detectable. The behavior is similar for other models although the difference is smaller. Thus, for Model C the maximum difference is $\approx 5$ $\mu$Hz, while for Model B it is $\approx 1$ $\mu$Hz (Fig. 6). It seems that a substantial overshoot at the base of the convection zone is perhaps not consistent with the observed eigenfrequencies. The difference in frequencies can be traced to difference in the sound speed as a function of depth. In particular, for models without overshooting the first derivative of the sound speed $c_s$ is continuous at the base of convection zone, while the second derivative is discontinuous. On the other hand, in models with overshoot the first derivative of sound speed is discontinuous at the edge of the overshoot layer. To illustrate this point we have shown in Figure 7 a plot of sound speed versus log $P$ for Models A and D, in the neighborhood of the overshoot layer. On the basis of a more detailed study of solar eigenfrequencies Monteiro et al. (1992) find no evidence for the existence of an overshoot region below the solar convection zone.

We have applied the arguments to demonstrate the consistency of the mixing length theory for the solar envelope model and tried to show that it is possible to combine the model approach with the modal analysis and get an estimate of the overshooting depth. For the most part, we find the overshoot distance to be about 10 percent of the local pressure scale height. Of course, our numerical results are derived by making several simplifying assumptions. To begin with, we have used a

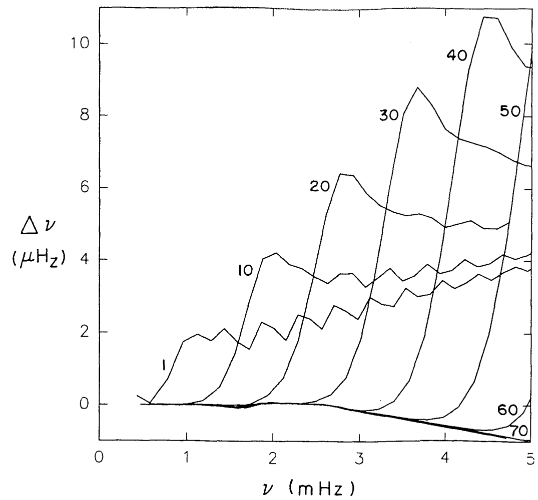


FIG. 5.—Difference in the frequencies between Models D and A ($\nu_D - \nu_A$) is plotted as a function of the frequency for different values of $l$.

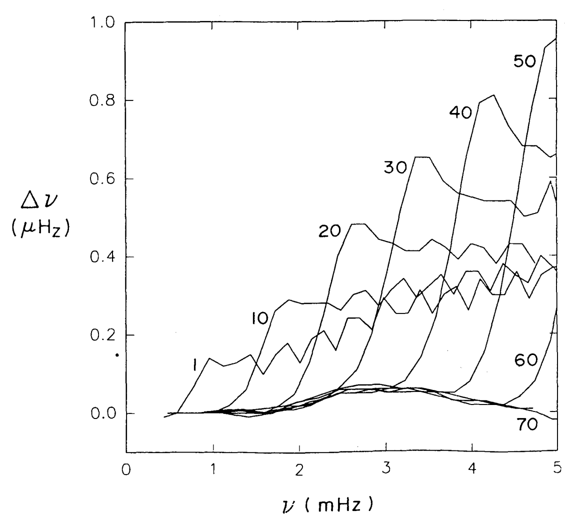


FIG. 6.—Difference in the frequencies between Models B and A ($\nu_B - \nu_A$) is plotted as a function of the frequency for different values of $l$.

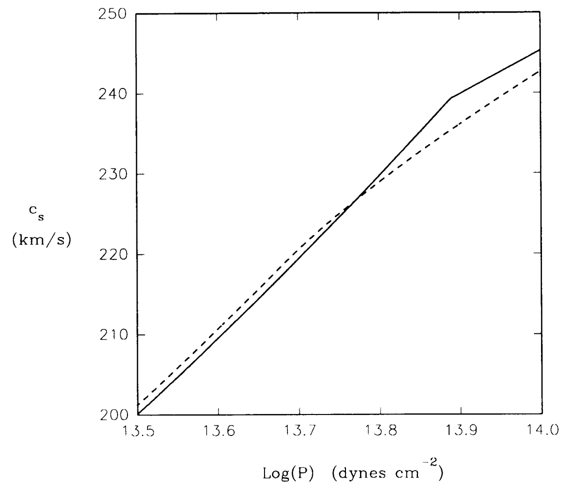


FIG. 7.—Sound speed $c_s$ is plotted as a function of log $P$ for Model A (*dashed curve*: no overshoot) and Model D (*solid curve*: with overshoot).

nonlocal mixing-length model which is highly idealized, but it is supposed to incorporate the basic physical feature of a nonlocal velocity field. Further, we have employed a local expression for the convective flux, $F^C$ and have neglected the contribution from the kinetic energy flux to $F^C$, which may be justifiable (cf. Zahn 1991). The role of turbulent stresses as manifested by the turbulent pressure (embodied by the parameter $C$) may be negligible in the present problem, but we certainly need a nonzero aerodynamic drag in the equation of motion of convective elements.

Roxburgh (1978) has included turbulent kinetic energy in the expression for convective flux and derived an integral constraint which requires the convective elements to penetrate into the surrounding stable layers. Using arguments based on the mean total energy equation for a static, inviscid fluid, Roxburgh concludes that it is impossible to satisfy the integral-constraint over the entire stellar convection zone, unless there is a negative entropy flow in part of the region. This can happen only in a stably stratified layer and hence there must occur overshooting into the neighboring stable zone. Roxburgh's criterion may be adequate for calculating penetration from the almost adiabatic convective cores which would increase the mass of the core and hence lengthen the main-sequence lifetime. But, it is probably not applicable to stellar convective envelopes because of the appreciable nonadiabatic effects arising from efficient thermal diffusion at the top of outer convection zones like the one in the Sun. Apart from this because of the neglect of viscous dissipation, Roxburgh's criterion only gives an upper limit on the extent of overshoot.

The present work has addressed the question of convective overshoot at the base of the solar convection zone. We hope that the diagnostic of the solar interior by the tools provided by helioseismology will shed light on the physics of the overshoot region by checking the profile of the temperature and sound speed at the base of the solar convection zone, which is liable to be influenced by the effects of penetration. The implication of overshoot for solar dynamo theories which store the magnetic field at the base of the convection zone are clearly profound.

We have not dealt with overshooting from convective cores in the present work, but evidently convective penetration in the stellar interior will have significant influence on the evolutionary characteristics of stars, particularly the time scales for hydrogen burning.

A distinguishing feature of our approach is that it is not based on any specific assumption about the degree of adiabaticity of stratification in the overshoot region. This enables us to estimate the penetration into the atmosphere overlying the solar photosphere, where damping due to radiative exchange is very strong for significant nonadiabatic effects to occur. From Figure 3 we can see that the superposed flux profile also penetrates above the photospheric level, where we get the negative convective flux which is less than 5% of the total flux, but the overshoot distance is a few scale heights.

We conclude that in the convective envelopes of stars there would certainly be penetration from the regions with superadiabatic temperature gradients into the bounding stable zones. By demanding the consistency of the mixing length theory as applied to the convection zone model and using the information available about the penetration of individual linear modes, we demonstrate that the penetration depth below the base of the solar convection zone is $\leq 20\%$ of the local pressure scale height. In the atmosphere overlying the solar photosphere, however, the overshoot distance could be several scale heights. It is hoped that accurate helioseismological data could enable us to estimate the extent of overshooting at the base of the convection zone.

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