ON AN INEQUALITY CONCERNING ORTHOGONAL POLYNOMIALS

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1. *Introduction.*—In a recent note,[1] G. Szegö has given several proofs of the following inequality of P. Turán for Legendre polynomials:—

$$\Delta_n(x) = P_n^2 - P_{n-1} P_{n+1} \geqslant 0 \tag{1}$$

and has pointed out that the inequality holds also for the Hermitian, the Laguerre and the ultraspherical polynomials. We show in this note that an elementary proof of the inequality may be given merely by considering the second derivative of $\Delta_n(x)$. This is done below for the Legendre, Hermitian and Laguerre polynomials. We follow the notation of G. Szegö:

Orthogonal Polynomials, 1939.

2. *Legendre Polynomials.*—By differentiating $\Delta_n(x)$ and using the relations

$$(n+1) P_{n+1} - (2n+1) xP_n + nP_{n-1} = 0 \tag{2}$$

$$(1 - x^2) P_n'(x) = n (P_{n-1} - xP_n) \tag{3}$$

we find, after some reduction,

$$\Delta''_n(x) = -\frac{2}{n(n+1)} \{P_n'(x)\}^2 \tag{4}$$

Thus $\Delta_n''(x) \leqslant 0$, and so $\Delta_n'(x)$ is a decreasing function of x. Now $\Delta_n(+1) = \Delta_n(-1) = 0$ and so, by Rolle's theorem, $\Delta_n'(x)$ must have at least one root in $(-1, +1)$; but, since $\Delta_n'(x)$ is a decreasing function in $(-1, +1)$, it can not have more than one root in that interval. Let $x = \xi$ be this root, so that $\Delta_n'(\xi) = 0$, $(-1 < \xi < +1)$. Then it is clear that $\Delta_n'(x) \geqslant 0$ in $-1 \leqslant x \leqslant \xi$ and $\Delta_n'(x) \leqslant 0$ in $\xi \leqslant x \leqslant 1$. Hence $\Delta_n(x)$ is an increasing function in $(-1, \xi)$ and a decreasing one in $(\xi, +1)$. Since $\Delta_n(-1) = \Delta_n(+1) = 0$, it follows that $\Delta_n(x) \geqslant 0$ in $(-1, +1)$, which establishes (1).

3. *Hermitian Polynomials.*—Since the interval of orthogonality for the Hermitian polynomials is $(-\infty, +\infty)$, it is convenient to introduce a new

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[1] G. Szegö, *Bull. Amer. Math. Soc.*, 1948, **54**, No. 4.

function $h_n(x)$ defined by

$$h_n(x) = e^{-x^2/2}\, H_n(x) \tag{5}$$

so that

$$h_n(x) \to 0 \text{ as } x \to \pm\infty \tag{6}$$

The relations

$$H_{n+1} - 2xH_n + 2nH_{n-1} = 0 \tag{7}$$

$$H_n'(x) = 2nH_{n-1} \tag{8}$$

expressed in terms of the h_n become

$$h_{n+1} - 2xh_n + 2nh_{n-1} = 0 \tag{9}$$

$$h_n'(x) = 2nh_{n-1} - xh_n \tag{10}$$

If we now set

$$f(x) = h_n^2 - h_{n-1}\, h_{n+1} = e^{-x^2}(H_n^2 - H_{n-1}\, H_{n+1}) = e^{-x^2}\, \Delta_n(x) \tag{11}$$

we find, by differentiating (11) and using (9) and (10),

$$f'(x) = -2h_{n-1}\, h_n \tag{12}$$

$$f''(x) = 2(h_n^2 - 2nh^2_{n-1}) \tag{13}$$

The relative maxima and minima of $f(x)$ are given by $f'(x) = 0$, *i.e.*, by the roots of $h_{n-1}(x) = 0$ and $h_n(x) = 0$. From (13) we see immediately that $h_n(x) = 0$ gives maxima, while $h_{n-1}(x) = 0$ gives minima. We further see that $f > 0$ at all the maxima and minima. This is obvious for the minima ($h_{n-1} = 0$); for the maxima ($h_n = 0$), $f > 0$ because then h_{n-1} and h_{n+1} are of opposite signs. Thus $f(x)$ has all its relative minima (and maxima) positive. In view of the fact that $f(x)$ and its derivatives are continuous, it is easy to conclude from this that f has a positive lower bound. Hence, $f(x) \geqslant 0$ for all x, which implies $\Delta_n(x) \geqslant 0$.

4. *Laguerre Polynomials.*—Here we set

$$l_n(x) = e^{-x/2}\, L_n(x),\ L_n(x) = \frac{e^x}{n!}\frac{d^n}{dx^n}(e^{-x}x^n) \tag{14}$$

The relations

$$(n+1)\, L_{n+1} - (2n+1-x)\, L_n + nL_{n-1} = 0 \tag{15}$$

$$xL_n'(x) = n(L_n - L_{n-1}) \tag{16}$$

then give

$$(n+1)\, l_{n+1} - (2n+1-x)\, l_n + nl_{n-1} = 0 \tag{17}$$

$$xl_n'(x) = \left(n - \frac{x}{2}\right) l_n - nl_{n-1} \tag{18}$$

If, as before, we write

$$f(x) = l_n^2 - l_{n-1}\, l_{n+1} = e^{-x}(L_n^2 - L_{n-1}\, L_{n+1}) = e^{-x}\, \Delta_n(x) \tag{19}$$

we find now the following results:—

$$f(0) = 0,\ f(x) \to 0 \text{ as } x \to +\infty \tag{20}$$

$$xf'(x) = (l_{n+1} - l_n)(l_n - l_{n-1}) \tag{21}$$

Thus the relative minima and maxima of $f(x)$ in $(0, \infty)$ are given by the positive roots of

$$l_{n+1} = l_n \text{ and } l_n = l_{n-1} \tag{22}$$

For $l_{n+1} = l_n$, we have $f = l_n^2 - l_{n-1}\, l_n = l_n(l_n - l_{n-1})$

$$\left.\begin{aligned} &= \frac{x}{n} l_n^2, \text{ by } (17) \\ \text{and similarly, for } l_n = l_{n-1},\quad f &= \frac{x}{n+1} l_n^2 \end{aligned}\right\} \tag{23}$$

so that all relative minima and maxima of $f(x)$ are positive for $x > 0$.

Actually we may easily verify by further differentiation that

$$\left.\begin{aligned} f''(x) &= \frac{1}{n+1} l_n^2 \text{ for } l_n = l_{n-1} \\ &= -\frac{1}{n} l_n^2 \text{ for } l_{n+1} = l_n \end{aligned}\right\} \tag{24}$$

so that $l_n = l_{n-1}$ gives minima and $l_{n+1} = l_n$ gives maxima of $f(x)$. It then follows as in the case of the Hermitian polynomials that

$$\Delta_n(x) = L_n^2 - L_{n-1}\, L_{n+1} \geqslant 0,\ x \geqslant 0.$$

5. The same method can be applied to show that the inequality in question holds also for the ultraspherical polynomials $P_n^{(\lambda)}(x)$ and the generalised Laguerre polynomials $L_n^{(\alpha)}(x)$ under suitable restrictions on the values of λ and α.