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The critical point of quantum chromodynamics through lattice and experiment

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Abstract. This talk discusses methods of extending lattice computations at finite temperature into regions of finite chemical potential, and the conditions under which such results from the lattice may be compared to experiments. Such comparisons away from a critical point are absolutely essential for quantitative use of lattice QCD in heavy-ion physics. An outline of various arguments which can then be used to locate the critical point is also presented.

Keywords. Quantum chromodynamics; lattice gauge theory; heavy-ion physics; phase transitions.

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1. Introduction

Particle physics progressed in the last century by looking for the simplest context in which a certain physics was observable. This usually means that two-particle interactions are used to measure each of the parameters entering a Lagrangian. By imposing tight constraints of this sort, various symmetries were worked out and the Standard Model was discovered.

Consistency tests of the standard model are in the application of the model to all possible processes with particle properties. This program fails for strong interactions. As a result, it affects all physics in the Standard Model. Here are some examples:

1. How large are hadronic contributions to the anomalous magnetic moment of the muon? For the electron this is computable to incredibly high accuracy within QED, and provides a very stringent test of that theory. Unfortunately, for the muon the contribution from strong interactions is large, and it turns out that the answer cannot be reliably computed in perturbation theory.
2. How good is the assumption of factorization in the extraction of CKM matrix elements? This is an assumption made in all current extractions of these very important parameters in the Standard Model. On the other hand, factorization is not yet proved, and, indeed, is known to fail in many processes at low-Q^2. Without factorization, perturbation theory cannot be used.

3. What is the magnitude of the cross-section of the semi-inclusive process $ep \rightarrow eX$ (with either large or small momentum transfer between the lepton and hadron subsystems) or the cross-section of the inclusive process $\nu p \rightarrow$ anything? How large are neutrino–proton or electron–proton total cross-sections at $\sqrt{S} = 10^{12}$ GeV? What is the partial width for the decay process $J/\psi \rightarrow K\pi\pi$? None of these results can be computed in perturbation theory.
4. Astrophysics is replete with questions which cannot be answered from the Standard Model in perturbation theory, although one expects that they are answerable in the Standard Model. A famous example is the Hoyle resonance in He burning stars, where a sequence of reactions $^4\text{He} + {}^4\text{He} \rightarrow {}^8\text{Be}$, $^4\text{He} + {}^8\text{Be} \rightarrow {}^{12}\text{C}$, $^4\text{He} + {}^{12}\text{C} \rightarrow {}^{16}\text{O}$, all seem to be fine-tuned to proceed with sufficient rapidity. Hoyle introduced the anthropic principle to 'explain' this. Similar spectacular fine-tunings are seen in a host of other nuclear reactions, including nucleon–nucleon scattering lengths. We believe that QCD can explain this. However, in QCD one can only tune three parameters, the u, d and s masses; how can that fine tune many different nuclear energy levels?
5. What are the phases of compressed baryonic matter? Are there phase transitions? Can some of these phases be found in compact stellar objects? What is the equation of state of such matter? What are the transport coefficients in such matter; do they transport momentum and energy efficiently?

A good number of the questions above can be addressed in lattice computations. The topic of this talk are the first couple of questions in point 5, their resolution in lattice QCD, and the prospect of testing these answers in experiment.

2. Avoiding the sign problem

Lattice computations are performed using Monte Carlo integration. At present such computations are not possible at finite baryon chemical potential, μ, because the integrand is not real and positive definite. However, this cannot stop us from developing methods to estimate thermodynamic averages using some kind of analytic continuation. The method which is now used is that of a Taylor series expansion around $\mu = 0$. In this talk we shall need the Taylor coefficients of the pressure

$$P(T,\mu) = P(T) + \frac{\mu^2}{2!}\chi^{(2)}(T) + \frac{\mu^4}{4!}\chi^{(4)}(T) + \frac{\mu^6}{6!}\chi^{(6)}(T) + \frac{\mu^8}{8!}\chi^{(8)}(T) + \cdots, \tag{1}$$

where the nonlinear susceptibilities (NLS), $\chi^{(n)}(T)$, are evaluated at $T = 0$ using lattice simulations [1], as is the pressure, $P(T)$. The series is even in μ due to the CP symmetry of the theory at $\mu = 0$. The successive derivatives of the pressure (the first is the baryon number, and the second is the baryon-number susceptibility) can also be evaluated using the same series coefficients. The baryon number susceptibility diverges at the critical point; as a result its series expansion around $\mu = 0$ should have finite radius of convergence. All these considerations are exact, and can be applied to any realization of QCD; in particular this formulation is neutral to the use of Wilson, staggered, overlap or any other formulation of quarks. The earlier approaches [2] now seem to be superseded by this method or its variants.

Until now the series coefficients have been evaluated with staggered quarks up to the eighth order with a lattice cut-off of $1/a \simeq 800$ MeV [3] and $1/a \simeq 1200$ MeV [4], and with P4 improved quarks to sixth order with a cut-off of $1/a \simeq 800$ MeV [5]. In both cases the bare quark mass is tuned so that the pion is light, with mass of around 220–230 MeV. Lattice simulations show that at some temperature, T^E, the series coefficients are all positive and their ratios are equal. This implies a breakdown of the series at a real value of the chemical potential, μ^E. T^E and μ^E are the position of the critical end point of QCD. The best estimate [4] is currently

$$\frac{T^E}{T_c} = 0.94 \pm 0.01 \quad \text{and} \quad \frac{\mu^E}{T^E} = 1.8 \pm 0.2. \tag{2}$$

Some shifts are expected when the quark mass is tuned to reproduce the physical pion mass and the lattice cut-off is removed. From the trends observable now, it seems possible that there will be little shift in T^E, and μ^E/T^E will lie in the range 1.5–2.5.

In the vicinity of μ^E, since the series diverges, it is not enough to sum a finite number of terms to get accurate predictions for physical quantities. One must develop methods for resumming the series. One common method of resummation is to use Padé approximants. In figure 1 we compare truncated series expansions with Padé approximants using the same number of coefficients. The Padé approximants are the rational functions

$$P^L_M(z) = \frac{a_0 + a_1 z + a_2 z^2 + \cdots + a_L z^L}{1 + b_1 z + b_2 z^2 + \cdots + b_M z^M}, \tag{3}$$

which have $L+M+1$ coefficients. These are fixed by equating $P^L_M(z)$ to $L+M+1$ terms of the Taylor series.

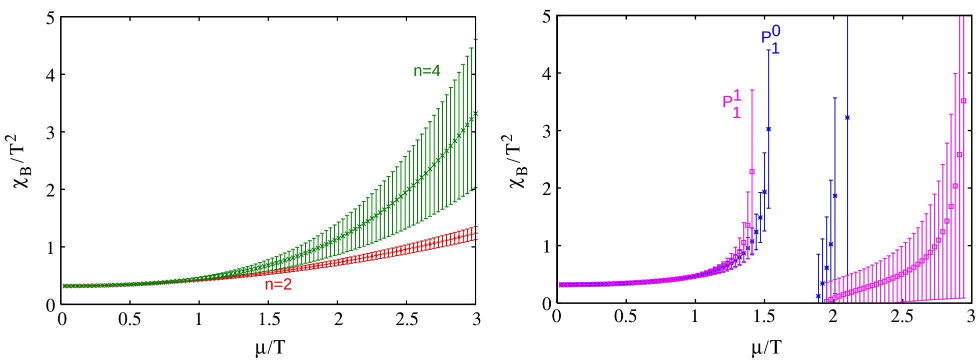

Figure 1. (Left) The baryon number susceptibility obtained by summing two and four terms of the Taylor expansion; the curves are smooth, although the radius of convergence lies in the centre of the range of the ordinates shown. (Right) The result of Padé resummation matched to the same polynomials. They diverge at the radius of convergence.

In figure 1 we show Taylor approximations to the baryon number susceptibility until the μ^n term for $n = 2$ and 4, and the Padé approximants P_1^0 and P_1^1 which are matched to these two [4]. Clearly, the truncated Taylor expansions show no signs of the breakdown of the series, whereas the Padé approximants diverge at that point. For each Padé, the plots show two branches of the function on either side of the singularity. We can use the Padé approximant to give an estimate of the width of the critical region, $\Delta_c\mu$. If one assumes that the critical region is that in which χ_B/T^2 exceeds unity, then one finds using P_1^0 that $\Delta_c\mu/T^E \simeq 0.5$. This corresponds to

$$\Delta_c\mu \simeq 80 \text{ MeV}. \tag{4}$$

As one can see from figure 1, this is merely a first estimate. Although this is all we have available at the moment, better estimators of the width can be devised.

3. Predicting the spectrum of fluctuations

At any normal point on the phase diagram of QCD, the longest correlation length, ξ, is finite. When a grand canonical system, of volume $V \gg \xi^3$, is allowed to come in equilibrium with a reservoir with which it can exchange energy or a conserved charge (such as the baryon number, B), then there are many independently fluctuating volumes in the system. As a result, the central limit theorem applies, and the spectrum of fluctuations about the mean thermodynamic value is Gaussian

$$P(\Delta B) \propto \exp\left(-\frac{(\Delta B)^2}{2VT\chi_B(T,\mu)}\right), \tag{5a}$$

where

$$\Delta B = B - \langle B(T,\mu)\rangle.$$

The thermodynamic average value of the baryon number is given by

$$\langle B(T,\mu)\rangle = V\frac{\partial P(T,\mu)}{\partial\mu}. \tag{5b}$$

These expressions for the thermodynamic means and variance remain intact in the thermodynamic (i.e., $V \to \infty$) limit. The ratio of the mean and variance, $[B^2]$, of the baryon number,

$$m_0 = \frac{\langle B(T,\mu)\rangle}{[B^2(T,\mu)]} \tag{6}$$

can be predicted from lattice QCD. As there are two thermodynamic parameters in the problem, the prediction of a single quantity is not of great significance. One must go beyond classic thermodynamics to sharpen tests of QCD.

This can be done on any finite volume by studying the higher cumulants of the distribution, since they are nonvanishing in general. With the notation $[B^n]$ for the nth cumulant, we find

$$[B^2] = (VT^3)\frac{\chi^{(2)}(T,\mu)}{T^2},$$

$$[B^3] = (VT^3)\frac{\chi^{(3)}(T,\mu)}{T},$$

$$[B^4] = (VT^3)\chi^{(4)}(T,\mu). \tag{7}$$

Volume-independent combinations are the ratios of cumulants. Experimentally, cumulants up to the fourth order have been measured. We define three possible measurements [6] by taking the ratios

$$m_1 = \frac{[B^3]}{[B^2]} = \frac{\chi^{(3)}(T,\mu)/T}{\chi^{(2)}(T,\mu)/T^2}, \quad m_3 = \frac{[B^4]}{[B^3]} = \frac{\chi^{(4)}(T,\mu)}{\chi^{(3)}(T,\mu)/T}, \tag{8}$$

and $m_2 = m_1 m_3$. Two of the three measurements are independent of each other. We may choose to compare the three measurements m_0, m_1 and m_2 to experimental data. The series expansions for these quantities are obtained by formal manipulation of series expansions for the numerator and denominator with the assumption that the expansion parameter ($z = \mu/T$) is small.

Note that at the critical point the susceptibility χ_B diverges with some critical exponent, ϕ. Then an NLS of order n will behave as $|\mu-\mu^E|^{\phi-n+2}$. As a result m_0 and m_1 will diverge at μ^E as a simple pole $1/|\mu-\mu^E|$, whereas the divergence of m_2 is as a double pole. Also, at small μ both m_0 and m_1 will go to zero linearly in μ, whereas m_2 will go to the constant value $T^2\chi^{(4)}(T)/\chi^{(2)}(T)$. We construct Padé approximants with such behaviour as

$$m_0 = zP_1^L(z), \quad m_1 = z\overline{P}_1^M(z), \quad m_2 = \underline{P}_2^N(z). \tag{9}$$

The unknown parameters in each of these three functions are determined, as usual, by equating them to the series expansion (the bars are meant to signify that the actual parameters entering the three functions could have different values). In the present work we choose $L = M = 1$ and $N = 0$. In future, as more terms are determined, it may be possible to increase the order of these approximants. Note that the leading powers of z (i.e., z^1 in the first two cases and z^0 in the third) are consequences of the CP symmetry that dictates $P(T,\mu) = P(T,-\mu)$, QCD manifests itself in the remaining factors of the Padé approximants, and information about the critical point of QCD are hidden in the pole structure of these functions.

Using the lattice measurements of the Taylor coefficients of χ_B, one may produce a table of values of these variables with varying T/T_c and μ/T. Then by comparing two pieces of experimental data with these tables one may extract T/T_c and μ/T appropriate to each $\sqrt{S_{NN}}$. Alternatively, if one believes that the values of T and μ at chemical freezeout temperatures have already been determined through the analysis of hadron yields, then one

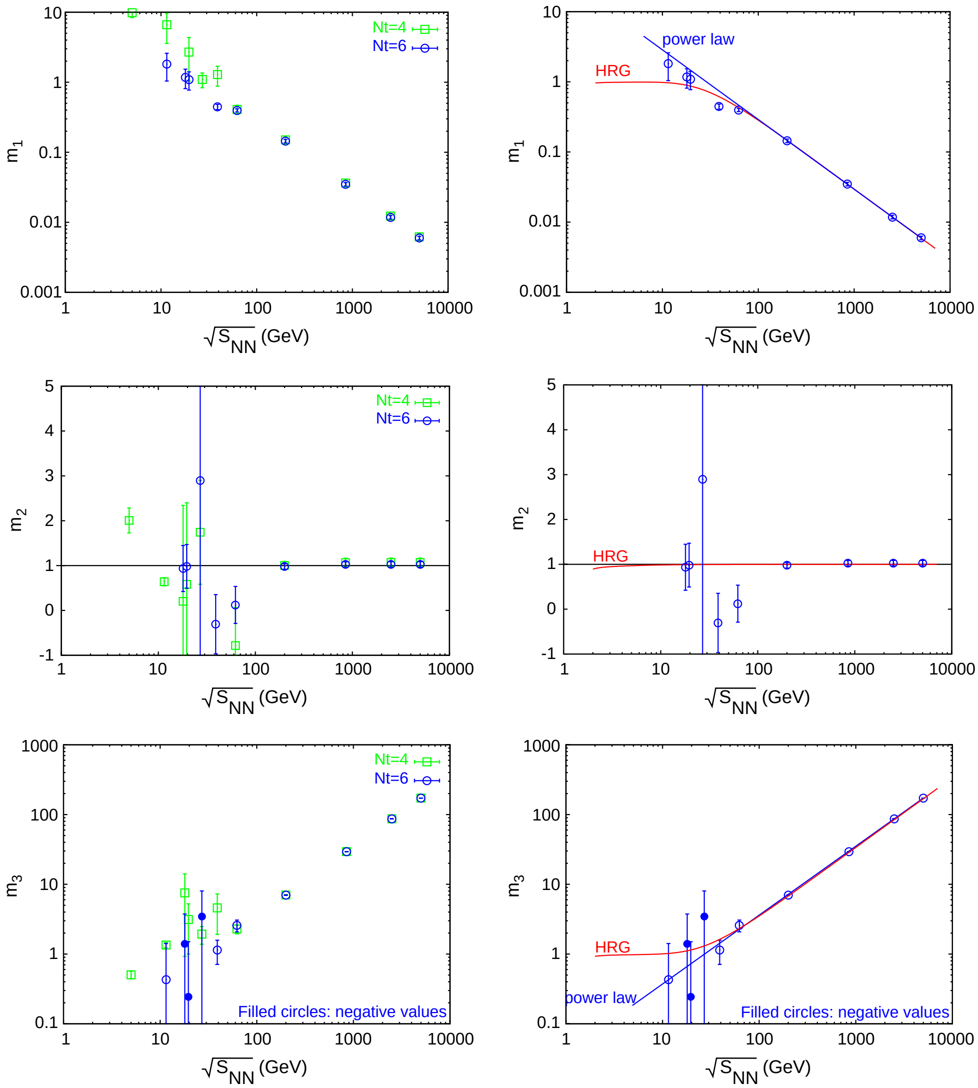

Figure 2. The energy dependence of the ratios m_1, m_2 and m_3, obtained by continuing the lattice measurements at $\mu = 0$ to the freezeout curve using the Padé approximants shown in eq. (9). The left column shows a comparison of measurements with two different lattice cut-offs. The right column compares the lattice results for $N_t = 6$ with a hadron resonance gas model and a power-law fit to the lattice results at large $\sqrt{S_{NN}}$.

may make QCD predictions for these measurements along the freezeout curve. Results from the latter approach [7] are shown in figure 2.

Deviations from a smooth behaviour near the critical point are visible in these extrapolations, although there are large errors. The reason is the following. With a choice of $T_c = 170$ MeV, the closest approach to the critical point (at least on the lattices available) occurs at $\sqrt{S_{NN}} = 18$ GeV, where $T/T_c \simeq 0.94$ and $z \simeq 1.39$. From the estimate of the critical point in eq. (2) and the width in eq. (4), it seems that this point is in the critical region. Now, the lattice results for the series coefficients have errors; although far from a pole they are under control, in the neighbourhood of a pole the errors are naturally magnified. In other words, to obtain accurate predictions close to a critical point, one needs to have very large statistics; this is how critical slowing down shows up in this method.

Interestingly, we see that the effect of a nearby critical point is not very pronounced in m_1. It is much more clearly visible in both m_2 and m_4, i.e., the kurtosis is a very sensitive measure of approach to criticality. With the combined behaviour of the three ratios, one can see that there is evidence, albeit statistically not very significant, that the kurtosis changes sign in the vicinity of the critical point while the skewness does not. It would be interesting to see what class of effective models which capture this leptokurtic behaviour is near the QCD critical point.

It has been argued that finite lifetime [8] and finite size [9] effects move the fireball away from equilibrium if the freezeout point is close to the critical point. In that case, after removing the nonthermal sources of fluctuations, one should expect an agreement between the lattice predictions and data at all $\sqrt{S_{NN}}$ except the ones in the vicinity of the critical point. This could be one strategy to locate the critical point in the experiment.

4. Connecting with experiments

It is impossible to see fluctuations of baryon number in experiments, since detectors are blind to neutrons. It has been proposed to use net proton number as a proxy

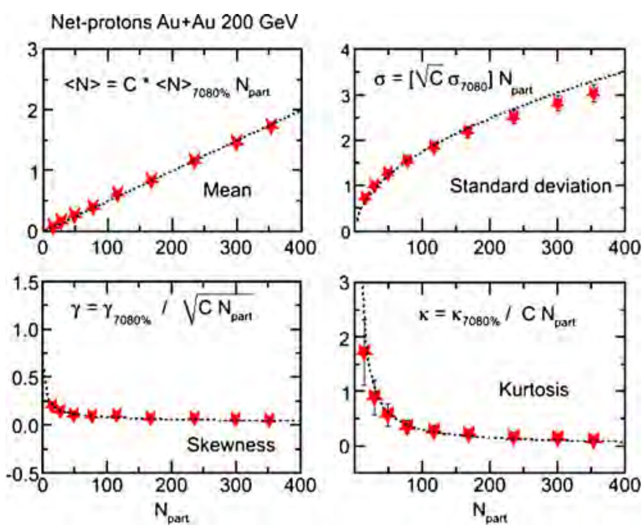

Figure 3. Experimental results [12] for the cumulants of the distribution of net proton number at the highest RHIC energy in Au–Au collisions. N_{part} is a proxy for the volume. The scaling of the cumulants is consistent with the central limit theorem, and hence, possibly, with thermodynamic behaviour.

measurement [10]. Since experiments have limited acceptance, measurements of conserved charges within the acceptance region may be treated in the grand canonical ensemble provided all nonthermal sources of fluctuations are first removed [11].

One year ago the STAR experiment presented the first data on measurements of several cumulants of the distribution of event-to-event fluctuations in the net proton number at the highest beam energy $\sqrt{S_{NN}} = 200$ GeV [12]. They measured the net number, $\langle B\rangle$, the variance, $\sigma^2 = [B^2]$, the skewness, $\mathcal{S} = [B^3]/\sigma^3$ and the kurtosis, $\mathcal{K} = [B^4]/\sigma^4$. All these quantities were measured as a function of the number of participant nucleons, $N_{\rm part}$, obtained using a Glauber picture of the heavy-ion collisions. It is believed that this can be taken as a proxy measure for the volume, V, of the fireball within the acceptance region at the time of freezeout. It was found that $\langle B\rangle \propto N_{\rm part}$, $\sigma \propto \sqrt{N_{\rm part}}$, $\mathcal{S} \propto 1/\sqrt{N_{\rm part}}$ and $\mathcal{K} = 1/N_{\rm part}$, in accord with the central limit theorem. These data are shown in figure 3.

While such scaling could indicate thermal behaviour away from a critical point, it is not the only possible explanation for Gaussian fluctuations. In order to check whether or not a thermal explanation holds, one has to compare with the predictions of QCD. Only if the comparison holds one can begin to have some confidence that the fluctuations are genuinely measuring thermodynamic quantities and not due to some other uncontrolled parameter.

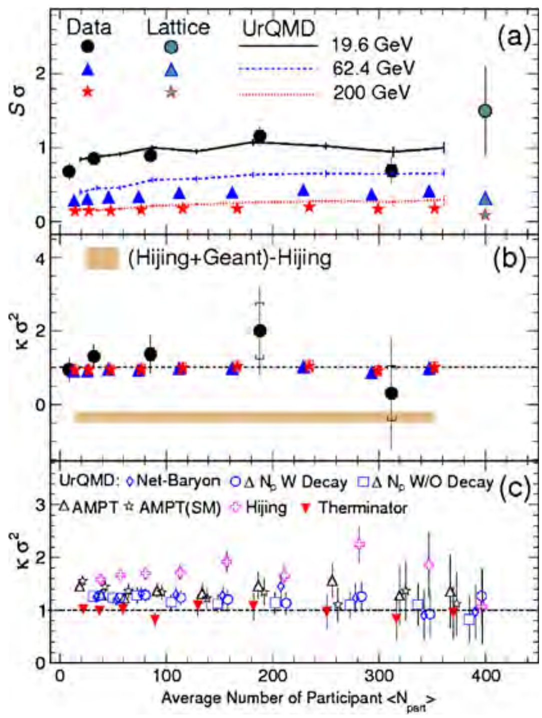

Figure 4. Experimental results [13] for $m_1 = \mathcal{S}\sigma$ compared with the lattice predictions. Also shown is the comparison of data and models for $m_2 = \mathcal{K}\sigma^2$.

The normalized cumulants by themselves are hard to compare with lattice QCD predictions, since they include incidental nonthermal quantities such as V. As discussed already, certain ratios of cumulants can be used to remove such extraneous parameters, rendering the result directly comparable to QCD. In fact, a comparison will tell us whether other possible sources of fluctuations are, or are not, responsible for the Gaussian fluctuations one observes in the limit of large N_{part}.

Such a comparison has been published very recently by the STAR Collaboration [13]. They present measurements of m_1 and m_2 at $\sqrt{S_{NN}} = 200$, 64.2 and 19.4 GeV (see figure 4). Contrary to previous expectations, it turns out that backgrounds are virtually nonexistent, since the data for m_1 agree very well with the predictions from lattice QCD shown in figure 2. Note that this has a strong dependence on $\sqrt{S_{NN}}$, and it would be hard for very different sources of other fluctuations to have precisely the same energy dependence. There also is reasonable agreement with the predictions for m_2.

While this agreement does not (by itself), say anything about the existence or location of the critical point of QCD, it does indicate that one is now able to compare experimental data with QCD predictions. New questions now arise: for example, can one convert this agreement into a measurement of fireball properties? If so, does this agree with alternative measures? Can one now begin to ask questions about the baryon diffusion constant? Does one see supporting evidence in charge and strangeness fluctuations? There are many questions, but it seems that one has entered a new epoch of quantitative thermal lattice QCD phenomenology. If these first questions are answered, a new generation of more accurate lattice computations will be required to pose fresh questions, and address new experimental challenges.

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