A LOWER BOUND FOR THE NUMBER OF TOPOLOGIES ON A FINITE SET

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An expression for the number of different topologies (not considering homeomorphisms) on a finite set of n elements is not known. However, lower and upper bounds for the same have been obtained. (*See* Evans *et al.* 1967 and Krishnamurthy 1966). Krishnamurthy has established a one-one correspondence between (1, 0) matrices of order n with certain properties and the topologies on a set of n elements. (These matrices turn out to be the adjacency matrices of labelled transitive digraphs with n vertices). This note examines these matrices and uses the observations to obtain a lower bound for the number of topologies.

First, a few useful observations are made regarding the structure of these matrices. It is noted that entries of 1's off the main diagonal play an important part. Hence the matrices are grouped into classes $S(n, k)$ viz., $(n\times n)$ matrices as above with 1's off the main diagonal in exactly k rows.

In the next section some of these classes are enumerated fully while estimates are obtained for the rest. This leads to a lower bound of

$$2+\sum_{k=1}^{n-1}\binom{n}{k}(2^{n-k}-1)^{k-2}\left[(2^{n-k}-1)^2+k(k-1)(3^{n-k}-2^{n-k})+\binom{k}{2}2^{n-k}\right] \quad (1)$$

for the number of topologies.

The next section suggests further refinements to this formula and shows how sharper bounds can be obtained.

Evans *et al.* (1967) have enumerated the number of digraphs in which no path is of length greater than 1. The concluding section obtains the same result through a consideration of a subclass of the above matrices.

INTRODUCTION

Let E be a finite set of n elements. Let $f(n)$ be the number of topologies on E (not taking homeomorphisms into account). No formula for $f(n)$ is known. However, many bounds for $f(n)$ have been obtained. In particular, lower and upper bounds can be found in the papers of Evans *et al.* (1967) and Krishnamurthy (1966). Krishnamurthy has established a one-one correspondence between certain (1, 0) matrices and topologies on E. In this note, we classify these matrices and obtain lower bounds for the number of members in each class. (Certain classes are enumerated fully). We establish the bound,

$$f(n) \geqslant 2 + \sum_{k=1}^{n-1} \begin{bmatrix} n \\ k \end{bmatrix} (2^{n-k}-1)^{k-2} \left\{ (2^{n-k}-1)^2 + k(k-1)\,(3^{n-k}-2^{n-k}) + \begin{bmatrix} k \\ 2 \end{bmatrix} 2^{n-k} \right\} \quad .. \quad (1)$$

and also show how this bound can be improved to include many more such terms and thus obtain as sharp a bound as possible by this process.

Classification of (1, 0) Matrices

We quote here the main theorem of Krishnamurthy (1966).

Theorem : "*On the finite set E of n elements there are as many topologies, (not taking homeomorphisms into account) as there are $n \times n$ matrices (a_{ij}) of zeroes and ones with $\mathbf{a}_{ii} = 1$ for all i, and with the following property :*

*(*) For every ordered pair $(\mathbf{R}_i\ \mathbf{R}_j)$ $(i, j = 1, \ldots, n)$ of rows of $(\mathbf{a}_{ij})$ and every index k, $\mathbf{a}_{ji} = 1 = \mathbf{a}_{ik}$ implies, $\mathbf{a}_{jk} = 1$.*"

We will hereafter say that a matrix with all the properties listed in the theorem has property**. Graph theorists will immediately recognize these as merely the adjacency matrices of labelled transitive digraphs. (cf. Evans *et al.* 1967).

It is convenient at this stage, to make a few observations on the structure or matrices with property (**). First of all, if $a_{ii} = 1$ for all i, in checking for property (*), we need only check for the nondiagonal elements. For, if $i = j$, $j = k$ or $i = k$, (*) is trivially satisfied.

Secondly, if $a_{ji} = 1$ for $j \neq i$, then by virtue of (*) $\mathbf{R}_j$ must have 1's wherever $\mathbf{R}_i$ has 1's. Also if $a_{ji} = 0$, then the distribution of 0's and 1's in $\mathbf{R}_j$ is independent of that in $\mathbf{R}_i$, since (*) does not apply to the pair $(\mathbf{R}_i, \mathbf{R}_j)$.

With the above comments in mind, we proceed to classify the matrices with property (**) as follows :

Let $S(n, k)$ be the class of all $n \times n$ matrices with property (**) such that in exactly k rows, we have 1's off the main diagonal also. Let $N(n,k)$ be the number of members in $S(n, k)$. [Clearly $N(n, 0) = 1$]. We then have,

$$f(n) = \sum_{k=0}^{n} N(n, k). \quad .. \quad (2)$$

Estimates for $N(n, k)$

Let us choose the rows $i_1, .., i_k$, to be the k rows with 1's off the main diagonal. This can be done in $\begin{bmatrix} n \\ k \end{bmatrix}$ ways. The remaining $n-k$ rows have 1 on the main diagonal and 0's elsewhere. As already mentioned we are interested in the nondiagonal elements of the k choosen rows. This again depends on the interrelation of these k

rows given by the $k\ (k-1)$ quantities $\mathbf{a}_{i_l i_m}$ $(l \neq m, l, m=1, \ldots, k)$. Our discussion will be centred round the values that these quantities assume and so we will call them 'pivots'.

Case (*i*) : All pivots take the value zero. In this case the k rows are all independent of each other. Since $(k-1)$ pivots and the diagonal element are fixed in each row we have $n-k$ free points in each row and these can be filled in $2^{n-k}-1$ ways in each row (we subtract one so that no row has a 1 only on the main diagonal). Thus we get $(2^{n-k}-1)^k$ matrices with the property (**). This case has an interpretation in diagraphs. (see conclusions.).

Case (*ii*) : Only one pivot is non zero. Let $\mathbf{a}_{i_l i_m} \neq 0$. This can be chosen in $k\ (k-1)$ ways. Also $\mathbf{R}_{i_l}$ is dependent on $\mathbf{R}_{i_m}$ and the remaining $(k-2)$ rows are independent of these as well as of each other and can be filled in $(2^{n-k}-1)^{k-2}$ ways. $\mathbf{R}_{i_m}$ can have r 1's fixed in $\begin{bmatrix} n-k \\ r \end{bmatrix}$ ways. This fixes r 1's together with the already fixed k points in $\mathbf{R}_{i_l}$. Thus $\mathbf{R}_{i_l}$ can now be filled in 2^{n-k-r} ways. Since r can range from 1 to $n-k$, we have that these two rows can together be filled in,

$$\sum_{r=1}^{n-k} \begin{bmatrix} n-k \\ r \end{bmatrix} 2^{n-k-r} = 3^{n-k} - 2^{n-k} \qquad \ldots (3)$$

ways. Thus this case yields, $k(k-1)\ (2^{n-k}-1)^{k-2}\ (3^{n-k}-2^{n-k})$ matrices with property (**).

Case (*iii*) : Two pivots are non-zero. This includes several subcases which are classified below :

$$(a) \quad \mathbf{a}_{i_l i_m} = \mathbf{a}_{i_m i_l} = 1.$$

$$(b) \quad \mathbf{a}_{i_l i_m} = \mathbf{a}_{i_n i_m} = 1.$$

$$(c) \quad \mathbf{a}_{i_l i_m} = \mathbf{a}_{i_l i_n} = 1.$$

$$(d) \quad \mathbf{a}_{i_l i_m} = \mathbf{a}_{i_p i_q} = 1.$$

Subcase (*a*) is the simplest and it terminates the case (iii) when $k = 2$. Here $\mathbf{R}_{i_l}$ and $\mathbf{R}_{i_m}$ are mutually dependent and hence are identical. This choice can thus be made in $\begin{bmatrix} k \\ 2 \end{bmatrix}$ ways and these two rows can together be filled in 2^{n-k} ways. Thus (*a*) yields, $\begin{bmatrix} k \\ 2 \end{bmatrix} (2^{n-k}-1)^{k-2}\ 2^{n-k}$ matrices. For reasons of length of the formula, we stop here for the upper bound. We have

$$N(n, k) \geqslant \begin{bmatrix} n \\ k \end{bmatrix} (2^{n-k}-1)^{k-2} \left[(2^{n-k}-1)^2 + k(k-1)(3^{n-k}-2^{n-k}) + \begin{bmatrix} k \\ 2 \end{bmatrix} 2^{n-k}\right] \quad .. \quad (4)$$

Since these cases do not contribute anything to $N(n, n)$ (for, then all non-diagonal elements are pivots and there is an upper limit of $n(n-2)$ for the number of zeroes in the matrix), we just consider the matrix with all entries unity for the class $S(n, n)$ and get the bound given by (1).

Further Refinements

To continue with the enumeration for the other subcases, we note that (*a*), (*b*) and (*c*) exhaust case (iii) when $k = 3$ and (*a*) to (*d*) exhaust case (iii) for $k \geqslant 4$. For cases (*b*) and (*c*) the choice of pivots is done in $k(k-1)(k-2)/2$ ways. The remaining $(k-3)$ rows are filled as usual in $(2^{n-k}-1)^{k-3}$ ways. In (*b*), $\mathbf{R}_{i_m}$ dictates to $\mathbf{R}_{i_n}$ and to $\mathbf{R}_{i_l}$. Thus filling r 1's in $\mathbf{R}_{i_m}$ we fix r 1's of $\mathbf{R}_{i_l}$ and $\mathbf{R}_{i_n}$ and in each, the remaining $(n-k)$ places can be independently filled in 2^{n-k-r} ways. Thus these three rows are together filled in

$$\sum_{r=1}^{n-k} \begin{bmatrix} n-k \\ r \end{bmatrix} 4^{n-k-r} = 5^{n-k} - 4^{n-k} \quad .. \quad (5)$$

ways. Thus (*b*) yields, $\frac{k(k-1)(k-2)}{2}(2^{n-k}-1)^{k-3}[5^{n-k}-4^{n-k}]$ matrices.

The subcase (*c*) is probably the most difficult of all. For, both $\mathbf{R}_{i_m}$ and $\mathbf{R}_{i_n}$ dictate to $\mathbf{R}_{i_l}$. So let us fix r 1's in one of them, say $\mathbf{R}_{i_m}$. Let us choose to have t overlaps between the 1's in $\mathbf{R}_{i_m}$ and $\mathbf{R}_{i_n}$ and let us have s more 1's in $\mathbf{R}_{i_n}$. Then it is readily seen that these three rows are together filled in,

$$\sum_{r=1}^{n-k}\left\{\sum_{s=0}^{n-k-r}\sum_{t=0}^{r}\begin{bmatrix} n-k \\ r \end{bmatrix}\begin{bmatrix} r \\ t \end{bmatrix}\begin{bmatrix} n-k-r \\ s \end{bmatrix} 2^{n-k-r-s} - \begin{bmatrix} n-k \\ r \end{bmatrix} 2^{n-k-r}\right\} = 5^{n-k} - 2.3^{n-k} + 2^{n-k} \quad .. \quad (6)$$

ways (since both s and t cannot be together zero). Thus (*c*) gives $\frac{k(k-1)(k-2)}{2}$ $(2^{n-k}-1)^{k-3}(5^{n-k}-2.3^{n-k}+2^{n-k})$ matrices.

Subcase (*d*) is case (ii) repeated for two sets of rows. The pivot choice is done in $k(k-1)(k-2)(k-3)$ ways. We have $(k-4)$ independent rows. Thus on the whole we have, $k(k-1)(k-2)(k-3)(2^{n-k}-1)^{k-4}(3^{n-k}-2^{n-k})^2$ matrices from (d). Thus to the coefficient of $\begin{bmatrix} n \\ k \end{bmatrix}$ in the bound for $N(n, k)$ we may add the terms got from subcases (iii) (b) through (d) also.

Upto now we have discussed the case when only two pivots are non-zero. Let us now have r non-zero pivots in the form,

$$a_{i_{l_1} i_{l_2}} = a_{i_{l_2} i_{l_3}} = \dots = a_{i_{l_r} i_{l_{r+1}}} = 1 \qquad \dots (7)$$

and of course all pivots arising out of the transitivity relations due to (7) are also equal to 1. Let all other pivots be zero. Then $\mathbf{R}_{i_{l_k}}$ dictates to $\mathbf{R}_{i_{l_j}}$ whenever $k > j$. Starting with s_{r+1} 1's in $\mathbf{R}_{i_{l_{r+1}}}$, and then s_r more 1's in $\mathbf{R}_{i_{l_r}}$ and so on we get the following r—fold summation, denoting the number of ways to fill the rows $\mathbf{R}_{i_{l_1}}, \dots, \mathbf{R}_{i_{l_{r+1}}}$ as,

$$\sum_{s_{r+1}, \dots, s_2} \begin{bmatrix} n-k \\ s_{r+1} \end{bmatrix} \begin{bmatrix} n-k-s_{r+1} \\ s_r \end{bmatrix} \cdots \begin{bmatrix} n-k-s_{r+1}-\dots-s_3 \\ s_2 \end{bmatrix} 2^{n-k-\dots-s_2}.$$

The range of s_{r+1} is from 1 to $n-k$, while s_j, the number of 1's fixed arbitrarily in $\mathbf{R}_{i_{l_j}}$, ranges from zero to $n-k-s_{r+1}-\dots-s_{j+1}$. The above summation can be evaluated step wise as $(r+2)^{n-k}-(r+1)^{n-k}$. The pivot choice is done in $k(k-1) \dots (k-r)$ ways. We have $k-r-1$ independent rows. So such a choice leads to $\frac{k!}{(k-r-1)!}(2^{n-k}-1)^{k-r-1} \times [(r+2)^{n-k}-(r+1)^{n-k}]$ matrices.

Thus when p pivots are non-zero we can break it up into such chains and enumerate them. Also when we have circuits within such a chain we can use a similar

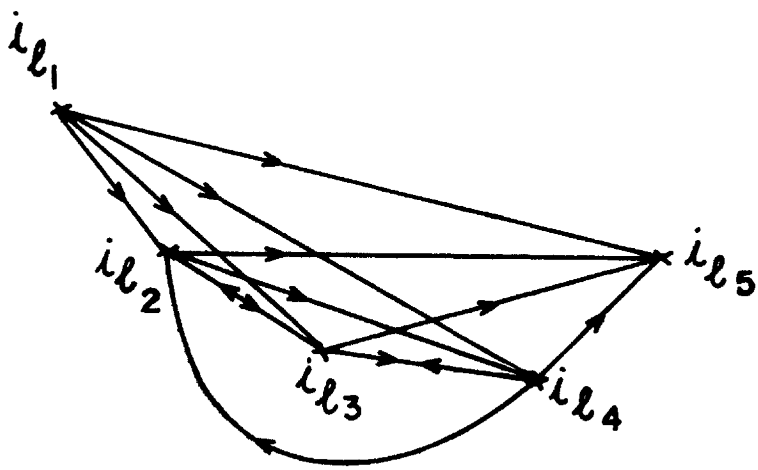

FIG. 1

formula. Going to the corresponding graph, we see that transitivity helps us to bypass the cycle and consider a chain of the above form but of reduced length (see Fig. 1).

From Fig. 1, $\mathbf{R}_{i_{l_2}} = \mathbf{R}_{i_{l_3}} = \mathbf{R}_{i_{l_4}}$ and we bypass the circuit $(i_{l_2}, i_{l_3}, i_{l_4})$ and consider the shorter chain given by

$$\mathbf{a}_{i_{l_1} i_{l_2}} = \mathbf{a}_{i_{l_2} i_{l_5}} = 1 \qquad \ldots (8)$$

and apply the formula obtained above.

Thus when more than two pivots are zero, we can split it up into forms discussed above and get a bound as sharp as we please.

Conclusions

Finally, we give an interpretation of case (i) in $S(n, k)$ and of the class $S(n, 0)$. Considering these contributions alone, we have,

$$f(n) \geqslant 1 + \sum_{k=1}^{n-1} \begin{bmatrix} n \\ k \end{bmatrix} (2^{n-k} - 1)^k$$

$$= \sum_{s=1}^{n} \begin{bmatrix} n \\ s \end{bmatrix} (2^s - 1)^{n-s}$$

$$= \delta(n), \text{ say.}$$

The expression $\delta(n)$ is precisely what is given in Evans *et al.* (1967) as the number of digraphs in which no path is of length greater than 1, (which are necessarily transitive). This can be deduced from matrix considerations also. When there are no pivots (as in the case of $S(n,o)$ or when no pivot is unity, there is no path of length greater than 1. Conversely, if one pivot is unity, then since at least one non-diagonal element is unity in each of the k chosen rows, we have a path of length at least 2. Thus $S(n, o)$ and case (i) of $S(n, k)$ cover all digraphs in which no path is of length greater than 1. Thus, the total number of such digraphs is,

$$\delta(n) = \sum_{s=1}^{n} \begin{bmatrix} n \\ s \end{bmatrix} (2^s - 1)^{n-s} \qquad .. (9)$$

References

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