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An Introduction To Continuum Non-Critical Strings

Sunil Mukhi

Tata Institute of Fundamental Research
Homi Bhabha Rd, Mumbai 400 005, India

Abstract

String theory with $c \leq 1$ matter is described in the path-integral, conformal-gauge approach. It is shown how to obtain the gravitational scaling dimensions and string susceptibility in this framework. This is followed by a discussion of the operator quantization of the theory, the nature of Liouville theory Fock space, the spectrum of physical states in the BRS framework, and the existence of extra physical states of nontrivial ghost number. Finally, I return to the path-integral to discuss the partition function on the torus and the tree-level correlation functions.

1. 1. Introduction

In 1981, Polyakov[1] wrote down a path integral formalism for bosonic string theory, in which the action depends on a two-dimensional metric in addition to the coordinates of the string, and both sets of fields are quantized independently. He observed that the symmetries of the action (reparametrization invariance on the world sheet, and invariance under Weyl rescalings) are such as to permit the elimination of all the local degrees of freedom contained in the two-dimensional metric, so long as these symmetries can be maintained upon quantization.

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Unfortunately, the Weyl symmetry is generically anomalous in quantization schemes which preserve reparametrization invariance. This means that in general, one of the three modes of the metric, which by convention can be chosen to be the scale factor, cannot be gauged away and needs to be quantized. Polyakov observed that the action for this mode of the metric takes the form of a two-dimensional field theory of a single scalar field with an exponential (Liouville) interaction. The coefficient of this term is proportional to $D - 26$, so that it drops out when the number of spacetime dimensions is 26. In this critical dimension, the anomaly in the Weyl symmetry cancels and the Liouville-like mode decouples, leading to a straightforward quantization of the theory. In dimensions $D < 26$, for which the string theory is said to be non-critical, the anomaly forces us to quantize the Liouville theory along with the matter conformal field theory of D free bosonic fields. This simple observation lies at the heart of the continuum approach to non-critical string theory.

The difficulty in quantizing the Liouville mode led to somewhat limited progress in this field over the next few years. Indeed, in a certain sense this difficulty persists to the present day. In what follows I will discuss the simplest, although possibly least rigorous, approach to the quantization of the noncritical Polyakov string. This discussion, like much of the work that followed it, is based on a sort of self-consistent approach, in which numerical values of critical exponents, and ultimately even correlation functions, are determined from such things as scaling arguments and free-field representations, rather than explicit resolution of the Liouville theory. Since the intent of this article is primarily pedagogical, I will refrain from discussing alternate approaches where a single one suffices to display the main results.

A powerful approach to non-critical string theory, in terms of random-matrix models[2] has been very successful in providing results to all orders in string perturbation theory. In what follows, we will occasionally note the correspondence between the continuum results and those coming from matrix models.

The first part of this article will deal with the functional integral approach (in conformal gauge), in which critical exponents may be derived. The second part deals with the operator quantization a la BRS, which is the most logical and complete framework in which to obtain the spectrum of physical states. Finally, I discuss the partition function on the torus, and the efforts that have been made to extract at least the tree-level correlation functions in this approach, for which the path-integral will again prove to be a convenient starting point.

2. 2. The Polyakov Path Integral

Consider a string propagating in a flat, Euclideanized D-dimensional spacetime. The coordinates of the string are parametrized by maps $X^\mu(\xi_1, \xi_2)$ from a two-dimensional surface to spacetime. Let the surface be compact and of genus h, and denote the metric on this surface by $g_{ab}(\xi_1, \xi_2)$. The basic object of relevance in the first-quantized approach to string theory is the path integral[1]

$$Z_h = \int \frac{\mathcal{D}_g g \mathcal{D}_g X}{\text{vol (Diff)}} \; e^{-S_M - S_c} \tag{1}$$

where

$$\begin{aligned} S_M &= \frac{1}{8\pi} \int d^2\xi \sqrt{g} g^{ab} \; \partial_a X^\mu \partial_b X^\mu \\ S_c &= \frac{\mu_0}{2\pi} \int d^2\xi \sqrt{g} \end{aligned} \tag{2}$$

The first term is the "matter" action, describing D free scalar fields coupled to two-dimensional gravity, while the second is a cosmological term in the two-dimensional sense, whose coefficient is an arbitrary parameter, the bare cosmological constant. The rationale for this action is that the first term alone, on elimination of the metric via its equations of motion, gives rise (classically) to the famous "area law" action of Nambu and Goto, analogous to the classical action for a free relativistic scalar particle, which is proportional to the invariant length of its trajectory. The second term is included to accommodate possible renormalization effects. The measures are formally divided by the volume of the diffeomorphism group, which should cancel out after gauge fixing.

On a surface with two discs removed, Z_h represents the amplitude, in order h, for a free string to propagate from one of the boundaries to the other. We will, however, generally choose the surface to be compact and without boundary. Scattering amplitudes of string states will then arise as correlation functions of physical operators, which can be studied in this formalism once the spectrum has been understood.

To examine Eq.(1) in more detail, it is necessary to give a meaning to the path integration measures. This done by first defining a reparametrization-invariant norm on infinitesimal variations of the fields appearing in the path integral, and then defining the functional integration over the squared norm. The norms are given by

$$\begin{aligned} \|\delta X\|_g^2 &\equiv \int d^2\xi \sqrt{g} \; \delta X^\mu \delta X^\mu \\ \|\delta g\|_g^2 &\equiv \int d^2\xi \sqrt{g} g^{ab} g^{bd} \; \delta g_{ab} \delta g_{cd} \end{aligned} \tag{3}$$

They carry the subscript g because they are metric-dependent. The functional measure is now defined implicitly via

$$\begin{aligned} \int \mathcal{D}_g(\delta X)\ e^{-\|\delta X\|_g^2} &= 1 \\ \int \mathcal{D}_g(\delta g)\ e^{-\|\delta g\|_g^2} &= 1 \end{aligned} \tag{4}$$

Because of the manifest diffeomorphism invariance of the measures, we can be sure that they will descend to measures on the space of metrics and coordinates modulo diffeomorphisms, so that the volume factor in the denominator will indeed cancel out.

Now let us examine the behaviour under the Weyl rescalings

$$g_{ab}(\xi) \to e^{\sigma(\xi)} g_{ab}(\xi) \tag{5}$$

Clearly the matter action S_M is invariant, although the cosmological term S_c is not. As for the measure, it is not invariant either, as is clear from the definitions Eqs.(3),(4)above. The choices for the norms above were dictated by reparametrization invariance, which is evidently incompatible with Weyl invariance.

The Weyl transformation of the bosonic coordinate measure is computable by a variety of methods (see for example Ref.[3]). One finds

$$\mathcal{D}_{e^\sigma g}\ X = e^{(D/48\pi) S_L^0(\sigma,g)} \mathcal{D}_g X \tag{6}$$

where the prefactor appearing is the exponential of a local action for the scale mode $\sigma(\xi)$ coupled to the two-dimensional metric:

$$S_L^0(\sigma,g) = \int d^2\xi \sqrt{g} \left(\frac{1}{2} g^{ab} \partial_a\sigma\partial_b\sigma + R\sigma + \mu e^\sigma \right) \tag{7}$$

Here, μ is an arbitrary constant, which has the effect of renormalising the bare cosmological constant μ_0. The relative coefficients of the first two terms are, however, not arbitrary.

Although we will not derive Eqs.

$$, \tag{8}$$

$$h \tag{9}$$

ere, it is instructive to check the relative coefficient of the first two terms in the above expression. Performing two successive Weyl transformations on the metric, with factors $\sigma(\xi)$ and $\sigma'(\xi)$, we have on the one hand

$$\mathcal{D}_{e^{\sigma+\sigma'} g} X = e^{\frac{D}{48\pi}\ S_L^0(\sigma+\sigma',g)} \mathcal{D}_g X \tag{10}$$

and on the other hand

$$\begin{aligned}\mathcal{D}_{e^{\sigma+\sigma'}g}X &= e^{(D/48\pi)S_L^0(\sigma,e^{\sigma'}g)}\ \mathcal{D}_{e^{\sigma'}g}X \\ &= e^{(D/48\pi)[S_L^0(\sigma,e^{\sigma'}g)+S_L^0(\sigma',g)]}\ \mathcal{D}_g X\end{aligned} \tag{11}$$

This gives the linear relation

$$S_L^0(\sigma+\sigma',g) = S_L^0(\sigma,e^{\sigma'}g) + S_L^0(\sigma',g)$$

Supposing that $\mu = 0$, the above relation is equivalent to

$$\int d^2\xi\sqrt{g}\left(g^{ab}\partial_a\sigma\ \partial_b\sigma' + R(g)\sigma - e^{\sigma'}R(e^{\sigma'}g)\sigma\right) = 0$$

which is true because of the Weyl-transformation law for the Ricci scalar:

$$R\left(e^{\sigma'}g\right) = e^{-\sigma'}\left[R(g) - g^{ab}D_a\partial_b\sigma'\right]$$

This shows that the relative coefficient of the first two terms is not arbitrary, and also that the form of S_L^0 (except the cosmological term, which is arbitrary anyway) is consistent with the required property.

It is appropriate to make one more comment about the Weyl anomaly here. It has been argued[4] that for any classically Weyl-invariant, reparametrization invariant theory in two-dimensions, one can compute the Weyl anomaly as follows. At the classical level, the stress-energy tensor T_{ab} will be conserved, by virtue of reparametrization invariance, and traceless, because of Weyl invariance. This implies that the component T_{zz} in a system of complex coordinates is analytic, from which the deep and elegant structure of two-dimensional conformal field theory can be worked out. In particular, conformal field theories are characterized by the value of the "central charge" in the commutation relations of two stress-energy tensors. Now in a curved background metric, tracelessness is violated at the quantum level by a term proportional to the scalar curvature $R(\xi)$ (which is one way of seeing the Weyl anomaly), which implies that analyticity of T_{zz} is also violated by a similar term. One can then show that the coefficient of the Weyl anomaly is proportional to the central charge of the corresponding flat-space conformal field theory.

Since we will not go into the details here, it is sufficient to write down the result: any conformal field theory of central charge c, when coupled to gravity, has a Weyl anomaly given by

$$e^{(c/48\pi)S_L^0(\sigma,g)} \tag{12}$$

where S_L^0 is the Liouville action written in Eq.(9). This provides the most practical way to determine Weyl anomalies in functional integrals. In the case of D free bosons, for example, one obtains Eq.(8)by noting that each free boson has a central charge $c = 1$.

Armed with the above results, we now proceed to study the full theory of D two-dimensional bosons coupled to gravity. We have seen that the entire theory (action as well as measure) is invariant under reparametrizations, which vary the metric as follows:

$$\delta g^{ab} = g^{ac}\nabla_c V^b + g^{bc}\nabla_c V^a - g^{ab}\nabla_c V^c \tag{13}$$

where $V^a(\xi)$ is the infinitesimal vector field which defines the reparametrization $\delta\xi^a = V^a$ of the coordinates.

We now pick a gauge in order to fix these degrees of freedom. A convenient choice is the conformal gauge, which amounts to fixing the metric to be conformal to some fixed reference metric:

$$g_{ab}(\xi) = e^{\phi^{(0)}(\xi)}\hat{g}_{ab(\tau)}$$

The reference metric is labelled by a set of complex parameters described for the sake of brevity by τ in the above. These parameters describe the moduli space of the chosen Riemann surface, which is the space of conformally inequivalent metrics on this surface.

Next we must replace the integration measure over all metrics by the measure over the scale factor $\phi^{(0)}(\xi)$, the measure over reparametrizations (the infinitesimal vector fields V^a) and the measure on moduli space. The replacement will involve the Jacobian of the transformation in Eq.(13)above. We will assume that the reference metric is just the identity δ_{ab} in the neighbourhood of some chosen point, in which situation it is convenient to go to a system of complex coordinates $z = \xi^1 + i\xi^2$, $\bar{z} = \xi^1 - i\xi^2$. near that point. In these coordinates, the components of the reference metric, and hence the full metric (from the above equation), satisfy $g_{zz} = g_{\bar{z}\bar{z}} = 0$.

Making an infinitesimal change of coordinates near the chosen point, one easily finds that the Jacobian $\mathcal{J}$ between variations of the components $g^{zz}, g^{\bar{z}\bar{z}}$ and the vector field V^z is

$$\mathcal{J} = \det\left(\nabla^z \nabla^{\bar{z}}\right)$$

where the differential operators inside the determinant are just the covariant derivatives on vector fields. Then we can cancel out the formal expression "vol(Diff)" and exponentiate the Jacobian determinant in the standard Faddeev-Popov procedure. Because the operators inside the determinant act on vector fields (which can be thought of as having conformal spin -1, by virtue of their single holomorphic upper index), we must introduce anticommuting ghost fields c^z of spin $j = -1$, and conjugate antighost fields b_{zz} of spin $1 - j = 2$, as well as their antiholomorphic counterparts. The action for these fields in conformal gauge will be

$$S_{gh} \sim \int d^2z \; (b_{zz}\nabla^z c^z + \text{c.c})$$

One can now rewrite this in a general coordinate system and general metric background:

$$S_{gh}(b, c; g) = \frac{1}{4\pi}\int d^2\xi\sqrt{g}\left(b_{ab}\left(\nabla_c c^b g^{ac} + \nabla_c c^a g^{bc} - g^{ab}\nabla_c c^c\right)\right) \tag{14}$$

Evidently the ghost is a vector field c^a, while the antighost is a traceless symmetric tensor b_{ab}.

Now one can check that the ghost action is also Weyl invariant:

$$S_{gh}(b, c, e^{\sigma}g) = S_{gh}(b, c, g)$$

but the measures defined through the norm

$$\begin{aligned}
\|\delta b\|_g^2 &= \int d^2\xi\sqrt{g}\left(g^{ac}g^{bd} + g^{ad}g^{bc} - g^{ab}g^{cd}\right)\; \delta b_{ab}\delta b_{cd} \\
\|\delta c\|_g^2 &= \int d^2\xi\sqrt{g}g_{ab}\; \delta c^a \delta c^b
\end{aligned}$$

are not. In fact, one finds that

$$\mathcal{D}_{e^{\sigma}g}b\; \mathcal{D}_{e^{\sigma}g}c = e^{-(26/48\pi)S_L^0(\sigma;g)}\; \mathcal{D}_g b\; \mathcal{D}_g c \tag{15}$$

The simplest way to derive this is to note that the system of ghosts desribed above, when studied as a conformal field theory in a flat metric, has central charge -26, so that the above result follows from Eq.(12).

Thus after gauge fixing and the introduction of ghosts, the Polyakov path integral becomes

$$\begin{aligned}
Z = \int [d\tau]\; \mathcal{D}_g\phi^{(0)}\; \mathcal{D}_g b\; \mathcal{D}_g c\; \mathcal{D}_g X\; \exp\big[&- S_M(X; g) \\
&- S_{gh}(b, c; g) - \frac{\mu_0}{2\pi}\int \sqrt{g}d^2\xi\big]
\end{aligned}$$

The integration over metrics has been replaced by an integral over moduli and over the scale or Liouville mode $\phi^{(0)}$, the factor "vol(Diff)" has been cancelled and the ghost terms have appeared to take care of the change of variables between metric variations and reparametrizations.

Now the only measure that has not yet been studied is that for the Liouville mode $\phi^{(0)}$. At this point the distinction between critical and noncritical string theory appears. We should first check whether the ghost and matter sectors contain a dependence on $\phi^{(0)}$ or not. We have noted that both the actions are Weyl invariant. The measures are not invariant, as we have seen, so the passage from a general metric to the reference metric will produce the Liouville action $S_L^0(\phi^{(0)}, g)$ defined in Eq.(9), with coefficient $\frac{(D-26)}{48\pi}$ as one can see by combining the anomalous transformation laws given in Eqs.(8) and (15). Therefore if we choose $D = 26$, corresponding to a string propagating in a 26-dimensional flat spacetime, the Weyl anomaly cancels between ghosts and matter coordinates. In this situation we can also set the cosmological constant to zero, and the entire theory becomes independent of the Liouville mode $\phi^{(0)}$. Then we must drop the integration over the Liouville mode, and what remains is the critical string theory.

If on the other hand D is not 26, then the integrand depends on the Liouville field through the local action $S_L^0(\phi^{(0)}, g)$. The coefficient will have the correct sign for a scalar field action if $D < 26$, and we confine ourselves to this case. Now we must indeed address the question of the measure for $\phi^{(0)}$. This should be induced from the norm that we had defined in Eq.(3) for variations of the full metric. It follows that

$$\begin{aligned}\|\delta\phi^{(0)}\|_g^2 &= \int d^2\xi\sqrt{g}\ (\delta\phi^{(0)})^2 \\ &= \int d^2\xi\sqrt{\hat{g}}\ e^{\phi^{(0)}}(\delta\phi^{(0)})^2\end{aligned}$$

This displays the crucial problem which made it difficult to study the noncritical string in the path integral formalism for several years. The measure for the Liouville field depends in a highly nonlinear way on the Liouville field itself, because of the exponential factor in the integrand. This makes it impossible to explicitly perform this part of the functional integration.

What we would like to do would be to transform the entire functional measure for matter, ghosts and the Liouville mode in the background of a general metric (which has the nonlinearity described above) into a measure for some set of fields in the background of the *reference* metric, in which case no such nonlinearity can be present. Accordingly, we make the ansatz[5] that there is such a transformation, and try to determine everything by self-consistency of the resulting theory. It is quite remarkable that such a bold and simple ansatz will lead to unambiguous and useful results.

We postulate that the Liouville field $\phi^{(0)}$ can be replaced by another scalar field ϕ, in terms of which the following equivalence holds:

$$\mathcal{D}_g\phi^{(0)}\ \mathcal{D}_g b\ \mathcal{D}_g c\ \mathcal{D}_g X = e^{-S_L(\phi,\hat{g})}\mathcal{D}_{\hat{g}}\phi\ \mathcal{D}_{\hat{g}}b\ \mathcal{D}_{\hat{g}}c\ \mathcal{D}_{\hat{g}}X$$

where ϕ has the simple norm (free of the exponential factor) given by

$$\|\delta\phi\|^2_{\hat{g}} = \int d^2\xi\sqrt{\hat{g}}\ (\delta\phi)^2.$$

and the prefactor $e^{-S_L(\phi,\hat{g})}$ is some *local, renormalizable* action for the new field $\phi(\xi)$. It is clear that if this ansatz is correct, we will be able to treat the new Liouville field ϕ on essentially the same footing as the string coordinates and the ghosts.

We now assume that the local action $S_L(\phi,\hat{g})$ has the same general form as the Liouville action $S^0_L(\phi^{(0)},g)$ introduced in Eq.(9), but with arbitrary coefficients. These coefficients will then be fixed by self-consistency. Thus we have

$$S_L(\phi,\hat{g}) = \frac{1}{8\pi}\int d^2\xi\sqrt{\hat{g}}\ \left(\frac{1}{\alpha^2}\ \hat{g}^{ab}\partial_a\phi\partial_b\phi + \frac{Q}{\alpha}\ R(\hat{g})\phi + \mu_1 e^{\phi}\right)$$

and it remains to fix the parameters Q and α. The third coefficient μ_1 multiplies the cosmological term, hence it remains arbitrary like the cosmological constants which appeared earlier, and will be a free parameter of the theory.

It is useful to first change the normalization of the Liouville field by the transformation

$$\phi \to \alpha\phi$$

so that the kinetic term takes the canonical form:

$$S_L[\phi,\hat{g}] = \frac{1}{8\pi}\int d^2\xi\sqrt{\hat{g}}\ \left(\hat{g}^{ab}\partial_a\phi\partial_b\phi + Q\ R(\hat{g})\phi + \mu_1 e^{\alpha\phi}\right) \tag{16}$$

Before proceeding further, it is worth commenting that by our ansatz we have replaced the original Liouville field $\phi^{(0)}$ and action $S^0_L(\phi^{(0)},g)$ by a new field ϕ and a new action $S_L(\phi,\hat{g})$, so it will be convenient in what follows to refer to this new field as the Liouville field, and the new action as the Liouville action. The process of going from $\phi^{(0)}$ to ϕ can be thought of as a kind of renormalization of the Liouville field, so that from the point of view of physical interpretation, it is the new field that should be thought of as the scale factor of the metric in the quantum theory. Since a rescaling of ϕ was also performed at the end, this means that

$$g_{ab} = e^{\alpha\phi}\hat{g}_{ab} \tag{17}$$

although admittedly this equation has a rather heuristic meaning.

Now we obtain the conditions coming from the self-consistency assumption. The first observation is that the reference metric $\hat{g}_{ab}$ was arbitrary. Since all the degrees of freedom inherent in the original metric have either been divided out or accommodated in the Liouville mode, the final path integral cannot depend on the choice of $\hat{g}_{ab}$. The way to see this is that a Weyl transformation on the reference metric can be compensated by a shift in the Liouville field, since the transformations

$$\begin{aligned}\hat{g}_{ab}(\xi) &\to e^{\sigma(\xi)}\hat{g}_{ab}(\xi)\\ \phi &\to \phi - \frac{\sigma}{\alpha}\end{aligned}$$

leave the original metric unchanged, from Eq.(17). But since the Liouville field is an integration variable in the path integral, it can be shifted back without changing anything, so the theory as a whole is invariant under Weyl transformations of the reference metric. Of course, since the cosmological constant μ_1 is an arbitrary parameter, we should only require invariance upto changes in μ_1, which is most simply examined by setting $\mu_1 = 0$.

Thus, defining

$$S_{total}(\phi, b, c, X, \hat{g}) = S_L(\phi, \hat{g}) + S_{gh}(b, c, \hat{g}) + S_M(X, \hat{g})$$

and setting the total cosmological term to zero, we have the requirement

$$\mathcal{D}_{e^\sigma\hat{g}}\phi\ \mathcal{D}_{e^\sigma\hat{g}}b\ \mathcal{D}_{e^\sigma\hat{g}}c\ \mathcal{D}_{e^\sigma\hat{g}}X\ e^{-S_{total}(\phi,b,c,X,e^\sigma\hat{g})} = \mathcal{D}_{\hat{g}}\phi\mathcal{D}_{\hat{g}}b\ \mathcal{D}_{\hat{g}}c\ \mathcal{D}_{\hat{g}}X\ e^{-S_{total}(\phi,b,c,X,\hat{g})} \tag{18}$$

This equation has a simple and beautiful meaning: the combined system of matter coordinates, ghosts and the Liouville field, has complete Weyl invariance even including measure factors. One may imagine that coupling the system to two-dimensional gravity has restored the Weyl invariance which was broken by the anomalies in the matter and ghost sectors. This has to happen, since after integrating over metrics there is nothing left to which the Weyl anomaly could be proportional!

This is enough to determine the arbitrary parameter Q, as follows. In absence of the cosmological term, the Liouville action Eq.(16)describes a conformal field theory. The

Weyl anomaly coming from this sector will be proportional to the central charge c_L of this theory, and Eq.(18)above merely implies that the total Weyl anomaly cancels out between string coordinates, ghosts and the Liouville mode, i.e.

$$D - 26 + c_L = 0 \tag{19}$$

Now to find c_L, we compute the analytic stress-energy tensor by varying Eq.(16)(with $\mu_1 = 0$), to get

$$T_{zz}^{(L)} = -\frac{1}{2}\left(\partial_z\phi\partial_z\phi + Q\partial^2\phi\right) \tag{20}$$

By explicitly computing the operator-product expansion for two stress-energy tensors, using Wick contraction for the free bosonic field, one gets

$$c_L = 1 + 3Q^2$$

and solving Eq.(19)above we end up with

$$Q = \sqrt{\frac{25 - D}{3}} \tag{21}$$

where we choose the positive sign of the square root by convention.

Next we determine α. The cosmological term, which is the only term that depends on α, is

$$\mu_1 \int d^2\xi\sqrt{\hat{g}}\; e^{\alpha\phi}$$

Now the vertex operator being integrated should, in the quantum theory, be a conformal primary field of dimension $(1,1)$ since only in that case does its integral over the surface have an invariant meaning. In the Liouville field theory at zero cosmological constant, defined by the stress-energy tensor in Eq.(20), we can compute conformal dimensions of vertex operators using Wick contractions, and one easily finds that the dimension Δ is given in general by

$$\Delta(e^{k\phi}) = -\tfrac{1}{2}k(k - Q)$$

Thus choosing $k = \alpha$ and setting the dimension equal to 1, we find $\alpha(\alpha - Q) + 1 = 0$, from which

$$\begin{aligned}\alpha &= \frac{Q}{2} \pm \tfrac{1}{2}\sqrt{Q^2 - 8} \\ &= \frac{1}{2\sqrt{3}}\left(\sqrt{25 - D} \pm \sqrt{1 - D}\right)\end{aligned} \tag{22}$$

Later we will show that requiring the correct semi-classical limit fixes the sign in the above expression to be negative.

At this point we may note that α is real for $D \leq 1$, imaginary for $D \geq 25$ and *complex* in between. Of course, the way we have defined string theory, the number D is an integer describing the number of dimensions of the flat spacetime in which the string propagates. But we can generalize the notion of string propagation by replacing the matter action S_M for D spacetime coordinates, in Eq.(2), by an *arbitrary* conformal field theory with some central charge c_M. In that case, although the physical interpretation of a string propagating in spacetime is lost, the manipulations above go through with D replaced everywhere by c_M. Now it makes sense to consider any value of c_M, in particular values less than 1.

We conclude that the region $1 < c_M < 25$ is problematic as it leads to complex values for the exponent α in the cosmological term. Actually, we will see shortly that several other quantities, which have some kind of physical interpretation, will also be complex in the same region of c_M. As a result, the present quantization of the Polyakov string may be thought of as inconsistent for these values of c_M.

In what follows we will therefore confine ourselves to the region $c_M < 1$. We could also consider $c_M = 1$, which is on the boundary of the forbidden region, but will not do so here for reasons of space. Many interesting phenomena occur only at the specific value $c_M = 1$, and the interested reader is urged to consult the relevant literature on this subject, or the excellent review by Klebanov[6].

3. 3. Scaling and Critical Exponents

In this section, we show that a good deal of information can be extracted from the Polyakov path integral at $c_M < 1$ just by scaling arguments[5]. It is impressive that this information agrees, in the domain of overlap, with a number of scattered results obtained from numerical simulations on the discretized theory of random surfaces, which is believed to become string theory in the continuum limit[7], and also with a continuum approach to two-dimensional gravity in which a certain $SL(2, R)$ current-algebra symmetry is exploited[8].

We start by recalling the expression for the Polyakov path integral that we will be studying:

$$Z_h(\mu) = \int [d\tau]_h \; \mathcal{D}_{\hat{g}}\phi \; \mathcal{D}_{\hat{g}}b \; \mathcal{D}_{\hat{g}}c \; \mathcal{D}_{\hat{g}}X \; e^{-S_{total}(\phi,b,c,X,\hat{g})}$$

where

$$S_{total}(\phi, b, c, X, \hat{g}) = S_L(\phi, \hat{g}) + S_{gh}(b, c, \hat{g}) + S_M(X, \hat{g})$$

and

$$S_L(\phi, \hat{g}) = \frac{1}{8\pi} \int d^2\xi \sqrt{\hat{g}} \left(\hat{g}^{ab} \partial_a \phi \partial_b \phi + QR(\hat{g})\phi + \mu e^{\alpha\phi} \right)$$

Here, the ghost action is the same as in Eq.(14), and we will assume in what follows that ghost zero-modes have been absorbed by appropriate insertions. For the matter sector we will not postulate any specific action, but will simply assume that it describes a conformal field theory of central charge $c_M < 1$. The symbol X above is therefore just a generic name for the fields that may appear in such an action. The parameters Q and α are fixed (the latter upto a sign) by Eqs.(21) and (22), while μ is arbitrary and the dependence of the path integral Z on it is displayed explicitly. The subscript h on Z displays the fact that it is to be computed each time for a fixed genus h of the two-dimensional surface. $[d\tau]_h$ denotes the integration measure over the moduli space of this surface.

We now start by examining the physical role played by μ and computing the μ-dependence of Z. Suppose first that the Liouville field ϕ takes a positive constant value ϕ_0. Then, examining only the second term in the Liouville action, we find

$$\begin{aligned} \frac{1}{8\pi} Q \int d^2\xi \sqrt{\hat{g}}\; R(\hat{g})\phi \to Q\phi_0 \left(\frac{1}{8\pi} \int d^2\xi \sqrt{\hat{g}} R(\hat{g}) \right) \\ = Q\phi_0(1-h) \end{aligned}$$

where h is the genus of the surface. The second step involves the standard relation which states that the scalar curvature integrated over a compact two-dimensional manifold is proportional to its Euler characteristic $\chi = 2 - 2h$. Now when $\phi_0 \to \infty$, this term tends to $-\infty$ if the genus is greater than 1, and e^{-S_L} diverges exponentially. However, the presence of a cosmological term saves us from disaster, since α is always positive and $\exp(-\mu e^{\alpha\phi} \int d^2\xi \sqrt{\hat{g}})$ tends to zero even more rapidly. Thus a (positive) cosmological constant is needed to stabilize the action, and we can expect singular behaviour when $\mu \to 0$. On the other hand, if the genus is 0, then the Q-term causes the integrand to diverge as $\phi_0 \to -\infty$, and in this case the cosmological term does not help to stabilize the theory. So we should be particularly careful about the genus-0 contribution, which might turn out to give unphysical results in formal manipulations.

We can easily extract the μ-dependence of $Z_h(\mu)$. Let us shift the Liouville field by a constant:

$$\phi \to \phi - \frac{1}{\alpha}\, \ln \mu$$

Then,

$$\mu e^{\alpha\phi} \to e^{\alpha\phi}$$

so that this shift has the effect of setting $\mu = 1$. The measure is of course invariant under this shift, and so is the kinetic term, but the Q-term changes, so the total action changes as follows:

$$\begin{aligned} S_{total} &\to S_{total} + \frac{Q}{8\pi}\int d^2\xi\sqrt{\hat{g}}\; R(\hat{g})\left(-\frac{1}{\alpha}\ln\mu\right) \\ &= S - \frac{Q}{\alpha}\ln\mu(1-h) \end{aligned}$$

So in the path integral the change is

$$\begin{aligned} e^{-S_{total}} &\to e^{-S_{total} + \frac{Q}{\alpha}\;(\ln\mu)(1-h)} \\ &= \mu^{\left(\frac{Q}{\alpha}\right)(1-h)}\; e^{-S_{total}} \end{aligned}$$

and we conclude that for any nonzero μ,

$$Z_h(\mu) = \mu^{\left(\frac{Q}{\alpha}\right)(1-h)}\; Z_h(\mu = 1) \tag{23}$$

This exhibits the fact that for genus $h > 1$, the path integral blows up for $\mu \to 0$, as we predicted above, but it also shows that the behaviour near $\mu = 0$ is singular in the sense that there is a branch cut, even in genus 0.

Recall that the cosmological term $\int d^2\xi\sqrt{\hat{g}}\exp(\alpha\phi)$ is actually the area of the surface, $\int d^2\xi\sqrt{g}$ in the original metric. Thus the cosmological constant μ, which multiplies it, is "dual" to the area, in the same sense as the chemical potential to the number of particles in a statistical system. Thus the expectation value of the area in a given genus is

$$\begin{aligned} \langle A\rangle &\equiv \frac{1}{Z_h(\mu)}\int \mathcal{D}_{\hat{g}}\phi\;\mathcal{D}_{\hat{g}}(X,b,c)\left(\int d^2\xi\sqrt{\hat{g}}e^{\alpha\phi}\right)\; e^{-S_{total}} \\ &= -\frac{\partial}{\partial\mu}\ln Z_h(\mu) \\ &= -\frac{Q}{\alpha}(1-h)\frac{1}{\mu}Z_h(\mu) \end{aligned}$$

This is singular near $\mu = 0$ just like the partition function itself. But note that for positive μ, the average area is positive only for $h > 1$, while on the sphere ($h = 0$) it is *negative.* This is a consequence of the singular behaviour of the action for large negative ϕ, which was noted earlier.

A more physically sensible quantity is the partition function at fixed area. We define this by inserting a delta-function in the path integral:

$$Z_h(A) = \int \mathcal{D}_{\hat{g}}\phi\ \mathcal{D}_{\hat{g}}(X,b,c)\delta\left(\int d^2\xi\sqrt{\hat{g}}e^{\alpha\phi} - A\right) e^{-S_{total}(\phi,X,b,c;\mu=0)} \tag{24}$$

where now we have set $\mu = 0$, since it effectively multiplies a constant and hence factors out of the path integral. One can also think of the fixed-area partition function as the Laplace transform in μ of the original partition function.

We can extract the area-dependence again by a scaling argument on the partition function. Even simpler is to extract the dependence from Eq.(23)by dimensional arguments, noting that a formal Laplace transform will produce the result

$$\begin{aligned} Z_h(A) &\sim K\ A^{\frac{Q}{\alpha}(h-1)-1} \\ &\sim K\ A^{\Gamma(h)-3} \end{aligned} \tag{25}$$

where K is some constant, and $\Gamma(h)$, called the string susceptibility, is given by

$$\begin{aligned} \Gamma(h) &= \frac{Q}{\alpha}(h-1) + 2 \\ &= \frac{1}{12}(h-1)\left(25 - c_M \mp \sqrt{(25-c_M)(1-c_M)}\right) + 2 \end{aligned}$$

Here we have used the expressions for Q and α derived earlier.

This expression can be compared with results obtained in specific situations via other approaches. In particular, the susceptibility has been computed in the semiclassical limit $c_M \to -\infty$, [9], giving

$$\Gamma(h) \to (1-h)\frac{(c_M - 19)}{6} + 2$$

which determines the sign in the above expression to be the lower one, which translates into the choice of the minus sign in Eq.(22). We will nevertheless find it convenient to keep both signs in some of the expressions that follow.

Finally, we can consider the gravitational scaling behaviour for non-trivial operators of the theory. To be systematic, we should first extract the full operator content of the theory, but we postpone this task to the next section. Here we simply make an ansatz for a class of physical operators, which will be justified subsequently in the BRS formalism. Suppose Ψ_M is some primary field of the matter theory. We assume that gravity "dresses" this by multiplying with a Liouville vertex operator, such that the result is a (1,1) primary field of the combined theory, which is then integrated over the entire surface. This assumption is in accord with what we know about critical string theory, where most of the physical operators can indeed be described as integrated $(1,1)$ primaries of the matter sector. Thus we have

$$\Psi_M(\xi) \to \int d^2\xi \sqrt{\hat{g}}\; e^{k_L \phi} \Psi_M$$

where the number k_L is to be determined. If Ψ_M has conformal dimension Δ_M, then the condition that the total dimension is 1 gives

$$\Delta_M - \frac{1}{2}\, k_L(k_L - Q) = 1$$

so that

$$k_L^{\pm} = \frac{1}{2\sqrt{3}} \left(\sqrt{25 - c_M} \pm \sqrt{1 - c_M + 24\Delta_M} \right)$$

Note that the two solutions $k_L^{\pm}$ satisfy $k_L^+ + k_L^- = Q$, so we adopt the convention that $k_L^- < \frac{Q}{2}$ and $k_L^+ > \frac{Q}{2}$. The semiclassical limit again imposes a particular choice of sign, namely the one corresponding to k_L^-. We will return to this point in the subsequent analysis.

The path integral with n insertions of these physical operators, which gives the string theory n-point amplitudes in genus h (without a fixed-area constraint, but with a cosmological constant) is written

$$\begin{aligned} &\langle \int \Psi_M^{(1)} e^{k_l^{(1)}\phi} \ldots \int \Psi_M^{(n)} e^{k_L^{(n)}\phi} \rangle_{h,\mu} \\ &\equiv \int \mathcal{D}_{\hat{g}}\phi\; \mathcal{D}_{\hat{g}}(X,b,c) \int \Psi_M^{(1)} e^{k_l^{(1)}\phi} \ldots \int \Psi_M^{(n)} e^{k_L^{(n)}\phi}\; e^{-S_{total}(\phi,b,c,X)} \end{aligned}$$

We can again examine when the potential is stable. A similar analysis as was carried out for the vacuum amplitude tells us that if

$$\sum k_i - Q(1-h) > 0$$

the cosmological term again succeeds in stabilizing the theory. If this condition is not satisfied then again we must constrain the area to get a sensible result.

Let us define the "gravitational scaling dimension" of the physical operator by examining the area-dependence of the *normalized* one-point functions of this operator at fixed area. Define

$$\begin{aligned} &\langle \int \Psi_M e^{k_L\phi} \rangle_A \\ &\equiv \frac{1}{Z_h(A)} \int \mathcal{D}_{\hat{g}}\phi\; \mathcal{D}_{\hat{g}}(X,b,c) \int \Psi_M e^{k_L\phi} \delta\left(\int d^2\xi \sqrt{\hat{g}} e^{\alpha\phi} - A \right) e^{-S_{total}(\phi,b,c,X;\mu=0)} \end{aligned}$$

The effect of dividing out by the vacuum amplitude is to remove the genus-dependence of this expression when we examine the scaling with area. Thus we will get a scaling behaviour which describes a purely local property of the physical operator. Using the now-familiar procedure of shifting the Liouville field by a constant, one can extract the result

$$\langle \int \Psi_M e^{k_L \phi} \rangle_A \sim K \; A^{1-\delta}$$

with

$$\begin{aligned} \delta^{\pm} &= 1 - \frac{k_L^{\pm}}{\alpha} \\ &= \frac{\sqrt{1 - c_M + 24\Delta_M} \pm \sqrt{1 - c_M}}{\sqrt{25 - c_M} - \sqrt{1 - c_M}} \end{aligned}$$

This equation also agrees perfectly with the result derived in Ref.[8] in the $SL(2, R)$ approach.

Let us note at this point that the various formulae derived above for the susceptibility and for various scaling exponents are not quite correct for a general matter conformal field theory background. In the case when this background is a non-unitary theory, certain modifications are required. The reason for this is that we had made an implicit assumption that the cosmological term is a pure Liouville vertex operator, which can be thought of as the "dressed" identity field of the matter theory. In a unitary matter theory, the identity is the operator of minimum conformal dimension, but if the matter theory is non-unitary then it will generally have operators of negative scaling dimension. The dominant perturbation of the theory will then be by the dressed version of the operator among these whose dimension is most negative. Let us call this operator $\Psi_M^{(min)}$, and denote its dimension by $\Delta^{(min)}$. It is the dressed version of this operator which we will *define* as the area operator for non-unitary matter theories, and which will appear in the action multiplied by the cosmological constant. This physically motivated assumption is supported by other calculations in different approaches.

Thus the cosmological term in the action is replaced by

$$\int d^2\xi \sqrt{\hat{g}} \; \Psi_M^{(min)} e^{\alpha\phi}$$

so that repeating the earlier arguments leads to

$$
\begin{aligned}
\Gamma &= \frac{2(h-1)\sqrt{25-c_M}}{\sqrt{25-c_M}-\sqrt{1-c_M+24\Delta_{\min}}} \\
\alpha &= \frac{1}{2=\sqrt{3}}\ \left(\sqrt{25-c}-\sqrt{1-c+24\Delta^{(min)}}\right) \\
\delta^{\pm} &= 1-\frac{k_L^{\pm}}{\alpha} \\
&= \frac{\sqrt{1-c_M+24\Delta_M}\pm\sqrt{1-c_M+24\Delta^{(min)}}}{\sqrt{25-c_M}-\sqrt{1-c_M+24\Delta^{(min)}}}
\end{aligned}
\tag{26}
$$

In particular, the matter CFT, for $c_M < 1$, is most naturally chosen to be a member of the minimal series obtained by Belavin, Polyakov and Zamolodchikov, for which the possible values are

$$
c_M = 1 - \frac{6(p-q)^2}{pq}
$$

The spectrum of conformal dimensions of primary fields in these theories is given by the Kac formula

$$
\Delta_{rs} = \frac{(ps-qr)^2-(p-q)^2}{4pq} \tag{27}
$$

from which one obtains the most negative dimension

$$
\Delta^{(min)} = \frac{1-(p-q)^2}{4pq}
$$

Inserting these in the expressions Eq.(26)one finds

$$
\begin{aligned}
\Gamma(h) &= \frac{2h(p+q)-2}{p+q-1} \\
\alpha &= \frac{1}{\sqrt{2}}\ \frac{p+q-1}{\sqrt{pq}} \\
\delta_{(r,s)} &= \frac{|ps-qr|-1}{p+q-1}
\end{aligned}
$$

This completes the formulation of the noncritical string as a path integral. We have extracted various scaling dimensions by formal manipulations on this path integral, without actually trying to evaluate it explicitly. In the next section, we will change over to the operator formalism for this theory and describe the physical states and operators of the theory.

4. 4. Spectrum of Physical States

We now turn to the operator formalism for the combined theory of minimal conformal matter, the Liouville field and ghosts, which was formulated above in the functional integral language.

From the above discussion, we conclude that the matter sector is described by a minimal conformal field theory with some analytic stress-energy tensor $T_M(z)$ with central charge and conformal dimensions given by Eqs. and (27). The Liouville theory, at zero cosmological constant, is described by the conformal field theory of a single free boson field ϕ with stress-energy tensor and central charge

$$T_L(\phi) = -\frac{1}{2}(\partial\phi_L\partial\phi_L + Q\partial^2\phi_L)$$
$$c_L = 26 - c_M = 1 + 3Q^2$$

Vertex operators $\exp(k_L\phi)$ in this theory have conformal dimensions

$$\Delta_L = -\frac{1}{2}\, k_L(k_L - Q)$$

These vertex operators create Liouville momentum states from the vacuum by

$$|k_L\rangle =: e^{k_L\phi} : (0)|0\rangle$$

Decomposing the Liouville field into its oscillator modes via

$$\partial\phi(z) = \sum_{n\epsilon \mathbb{Z}} \alpha_n^{(L)} z^{-n-1}$$

we find that the momentum states are eigenstates of $\alpha_0^{(L)}$:

$$\alpha_0^{(L)}|k_L\rangle = -ik_L|k_L\rangle$$

and the conformal dimension (L_0-eigenvalue) of these states is

$$\Delta_{k_L} = -\tfrac{1}{2}k_L(k_L - Q)$$

Thus for each dimension, there are two possible momenta, a fact which we have encountered earlier. Because of the background charge (Q-term) in the Liouville stress-energy, the inner product between momentum states is given by

$$\langle k_L|k'_L\rangle = \delta(k_L + k'_L - Q) \tag{28}$$

Finally, the ghosts are described by a CFT of anticommuting fields (b,c) of spins $(2,-1)$ respectively, with a stress-energy tensor T_{gh} and central charge $c_{gh} = -26$. We decompose the ghost fields into their modes by

$$
\begin{aligned}
c(z) &= \sum_{n \epsilon \mathbb{Z}} c_n z^{1-n} \\
b(z) &= \sum_{n \epsilon \mathbb{Z}} b_n z^{-n-2}
\end{aligned}
$$

Denoting by $|0\rangle$ the $SL(2,C)$-invariant ghost vacuum, we find that it must be annihilated by a semi-infinite subset of ghost modes as follows:

$$
\begin{aligned}
c_m|0\rangle &= 0, \quad m \geq 2 \\
b_m|0\rangle &= 0, \quad m \geq -1
\end{aligned}
$$

As a consequence, the true ghost vacuum has L_0-eigenvalue -1, and is given by $c_1|0>$.

We should keep in mind that from the outset, the Liouville mode and the ghosts are described by the bosonic and fermionic Fock spaces described above, but the matter theory can be described completely by its irreducible module under the Virasoro algebra, as in Ref.[10].

Thus the full Hilbert space of the theory may be denoted

$$
\mathcal{H} = \mathcal{H}_L \otimes \mathcal{H}_M \otimes \mathcal{H}_{gh}
$$

which is the tensor product of the individual Liouville, matter and ghost Hilbert spaces. To construct the physical states, we will follow the BRS procedure. Define the operator

$$
\begin{aligned}
Q_{BRS} &= \oint : c(z)\left(T_M(z) + T_L(z)\right) + \frac{1}{2}\oint : c(z) T_{gh}(z) : \; + \; \text{antiholomorphic part} \\
&= \sum_{n \epsilon \mathbb{Z}} c_{-n}\left(L_n^{(M)} + L_n^{(L)}\right) - \frac{1}{2}\sum_{m,n \epsilon \mathbb{Z}} (m-n) : c_{-m} c_{-n} b_{m+n} : \; + \; \text{c.c}
\end{aligned} \tag{29}
$$

where

$$
L_n^{(L)} = \frac{1}{2}\sum_{m=-\infty}^{+\infty} : \alpha_{n-m}^{(L)} \alpha_m^{(L)} : + \frac{iQ}{2}\,(n+1)\alpha_n^{(L)}
$$

One can check by explicit calculation that the BRS charge Q_{BRS} is nilpotent: $Q_{\text{BRS}}^2 = 0$. Now in this framework, the physical states of the theory are defined as the cohomology of this operator on the full Hilbert space. This is the space of states annihilated by the operator (the kernel) modulo those created by the action of the operator on other states (the image). One can write this as

$$\{|\Phi\rangle \ : \ Q_{\rm BRS}|\Phi\rangle = 0 / |\Phi\rangle \sim |\Phi\rangle + Q_{\rm BRS}|\Lambda\rangle\}$$

Thus, physical states are annihilated by $Q_{\rm BRS}$ and are not produced by acting with $Q_{\rm BRS}$ on anything else. States not satisfying both these conditions are not in the spectrum: they can be classified into two types, those which are not annihilated by $Q_{\rm BRS}$ (which we will refer to as "unphysical") and those which are produced by acting with $Q_{\rm BRS}$ on some other state (which we will refer to as "pure gauge").

Although $Q_{\rm BRS}$ is actually the sum of holomorphic and antiholomorphic components, it turns out that one can extract the full cohomology from the cohmology of the purely chiral BRS operator, corresponding to the first, or holomorphic, term in Eq.(29). Hence in what follows we will study only the chiral cohomology, and we will use the notation $Q_{\rm BRS}$ to mean only the holomorphic part.

Classifying physical states amounts to finding this cohomology. Before starting to examine this problem, let us note the following. The anticommutator of $Q_{\rm BRS}$ with the b-ghost zero mode b_0 is

$$\{Q_{\rm BRS}, b_0\} = L_0^{\rm tot} \equiv L_0^{(M)} + L_0^{(L)} + L_0^{(G)}$$

Suppose we have a state $|\Phi\rangle$ such that $Q_{\rm BRS}|\Phi\rangle = 0$, and $L_0^{\rm tot}|\Phi\rangle = \Delta|\Phi\rangle$ with $\Delta \neq 0$. Then one can immediately show that $|\Phi\rangle$ is a pure gauge state:

$$Q_{\rm BRS}\ b_0|\Phi\rangle + b_0\ Q_{\rm BRS}|\Phi\rangle = \Delta|\Phi\rangle$$

implies

$$|\Phi\rangle = Q_{\rm BRS}\left(\frac{1}{\Delta}\ b_0|\Phi\rangle\right)$$

Thus, we can restrict to states with $L_0^{\rm tot}|\Phi\rangle = 0$ in studying the cohomology.

Next, let us consider an arbitrary state in the cohomology and expand it in c_0:

$$|\Phi\rangle = |\Phi_1\rangle + c_0|\Phi_2\rangle$$

Since $|\Phi\rangle$ is annihilated by $Q_{\rm BRS}$ we have

$$Q_{\rm BRS}|\Phi_1\rangle + Q_{\rm BRS}(c_0|\Phi_2\rangle) = 0$$

Now let us similarly decompose $Q_{\rm BRS}$ in terms of its c_0 content:

$$Q_{\rm BRS} = c_0 L_0^{\rm tot} + \tilde{Q}_{\rm BRS}$$

Since we have already decided that $L_0^{\rm tot}$ must annihilate $|\Phi\rangle$, we find

$$\tilde{Q}_{\rm BRS}|\Phi_1\rangle + \tilde{Q}_{\rm BRS} c_0 |\Phi_2\rangle = 0$$

from which we find the two equations

$$\begin{aligned} \tilde{Q}_{\rm BRS}|\Phi_1\rangle + \kappa|\Phi_2\rangle &= 0 \\ \tilde{Q}_{\rm BRS}|\Phi_2\rangle &= 0 \end{aligned} \tag{30}$$

To derive this, we have commuted $\tilde{Q}_{\rm BRS}$ through c_0 and used the relation

$$\{Q_{\rm BRS}, c_0\} = \{\tilde{Q}_{\rm BRS}, c_0\} = \kappa \equiv \sum_{n=1}^{\infty} n c_{-n} c_n$$

which defines the operator κ. The two equations then follow from separately equating to zero the terms proportional to and independent of c_0. One can rewrite Eq.(30)after dropping the tilde on $Q_{\rm BRS}$, since again the extra term is just proportional to $L_0^{\rm tot}$ which annihilates the states. Thus, the components of $|\Phi\rangle$ separately satisfy

$$\begin{aligned} Q_{\rm BRS}|\Phi_1\rangle + \kappa|\Phi_2\rangle &= 0 \\ Q_{\rm BRS}|\Phi_2\rangle &= 0 \end{aligned}$$

In a similar fashion, one can examine the condition for $|\Phi\rangle$ to be a pure gauge state, in terms of its components, and one finds

$$\begin{aligned} |\Phi_1\rangle &= Q_{\rm BRS}|\Lambda_1\rangle - \kappa|\Lambda_2\rangle \\ |\Phi_2\rangle &= Q_{\rm BRS}|\Lambda_2\rangle \end{aligned}$$

We see that in some sense the role of c_0 is to cause a "doubling" of all the considerations involved in studying the cohmology of $Q_{\rm BRS}$. Thus it makes sense to restrict one's attention to states $|\Phi\rangle$ which have $|\Phi_2\rangle = 0$, which is the same as choosing the subspace

$$\{|\Phi\rangle \ : \ b_0|\Phi\rangle = 0\}$$

We will impose this condition on the Hilbert space in what follows. The cohomology of $Q_{\rm BRS}$ within this subspace is known as the "relative cohomology".

Now that we have established all the necessary notation to study the BRS cohomology of noncritical string theory, we can proceed with the analysis. The first step is to analyse the Liouville Fock space, whose unusual properties turn out to be responsible for the special characteristics of noncritical string theory.

In CFT we often hear the statement that all states in the Hilbert space are either primary or secondary. If they are both, we call them null vectors since they are orthogonal to both primaries and secondaries. In the Liouville Fock space this property does not hold. Let us start with an example. Consider the two states $\alpha_{-1}^{(L)}|k_L = 0\rangle$, $\alpha_{-1}^{(L)}|k_L = Q\rangle$. These have the same L_0 eigenvalue. Now let us ask if these are primary or secondary. It is a simple exercise to check that the first state is not primary (since it is not annihilated by L_1) but it is also not secondary (it cannot be created by L_{-1}, since this operator annihilates the vacuum). On the other hand, the second state is primary, since it is annihilated by L_1, and also secondary, since it is created by L_{-1} from the state $|k_L = Q\rangle$. It is convenient to divide the Liouville Fock space $\mathcal{H}_L$ into two components: the sector with momenta $k_L < \frac{Q}{2}$, which we call ${\mathcal{H}_L}^-$, and the one with momenta $k_L > \frac{Q}{2}$, which we call ${\mathcal{H}_L}^+$. Then the two states above are respectively in ${\mathcal{H}_L}^-$ and ${\mathcal{H}_L}^+$. This simple example is a special case of the following result:

Theorem:

(i) States in ${\mathcal{H}_L}^+$ are either primary or secondary or both.

(ii) States in ${\mathcal{H}_L}^-$ are either primary or secondary or neither.

We will examine this theorem below. More details, including the proof, can be found in Refs.[11],[12],[13]. One important point to note is that if we look first at the Verma module for the Liouville theory, then the projection from this to the Fock module has the following property: in ${\mathcal{H}_L}^+$, null states in the Verma module descend to non-vanishing states in the Fock module. Thus the projection to the Fock module is bijective. In ${\mathcal{H}_L}^-$, on the other hand, there are no null states in the Fock module, so the projection loses these null states (they vanish in the Fock space). At the same time, there are non-primary, non-secondary states in the Fock module, which cannot come from the Verma module, so that the projection in this sector is neither one-to-one nor onto. In fact, these results are the ingredients which go into the proof of the above theorem.

Let us parametrise the matter and Liouville central charges as follows:

$$c_M = 13 - \frac{6}{t} - 6t$$
$$c_L = 13 + \frac{6}{t} + 6t$$

Then the case of interest to us, $c_M < 1$, corresponds to $t > 0$. Now the Kac formula tells us that a primary state in the Liouville theory of conformal dimension

$$\Delta_L = \frac{(1-rs)}{2} - \frac{(r^2-1)}{4}\frac{1}{t} - \frac{(s^2-1)}{4}t$$

has a null vector at level rs above it. Suppose we look at Liouville momentum states of this dimension:

$$-\frac{1}{2}k_L(k_L - Q) = \Delta_L$$

Then we get two null vectors in Fock space, one over $k_L^+ > Q/2$, the other over $k_L^- < Q/2$. Now the first step is to prove that the projection from the Verma module to Fock space is bijective in ${\mathcal{H}_L}^+$, while in ${\mathcal{H}_L}^-$ it has a kernel corresponding to all the null vectors. This can be proved from a detailed construction of the null states in the Verma module, although we will not enter into those details here.

We can now see how this result implies the theorem above, by a state-counting argument. The Verma module has states above a primary of given dimension Δ, given by

$$L^{(L)}_{-p_1}L^{(L)}_{-p_2}\cdots L^{(L)}_{-p_n}|\Delta\rangle.$$

Similarly, the Fock module has states above a primary of given momentum k_L, given by

$$\alpha^{(L)}_{-p_1}\alpha^{(L)}_{-p_2}\cdots\alpha^{(L)}_{-p_n}|k_c\rangle.$$

The number of states at a given level is the same in both, since it is given by the number of partitions of the level. However, the Verma module may have states which vanish on reduction to the Fock space ${\mathcal{H}_L}^-$ (which is possible only if they are both primary and secondary, i.e. null). In that case, a new state must appear in the Fock space, which does not come from the Verma module. But one can also predict the existence of such a state from the fact that the Fock space is manifestly positive definite, and it has states in ${\mathcal{H}_L}^+$ which are both primary and secondary. This means that there must be states in the dual space ${\mathcal{H}_L}^-$ with which the original state has a nonzero inner product, in the norm induced by Eq.(28). Thus these new states in ${\mathcal{H}_L}^-$ must be neither primary nor secondary.

Now we can begin to classify physical states. The first thing to note is that all states in the full Hilbert space $\mathcal{H}$ which are of the form

$$|\text{primary}\rangle_{L,M} \otimes c_1|0\rangle_{gh}$$

are necessarily in the relative cohomology. Here we have simply taken an arbitrary primary of the matter-Liouville system of dimension 1, so that after adding the dimension of the true ghost vacuum, one finds total conformal dimension 0 as required. The proof of this result is straightforward: it is easy to check that such states are always annihilated by Q_{BRS}, and one can then show that there is no state on which the action of Q_{BRS} produces this state. This result is analogous to a well-known result in critical string theory. Thus, classifying the $(1,1)$ primaries of the matter-Liouville system will produce a class of physical states, although as we will see shortly, these are by no means all or even most of the physical states in the theory.

It has been shown, in Ref.[14], that given the tensor product of two conformal field theories with total central charge 26, a dimension $(1,1)$ primary of the combined theory must be a primary of each theory separately. However, a crucial ingredient in this proof is the assumption that all states in each CFT are either primary or secondary. If we now seek to apply this result to the Liouville-matter system, then this assumption holds only in the sector ${\mathcal{H}_L}^+$, as we have just seen. Thus in this sector, a $(1,1)$ primary of the combined matter- Liouville sector must be a primary of both matter and Liouville sectors separately. Moreover, our analysis above also implies that a (non-null) primary of Liouville in ${\mathcal{H}_L}^+$ can only be a pure momentum state, since a new primary in the Fock module above a momentum state would mean that the projection from the Verma module is not onto. As a result, all states of the form

$$|\mathrm{primary}\rangle_M \otimes |k_L^+\rangle_L \otimes c_1|0\rangle_{gh} \tag{31}$$

are in the relative cohomology. One can check that they are in the absolute cohomology as well. It is important to note that the lengthy discussion above was necessary to derive this first simple result, as otherwise we could not rule out two kinds of states: those which are primary in the combined matter-Liouville theory but not primary in one or the other, and those which are primary in both sectors but not pure vertex operator states in the Liouville sector. Moreover, all this holds only in the sector ${\mathcal{H}_L}^+$ so far, but we may now note that replacing k_L^+ by k_L^- in the above expression produces a state in ${\mathcal{H}_L}^-$ which is still annihilated by Q_{BRS}, and which has nonzero inner product with the above state. This means that all states of the form of Eq.(31)with k_L^+ replaced by k_L^- are also physical according to the BRS definition.

At this point we have discovered precisely the "dressed" matter primaries which were introduced as an ansatz in the previous section on functional integral techniques. We noted earlier that the semi-classical limit suggests a restriction to k_L^- only, but we worked with both values of k_L precisely because the BRS procedure finds both of them to give physical states. A physical distinction between operators dressed by k_L^+ and k_L^- is described in Ref.[15].

The final result seems intuitively appealing: a matter CFT coupled to gravity has a set of physical states in one-to-one correspondence with the original matter primaries, and they can be thought of as having been dressed by gravitational interactions.

Unfortunately, this intuitive result is misleading. We have only examined a special class of physical states, those of the form of Eq.(31)and their counterparts in ${\mathcal{H}_L}^-$. All these states are built on the true ghost vacuum, $c_1|0\rangle_{gh}$. But the cohomology analysis should be performed on all possible states, including in particular, states with ghost excitations (excluding of course c_0, since we are studying the relative cohomology).

The simplest indication that there are physical states in sectors of nontrivial ghost number with respect to the true ghost vacuum is the following. Let us assign ghost number +1 to the c-ghost, -1 to the b-antighost, and 0 to the true ghost vacuum $c_1|0\rangle_{gh}$. Now consider the state

$$|0\rangle_M \otimes |0\rangle_L \otimes |0\rangle_{gh}$$

corresponding to the $SL(2,C)$ invariant vacuum state. In our conventions this has ghost number -1, and it is easily verified that it is a physical state. Of course, an analogous state exists in the critical string, and seems to play no essential role there, so one could dismiss this example as not being very important. This might have been a reasonable point of view, but for the following theorem, due to Lian and Zuckerman[16]:

Theorem:

(i) For $c_M < 1$ minimal models coupled to gravity, there are infinitely many states in the relative cohomology of $Q_{\rm BRS}$ at every positive and negative ghost number. These occur in the following situation. Consider the Hilbert space above a matter primary state $|\psi_{rs}\rangle_M$ and a Liouville momentum state $|k_L\rangle_L$. If the conformal dimension associated to k_L is such that it could "dress" the *null vector* at level rs above the matter primary, then this sector of the Hilbert space contains a physical state of ghost number ± 1. The sign of the ghost number is positive or negative depending on whether the Liouville momentum is in ${\mathcal{H}_L}^+$ or ${\mathcal{H}_L}^-$.

The meaning of "dressing" in this context is that the sum of dimensions is 1. Thus the statement of the condition is that if k_L satisfies

$$\Delta_{rs} + rs - \tfrac{1}{2}k_L(k_L - Q) = 1$$

then there will be a physical state in the Hilbert space.

Now we know that in minimal models, there are two basic null vectors above a given primary, but these null vectors in turn have other null vectors above themselves and so on. Consider one such null vector in this chain, which is at a "distance" d from the original primary in this sense (the case of the basic null vectors considered above corresponded to $d = 1$). In this case we have:

(ii) If the conformal dimension associated to the Liouville momentum k_L is such as to dress a null vector at distance d in the chain above the matter primary, there is a physical state at ghost number $\pm d$, where the sign of the ghost number is again determined by the sector in which k_L lies, as above.

Since there are infinitely many null vectors above each primary of a minimal model, the theorem implies that noncritical string theory has infinitely many physical states, with finitely many at each value of the ghost number. The dressed matter primaries are only a small subset of these, which are the easiest to construct. An explicit construction of the physical states at ghost number ± 1 is also known[13], and involves a new BRS-like operator which interpolates between the positive and negative Liouville Fock spaces.

The physical meaning of the extra states at nonzero ghost number has been clarified to a certain extent by some recent developments, which we mention here in brief. In a Fock space description of minimal CFT models[17],[18],[19] it is useful to introduce a nilpotent BRS-like operator, which we will call Q_F, which projects out a very restricted set of Fock-space states corresponding to the irreducible modules. It is this operator which ultimately enforces the restriction to finitely many primary fields. Vertex operators with momenta outside a certain finite set turn out not to be in the cohmology of Q_F. Now, combining this Fock-space theory with the Liouville Fock space, one gets a rather symmetric version in which the matter and Liouville sectors appear as two space(-time) dimensions. However, one has to restrict to the cohomology of *two* different BRS-like operators, Q_F (which is built in to the description of the minimal model) and Q_B (which arises because of the gauge-fixing of two-dimensional gravity).

The extra states of nontrivial ghost number were studied in this formalism in[20] and [21]. The approach we described above clearly corresponds to passing first to the cohomology of Q_F and then to that of Q_B. Suppose, however, that we were to do it the other way. In this case, it turns out that the physical states are all the dressed vertex operators, including those with momenta outside the "minimal table". Moreover, one can make a correspondence between the "extra" dressed vertex operators arising in this approach, and the states of nontrivial ghost number arising in the former approach. Thus in a well-defined sense, they are the same thing. Moreover, they are related to each other by a series of "descent equations" in the double cohomology.

Thus it appears that on coupling to two-dimensional gravity, infinitely many extra states appear in the spectrum. In the Fock-space description, these can be interpreted as the quantized momentum states of a particle in two dimensions. It has been suggested some time ago[22] that a spacetime description of $c < 1$ string theory could be the origin of certain Virasoro and W-algebra constraints arising in matrix-model and topological approaches, and the existence of extra states seems to support this idea.

5. 5. Partition Function on a Torus

In this section we return to the Polyakov path-integral formulation, and discuss how to evaluate $Z_h(A)$, defined in Eq.(24), for the case $h = 1$, corresponding to a torus[23]. The area term is modified to accommodate non-unitary matter, as discussed above.

The first thing we can do on the torus is to choose the flat reference metric, $\hat{g}_{ab} = \delta_{ab}$. The torus is then described by a parallelogram with sides 1 and τ, where τ is a complex parameter with positive imaginary part, known as the modular parameter.

In this and other calculations that we will do in the path integral, it is convenient to separate out the Liouville zero mode ϕ_0 and carry out this integration first[24]. In genus 1, the Liouville action does not depend on the zero mode, and this part of the functional integration reduces to

$$\int d\phi_0 \delta \left(\int d^2\xi \ \Psi_M^{(min)} \ e^{\alpha\phi} - A \right) = \frac{1}{|\alpha| A}$$

For the remaining modes, the integration is straightforward. It decomposes into the product of three conformal field theory partition functions, for the matter, Liouville and ghost sectors, each of which can be defined in the operator formalism as

$$Z_{CFT} \equiv \mathrm{tr} \left(q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{c}{24}} \right)$$

The matter sector produces the minimal model partition function $Z_M(\tau, \bar{\tau})$ which can be found in the literature. The ghost and Liouville integrations are simpler since they involve fermionic and bosonic Fock-space oscillators respectively. It turns out that each ghost integration produces a factor of

$$\eta(q) \equiv q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n), \quad q \equiv e^{2\pi i \tau}$$

so that combining the result from b and c ghosts and their antiholomorphic counterparts gives a factor $|\eta(q)|^4$. The integration over the nonconstant modes of the Liouville field (the oscillators) produces $|\eta(q)|^{-2}$. In addition a few factors of $\tau_2 \equiv \mathrm{Im}(\tau)$ arise from the fact that the volume of the torus is τ_2. One eventually finds

$$Z_1(A) \sim \frac{1}{|\alpha|A} \int d^2\tau \; \tau_2^{-\frac{3}{2}} |\eta(q)|^2 Z_M(\tau, \bar{\tau}) \tag{32}$$

It is known[25] that the partition function for the (p, q) minimal model can be written as the difference of the partition functions for a scalar field compactified on two different radii:

$$Z_{p,q} = \tfrac{1}{2} \left(Z(R = \sqrt{pq}) - Z(R = \sqrt{\frac{p}{q}}) \right)$$

where

$$Z(R) \equiv \frac{1}{|\eta(q)|^2} \sum_{s,t} q^{(s/R+tR)^2/4} \bar{q}^{(s/R-tR)^2/4}$$

Inserting this expression into Eq.(32) above and performing the τ-integration over the fundamental region, one finds the torus partition function for string theory in the background of a (p, q) minimal model to be

$$Z_1(A) \sim \frac{(p-1)(q-1)}{(p+q-1)A}$$

where we have omitted a proportionality constant independent of p, q and A. The area-dependence of this exact expression was of course predicted in Eq.(25). The above equation agrees with the corresponding result coming from matrix models.

6. 6. Correlation Functions

In this section I will briefly outline an approach to the computation of correlation functions of certain physical operators. The operators that we will consider are (integrals of) the "dressed primaries" referred to earlier, which can be written

$$\Phi^{\pm}_{r,s}(z, \bar{z}) \equiv \int d^2\xi \sqrt{\hat{g}} \; \Psi_{r,s}(\xi) e^{k^{\pm}_{r,s}\phi(\xi)} \tag{33}$$

where the Liouville momentum $k^{\pm}_{r,s}$ gives the right conformal dimension to the vertex operator so that the total dimension of the dressed primary is $(1, 1)$. The label $\pm$ distinguishes the two values of k which are respectively greater or less than $\frac{Q}{2}$. One easily finds

$$k^{\pm}_{r,s} = \tfrac{1}{2} \left(\frac{(1 \pm r)}{\sqrt{t}} + (1 \pm s)\sqrt{t} \right)$$

Technically, one should have multiplied the dressed primaries by ghost factors $c(z)\bar{c}(\bar{z})$ to get physical states. But the ghost factors are absorbed independently and their only contribution is to enforce an integration of each of the operators above over the whole surface. Thus we can more conveniently work just with the integrated primaries. We will accordingly drop all reference to the ghost measures and action in what follows. In addition, we will assume that there is some well-defined prescription to compute the correlation functions in the matter theory, which is a well-studied subject in CFT. This means we only need to concentrate on the computation of correlators of vertex operators in the Liouville sector.

A useful approach in studying these correlation functions is to first perform the Liouville zero-mode integration, as was done for the case of the torus partition function. In the presence of the cosmological constant as well as operator insertions, this integration is rather more nontrivial than in the preceding case, but still can be done after making some continuation, as we will see. We have

$$
\begin{aligned}
&\langle \prod_{i=1}^{n} \int d^2\xi \sqrt{\hat{g}}\; e^{k_i^{\pm}\phi} \rangle \\
&= \int \mathcal{D}_{\hat{g}}\phi \prod_{i=1}^{n} \int d^2\xi \sqrt{\hat{g}}\; e^{k_i^{\pm}\phi} e^{-S_L(\phi,\hat{g})}
\end{aligned}
$$

where, as usual,

$$
S_L(\phi,\hat{g}) = \frac{1}{8\pi} \int d^2\xi \sqrt{\hat{g}} \left(\hat{g}^{ab}\partial_a\phi\partial_b\phi + QR(\hat{g})\phi + \mu e^{\alpha\phi} \right)
$$

Now decompose $\phi(\xi)$ into a constant part and a part which has no constant mode:

$$
\phi(\xi) = \phi_0 + \tilde{\phi}(\xi)
$$

The part $\tilde{\phi}$ must satisfy the condition that its integral over the whole surface vanishes.

In terms of these new independent variables we can rewrite the Liouville action:

$$
\begin{aligned}
S_L(\phi_0,\tilde{\phi},\hat{g}) &= \frac{1}{8\pi} \int d^2\xi \sqrt{\hat{g}} \left(\hat{g}^{ab}\partial_a\tilde{\phi}\partial_b\tilde{\phi} + QR(\hat{g})\phi_0 + QR(\hat{g})\tilde{\phi} + \mu e^{\alpha\phi_0} e^{\alpha\tilde{\phi}} \right) \\
&= S_0(\tilde{\phi},\hat{g}) + Q\phi_0(1-h) + \frac{\mu}{8\pi} e^{\alpha\phi_0} \int d^2\xi \sqrt{\hat{g}}\; e^{\alpha\tilde{\phi}}
\end{aligned}
$$

where

$$
S_0(\tilde{\phi},\hat{g}) = \frac{1}{8\pi} \int d^2\xi \sqrt{\hat{g}} \left(\hat{g}^{ab}\partial_a\tilde{\phi}\partial_b\tilde{\phi} + QR(\hat{g})\tilde{\phi} \right)
$$

Also the vertex operators factorise:

$$\prod_{i=1}^{n}\int d^2\xi\sqrt{\hat g}\; e^{k_i^{\pm}\phi} = \prod_{i=1}^{n} e^{k_i^{\pm}\phi_0}\prod_{i=1}^{n}\int d^2\xi\sqrt{\hat g}\; e^{k_i^{\pm}\tilde\phi}$$
$$= e^{(\sum_i k_i^{\pm})\phi_0}\prod_{i=1}^{n}\int d^2\xi\sqrt{\hat g}\; e^{k_i^{\pm}\phi_0}$$

As a result, the Liouville path integral for correlators becomes

$$\int \mathcal{D}_{\hat g}\phi\prod_{i=1}^{n}\int d^2\xi\sqrt{\hat g}\; e^{k_i^{\pm}\phi}e^{-S_L(\phi,\hat g)}$$
$$= \int \mathcal{D}_{\hat g}\tilde\phi\prod_{i=1}^{n}\int d^2\xi\sqrt{\hat g}\; e^{k_i^{\pm}\tilde\phi}e^{-S_0(\tilde\phi,\hat g)}\int d\phi_0\; e^{\left((\sum_i k_i^{\pm})-Q(1-h)\right)\phi_0}e^{-\frac{\mu}{8\pi}e^{\alpha\phi_0}\int d^2\xi\sqrt{\hat g}\; e^{\alpha\tilde\phi}}$$

Now the zero-mode integral is of the form

$$\int_{-\infty}^{\infty} dx\; e^{Ax-Be^{Cx}} = \frac{1}{C}\; B^{-\frac{A}{C}}\;\Gamma(\frac{A}{C})$$

where this answer holds if A, B, C are real and positive. For other values of these constants, the integral is not well-defined. Accordingly, we *define* the Liouville zero-mode integral to be given by the right hand side of the above equation, for all values of the constants, as a kind of analytic continuation. This gives the prescription

$$\int d\phi_0\; e^{\left((\sum_i k_i^{\pm})-Q(1-h)\right)\phi_0}e^{-\mu e^{\alpha\phi_0}\int d^2\xi\sqrt{\hat g}\; e^{\alpha\tilde\phi}} = \frac{1}{\alpha}\left(\frac{\mu}{8\pi}\int\sqrt{\hat g}e^{\alpha\tilde\phi}\right)^{s}\Gamma(-s)$$

with

$$s = -\sum_{i=1}^{n}\frac{k_i^{\pm}}{\alpha}+\frac{Q}{\alpha}(1-h) \tag{34}$$

Thus we are left with the integral over the nonconstant modes:

$$\langle\prod_{i=1}^{n}\int d^2\xi\sqrt{\hat g}\; e^{k_i^{\pm}\phi}\rangle = \frac{1}{\alpha}\left(\frac{\mu}{8\pi}\right)^{s}\Gamma(-s)\int \mathcal{D}_{\hat g}\tilde\phi\int\sqrt{\hat g}e^{\alpha\tilde\phi}\prod_{i=1}^{n}\int d^2\xi\sqrt{\hat g}\; e^{k_i^{\pm}\tilde\phi}e^{-S_0(\tilde\phi,\hat g)} \tag{35}$$

If s, defined in Eq.(34), were a positive integer, then this functional integral would just be a CFT correlation function for a product of vertex operators and screening charges in a free boson theory with a background charge iQ. Because of the absence of zero modes, the momentum-conserving delta-function is absent.

Unfortunately, s is never a positive integer. Note that the quantities Q and α which enter the definition of s are fixed by the central charge and minimum dimension operator of the matter CFT. Moreover, the $k_i^{\pm}$ are those which dress the matter CFT dimensions, which are also determined. One can thus compute s explicitly for the case of a (p,q) minimal model and a set of physical states of Liouville momentum $k^+_{r_i,s_i}$ and $k^-_{r'_j,s'_j}$:

$$s(p,q;\; r_1,s_1,\ldots,r_n,s_n;\; r'_1,s'_1,\ldots,r'_m,s'_m) =$$
$$\frac{1}{p+q-1}\left((p+q)(2-2h-m-n)-\sum_{i=1}^{n}(pr_i+qs_i)-\sum_{j=1}^{m}(pr'_j+qs'_j)\right)$$

where $1 \le r_i, r'_j \le q-1$ and $1 \le s_i, s'_j \le p-1$. We see that s is generically *negative* and non-integer.

At this point it should be clear what the prescription will be, even though it may be less clear how to implement it. We simply evaluate Eq.(35) for positive integer s, (which can be thought of as continuing the matter central charge away from its physical value) by treating the insertions of the area operator as screening charges, and performing a standard free-field computation[26]. If it is then possible to "continue" the result beck to negative fractional values of s, then this will be defined to be the answer. A class of three-point correlation functions in genus zero, of the diagonal operators $\int \Phi^-_{r_i,r_i}$ (see Eq.(33)) has been computed in this approach, in the *unitary* minimal series backgrounds, corresponding to $p = q+1$ in Eq., The calculation in the matter sector is carried out following the usual Feigin-Fuchs procedure[18], while the Liouville part is evaluated by similar techniques after performing the steps described above. After a straightforward if slightly tedious calculation, one obtains products of Γ-functions which depend on s, so that one can then restore s to its physical value. The final result is simplest after normalizing the correlators to cancel out any overall factors in the definition of the partition function and the physical fields:

$$\frac{\langle \Phi^-_{r_1,r_1}\Phi^-_{r_2,r_2}\Phi^-_{r_3,r_3}\rangle^2\; Z_0(\mu)}{\langle \Phi^-_{r_1,r_1}\Phi^-_{r_1,r_1}\rangle\langle \Phi^-_{r_2,r_2}\Phi^-_{r_2,r_2}\rangle\langle \Phi^-_{r_3,r_3}\Phi^-_{r_3,r_3}\rangle} = \frac{r_1\; r_2\; r_3}{(q+1)(2q+1)} \tag{36}$$

The limitations in this calculation, to three-point functions, diagonal operators and unitary models, appear to be purely technical. All of them have been overcome to some extent in subsequent calculations.

It is worth describing briefly one refinement of the above method. Instead of treating the area term as the only type of screening charge, one may introduce the other dimension

(1, 1) Liouville vertex operator, corresponding to the other choice of sign in Eq.(22), and allow both to be inserted in correlation functions[27]. This is again a sort of self-consistent assumption, and increases the similarity with the computation of minimal-model correlation functions in the Feigin-Fuchs approach, where also there are two kinds of screening charges. Once the Liouville and matter sectors have been brought onto a similar footing, it is natural also to make the continuation in central charge simultaneously in the Liouville and matter sectors, thereby always preserving the condition that the total central charge is 26. In this approach there is a certain computational simplification: one only needs to consider negative *integer* numbers of screening charge insertions, and it is relatively easy to give an unambiguous prescription for treating these values.

In addition to this simplification, there is a surprising and rather important result in this approach (this was also independently observed in Ref.[28]) - dressed primaries with indices (r, s) *outside* the "minimal table" (i.e., not satisfying $1 \le r \le q-1$, $1 \le s \le p-1$) give nonzero correlators. This is at variance with the fact that such states in the minimal model sector alone give vanishing correlators, which is the reason why minimal models have finite numbers of primary fields. The point is that on analytically continuing away from physical values of the parameters, both matter and Liouville sectors give generically nonzero contributions. For the special case of states outside the minimal table, the matter contribution vanishes linearly in some parameter as we return to physical values, but the Liouville part acquires a simple pole in the same parameter, so that the product remains finite. Thus, this heuristic prescription appears to give a nontrivial new piece of information about minimal models coupled to gravity: there are infinitely many physical states of the form of "dressed primaries". In fact, as we have seen in a previous section, these extra states appear naturally if one formulates the physical state problem as a double cohomology. The above results are also in agreement with matrix models, which is one more reason to take them seriously.

The genus-zero three-point function obtained in this approach, which generalizes Eq.(36)above to the case of operators which are not necessarily diagonal and need not lie in the minimal table, is

$$\frac{\langle \Phi^-_{r_1',r_1}\Phi^-_{r_2',r_2}\Phi^-_{r_3',r_3}\rangle^2\ Z_0(\mu)}{\langle \Phi^-_{r_1',r_1}\Phi^-_{r_1',r_1}\rangle\langle \Phi^-_{r_2',r_2}\Phi^-_{r_2',r_2}\rangle\langle \Phi^-_{r_3',r_3}\Phi^-_{r_3',r_3}\rangle} = \frac{(r_1'-r_1\rho)(r_2'-r_2\rho)(r_3'-r_3\rho)}{(1-\rho)\rho(1+\rho)}$$

Here the minimal model is still in the unitary series, with $p = q+1$, and we have introduced the parameter $\rho \equiv \frac{q+1}{q}$.

Finally, one can generalize to the case of non-unitary (p,q) models. Here one encounters the fact, noted above, that the cosmological constant operator contains not the identity field in the matter sector but the operator of lowest conformal dimension. This introduces a severe technical difficulty in doing the computation as above. Recently it has been suggested[29] that one can make an ansatz according to which screening in the Liouville sector is done by the dressed identity field, as in the unitary theories, with the difference that this time one cannot interpret this to be be the cosmological term. Again, this appears to be a kind of self-consistent approach. Fixing the coefficient of the new screening operators to be an appropriate power of the cosmological constant μ, one obtains a genus-zero three-point function which reduces to the correct result in the unitary case (by construction), but also gives the right answer when compared with tree-level correlators in the multicritical one-matrix model. In this case the result is

$$\frac{\langle \Phi^-_{r'_1,r_1}\Phi^-_{r'_2,r_2}\Phi^-_{r'_3,r_3}\rangle^2 \; Z_0(\mu)}{\langle \Phi^-_{r'_1,r_1}\Phi^-_{r'_1,r_1}\rangle\langle \Phi^-_{r'_2,r_2}\Phi^-_{r'_2,r_2}\rangle\langle \Phi^-_{r'_3,r_3}\Phi^-_{r'_3,r_3}\rangle} = -\frac{2(r'_1p-r_1q)(r'_2p-r_2q)(r'_3p-r_3q)}{(q+p)(q+p+1)}$$

This generalizes the two preceding results.

Some higher-point functions have also been calculated in the unitary case[30]. It is clear, however, that the continuum formulation of non-critical string theory gives very limited information after a lot of work, perhaps a disappointing conclusion if one believes that the continuum approach is the most basic one in string theory.

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