TIFR/TH/89-33
June 1989

MODULAR GEOMETRY AND THE CLASSIFICATION OF RATIONAL CONFORMAL FIELD THEORIES

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ABSTRACT

I review a recently developed procedure to classify all conformal field theories with a finite number of characters. This method involves writing the most general modular-invariant differential equation on the moduli space of the torus, and looking for solutions which satisfy the axioms of conformal field theory. On identifying these solutions with the genus-1 characters, one can then reconstruct the primary field content, the fusion rules, the correlation functions and the chiral algebra of the associated theory. Contour-integral representations of Feigin-Fuchs type are proposed for the characters.

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* To appear in the proceedings of the IIIrd Regional Conference on Mathematical Physics, Islamabad, Pakistan, February 1989.

** Based on work done in collaboration with Samir Mathur, Sudhakar Panda and Ashoke Sen.

1. Introduction

Rational conformal field theories (RCFT) were discovered by Belavin, Polyakov and Zamolodchikov (BPZ) [1], although the term "rational" was used in this context later. These authors constructed a class of RCFT with Virasoro central charge $c < 1$, the so-called minimal series. Their procedure starts from highest-weight representations of the Virasoro algebra, and the existence of null-vectors in these representations. The fields creating highest-weight states are called primaries, and their descendants under the Virasoro algebra, secondaries. It is shown that the constraints due to the presence of null vectors are strong enough to deduce the primary field content of the theory. Conformal Ward identities and null vectors together then provide differential equations for the correlation functions of the theory, from which the operator-product coefficients and fusion rules can be worked out. The characters can be obtained from the representation theory of the Virasoro algebra [2], and the partition function is constructed in terms of these. In this sense, the $c < 1$ RCFT's are completely classified and exactly solved on the plane.

The situation is rather different for $c > 1$ or for Riemann surfaces other than the plane. For Riemann surfaces of higher genus, the methods of [1] alone are not sufficient to give ordinary differential equations for the conformal blocks in terms of which the correlation functions are constructed. This is because insertion of the energy-momentum tensor brings about deformations in the moduli of these surfaces [3], and one gets partial differential equations in terms of locations and moduli. The question of how to solve RCFT (other than those described by free bosons and fermions) on Riemann surfaces of genus $g > 0$ was studied in [4–6]. Ref. 6, in particular, gives a general procedure to obtain ordinary differential equations for the correlators and characters of *any* RCFT, on a Riemann surface of *any* genus. This method does not rely in any direct way on the existence of null vectors, nor does it encounter any limitations on the value of the central charge c. The differential equations in [1] for correlators on the plane can be reproduced by this method.

So much for the solution of RCFT on general Riemann surfaces. As to the classification of RCFT, most of the progress in the last few years involved finding representations of extended chiral algebras which include the Virasoro algebra as a proper subalgebra. Knizhnik and Zamolodchikov [7] studied RCFT based on affine Kac-Moody algebras associated to compact simple Lie groups. The procedure of BPZ goes through, and produces a collection of infinite discrete series of RCFT, one for each compact Lie group G, labelled by the value of the Kac-Moody central charge k. As before, the solution of the theory on the plane relies on the existence of null vectors, this time for the combined Virasoro-Kac-Moody algebra. This procedure was also extended to superconformal, parafermion and W-algebras, among others. (References to these can be found in [8]).

Unfortunately, none of these procedures provided a general method to classify and construct all rational conformal field theories. It became clear that a different approach is needed for such a classification. Two important, and related, inputs became available more recently. One was the observation, by Friedan and Shenker [9], that the characters of RCFT can be thought of as holomorphic sections of a certain line bundle over moduli space. The other was the remarkable discovery, by Verlinde [10], that the fusion rules are diagonalized by the matrix S_{ij} which implements the modular transformation $\tau \rightarrow -\frac{1}{\tau}$ on the characters. Verlinde's result implies certain constraints on the values of the central charge and the conformal dimensions in an RCFT, given the number of characters and the fusion rules. These were explored in [10] and [11].

An important ingredient that was lacking in approaches to RCFT based on "modular geometry" was the fact that, besides forming a section of a line bundle on moduli space, the characters have another important property: their power-series expansion in the variable $q \equiv \exp(2\pi i\tau)$ involves coefficients which are positive integers. These integers count the number of secondaries at a given level, with respect to whatever chiral algebra is involved. Another relevant observation is that for the character over the identity field, the overall normalization is fixed by the axiom that there is a unique identity field in any quantum field theory. Thus the first term in the expansion of this character must have coefficient unity.

These simple facts about RCFT were used in the work of Mathur, Sen and myself [12–13], to provide a complete and systematic classification of all rational conformal field theories. This approach does not rely on specific chiral algebras and their null vectors. Instead, it simply addresses the question of how to find all possible sets of (multivalued) functions on moduli space which have the right properties to be the characters of an RCFT. This approach, and some of its consequences, will be discussed in the rest of this article.

2. Characters and Fusion Rules

The characters of a conformal field theory are defined by

$$\chi_i(\tau) \equiv \mathrm{tr}_i \;\; q^{L_0 - \frac{c}{24}}, \qquad q \equiv e^{2\pi i\tau} \tag{2.1}$$

where L_0 is the zero mode operator in the Virasoro algebra, c is the Virasoro central charge, and the trace is taken over all states above a given primary, generated by the action of some (as yet unknown) chiral algebra. (We require this chiral algebra to have only integrally moded operators. This not a serious limitation, since if there are generators of fractional spin, one can always take combinations of them which have integral moding, to form the "spectrum generating algebra").

The partition function of the theory is then constructed from bilinears in the characters:

$$Z(\tau,\bar{\tau}) = \sum_{i,j=0}^{n-1} \bar{\chi}_i(\bar{\tau})\, M_{ij} \chi_j(\tau) \tag{2.2}$$

Here M_{ij} is a constant matrix. In what follows, we confine our attention to the case when M_{ij} is diagonal†.

† It is known [14] that whenever it is possible to construct a modular-invariant partition function from a non-diagonal combination of characters, there also exists a modular-invariant diagonal combination.

Under modular transformations, the characters transform as

$$\begin{aligned} \chi_i(\tau+1) &= e^{2\pi i\left(h_i - \frac{c}{24}\right)} \\ \chi_i\left(-\frac{1}{\tau}\right) &= S_{ij}\chi_j(\tau) \end{aligned} \tag{2.3}$$

A modular-invariant diagonal partition function Z (Eq. (2.2)) will exist if S_{ij} leaves invariant the matrix

$$M = \begin{pmatrix} 1 & & & & & & \\ & M_1 & & & & & \\ & & M_2 & & & & \\ & & & \cdot & & & \\ & & & & \cdot & & \\ & & & & & \cdot & \\ & & & & & & M_{n-1} \end{pmatrix} \tag{2.4}$$

The numbers M_i appearing in the diagonal matrix M represent the number of distinct primaries in the theory with the same character. For example, a primary may be inequivalent to its charge-conjugate, but the two necessarily have the same character. Some theories even have symmetries which are responsible for assigning the same character to three or more primaries.

The fusion rules of a conformal field theory are defined to be positive integers N_{ijk} which count the number of distinct ways in which primaries i and j (or their descendants) can fuse to give the representation k. (For ease of notation we write all indices as subscripts, although this is strictly correct only when the primaries are self-conjugate). Verlinde [10,14] showed that, as matrices acting in the space of primary fields, the S_{ij} determine the fusion rules completely† via the formula

$$N_{ijk} = \sum_{a=0}^{m-1} \frac{S_{ia}S_{ja}S_{ka}}{S_{0a}} \tag{2.5}$$

† Actually, the S_{ij} which satisfy Verlinde's identity act in the space of primaries, which is generically larger than the space of characters. The precise relation between these matrices and the ones defined in Eq. (2.3) is explained in Ref. 13.

Finally, we examine the power-series expansion of the characters defined in Eq. (2.1). This must be of the form

$$\chi_i(\tau) = q^{h_i - \frac{c}{24}} \sum_{n=0}^{\infty} a_n^{(i)} q^n \tag{2.6}$$

where h_i is the conformal dimension of the primary field above which the character is built. The important point here is that the coefficients $a_n^{(i)}$ must all be integers ≥ 0, simply because they count the number of states at each level. In particular, $a_0^{(i)}$ counts the degeneracy of the ith primary, and the axiom that the identity be non-degenerate implies that $a_0^{(0)} = 1$. We will see below that in general our approach does not automatically determine the normalization of the characters, as it is based on linear homogeneous differential equations. As a result, the preceding statement, which fixes the normalization for the identity character, turns out to be rather crucial.

3. Modular-Invariant Differential Equations

We have seen that the characters of any RCFT form a set of holomorphic functions of the parameter τ. Now, given any set of n such functions $\chi_0, \chi_1, \cdots \chi_{n-1}$, we can always view them as the independent solutions of a linear homogeneous ordinary differential equation in τ:

$$\begin{vmatrix} \chi_0 & \cdots & \chi_{n-1} & \chi \\ \mathcal{D}_\tau \chi_0 & \cdots & \mathcal{D}_\tau \chi_{n-1} & \mathcal{D}_\tau \chi \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \mathcal{D}_\tau^n \chi_0 & \cdots & \mathcal{D}_\tau^n \chi_{n-1} & \mathcal{D}_\tau^n \chi \end{vmatrix} = 0 \tag{3.1}$$

Here we have used the covariant derivative on modular forms of weight r, defined in terms of the second Eisenstein series, which is a holomorphic connection on moduli space [15]:

$$\begin{aligned} \mathcal{D}_\tau^{(r)} &\equiv \frac{\partial}{\partial \tau} - \frac{i\pi r}{6} E_2(\tau) \\ E_2(\tau) &\equiv const. \sum_{\substack{m,n \,\in\, Z \\ m \neq 0, n \neq 0}} \frac{1}{(m + n\tau)^2} \end{aligned} \tag{3.2}$$

Clearly, Eq. (3.1) is satisfied if and only if χ is a linear combination of the given functions $\chi_0, \chi_1, \cdots \chi_{n-1}$. The equation can alternatively be written

$$\sum_{k=0}^{n-1} (-1)^k W_k(\tau) \mathcal{D}_\tau^k \chi(\tau) = 0 \tag{3.3}$$

where the functions $W_k(\tau)$ are the minors in the expansion of the determinant above, in terms of the last column.

We know that under modular transformations, the characters transform into linear combinations of themselves, as in Eq. (2.3), or in other words, they form finite-dimensional representations of the modular group, $SL(2, Z)$. It follows that the Wronskians W_k transform as modular forms of weight $n(n+1) - 2k$, as one can check using the fact that the covariant derivative $\mathcal{D}$ increases the weight of a modular form by 2. The fact that the W_k are actually modular forms, in the sense that they are holomorphic everywhere in the interior of moduli space, follows from the holomorphicity properties of the characters.

It is convenient to rewrite Eq. (3.3) in monic form. Dividing out by W_n, and defining $\Phi_k \equiv (-1)^{n-k}(W_k/W_n)$, we have

$$\mathcal{D}^n \chi + \sum_{k=0}^{n-1} \Phi_k \mathcal{D}_\tau^k \chi = 0 \tag{3.4}$$

Clearly, the Φ_k transform like modular functions of weight $2(n-k)$. However, unlike the W_k, the Φ_k need not be modular forms. In general they have poles, wherever W_n has zeroes. For the classification which follows, the number of zeroes of W_n will play a crucial role, and we compute it here.

This number can actually be fractional, since moduli space has orbifold points which can be encircled by traversing an angle of $\pi/3$ or π rather than 2π. However, precisely for this reason, it must be of the form $l/6$ where l is an integer. Now it is a standard theorem [15] that the number of zeroes and poles of a weight-k modular form (including the behaviour as $\tau \to \infty$) differ by $k/12$. We know that each character has the asymptotic behaviour

$$\chi_i(\tau) \to q^{-\frac{c}{24}+h_i} \quad \text{as} \quad q \to 0 \tag{3.5}$$

From this it follows that the number of zeroes of W_n is given by

$$\frac{l}{6} = \frac{n(n-1)}{12} + \frac{nc}{24} - \sum_i h_i \tag{3.6}$$

(Recall that n is the number of characters). From what was said above, $l/6$ is also the maximum number of poles that each Φ_k can have.

The classification procedure requires one to first choose a value for n and l. Then one finds the most general modular-invariant differential equation consistent with these values, and imposes the requirements that the q-expansion coefficients be integers, and that the identity be non-degenerate.

Suppose first that we choose $l = 0$. This means that the Φ_k have no poles. Then each Φ_k is a genuine modular form, of weight $2(n-k)$. It is a fundamental result in the theory of modular forms that the space of forms of a given weight is *finite dimensional.* In fact, the number d_k of independent modular forms of weight k is given by the formula [15]

$$\begin{aligned} d_k &= 0, \qquad k \text{ odd} \\ d_k &= \left[\frac{k}{12}\right], \qquad k \equiv 2 \bmod 12 \\ d_k &= \left[\frac{k}{12} + 1\right] \qquad \text{otherwise} \end{aligned} \tag{3.7}$$

where $[x]$ denotes the integer part of x. Moreover, a convenient basis for the space of modular forms of weight k is provided by the Eisenstein series E_4 and E_6, where

$$E_{2i} \equiv const. \sum_{\substack{m,n \,\epsilon\, Z \\ (m,n)\neq(0,0)}} \frac{1}{(m+n\tau)^{2i}} \tag{3.8}$$

The space of modular forms of weight k is generated by all monomials of the form $E_4^a E_6^b$ with the integers a and b satisfying $4a + 6b = k$.

Thus the most general modular-invariant differential equation does not contain arbitrary functions, but only a finite number of arbitrary constants. This is what ultimately makes our analysis tractable. A similar analysis goes through for $l \neq 0$, although that situation generally involves more parameters. As an illustration of the discussion above, we may write down the most general modular-invariant differential equation in the simplest non-trivial case, namely, $l = 0$ and $n = 2$:

$$\mathcal{D}_\tau^2 \chi(\tau) + \mu\pi^2 E_4(\tau)\chi(\tau) = 0 \tag{3.9}$$

This involves only a single arbitrary constant, μ. In the next section we show how to analyse this equation to extract all possible RCFT's with two characters (and $l = 0$).

4. Integrality Requirements and Classification

The first step in examining Eq. (3.9) is to rewrite it in terms of ordinary derivatives. From the definition of $\mathcal{D}_\tau$ (Eq. (3.2)) one gets

$$\partial_\tau^2 \chi(\tau) - \frac{i\pi}{3} E_2(\tau)\partial_\tau \chi(\tau) + \mu\pi^2 E_4(\tau)\chi(\tau) = 0 \tag{4.1}$$

Now if the solutions of this equation are to be characters, they must have power-series expansions of the form

$$\chi_i(\tau) = q^\alpha \sum_{n=0}^{\infty} a_n^{(i)} q^n \tag{4.2}$$

for some integral coefficients $a_n^{(i)}$, and some number α, which must in fact be rational, since it corresponds to $-\frac{c}{24}+h_i$. We insert this power-series in the differential equation, along with the power-series expansion of the Eisenstein series

$$E_{2i}(\tau) = 1 + \sum_{n=1}^{\infty} E_{2i,n} q^n \tag{4.3}$$

To lowest order, the differential equation gives

$$4\alpha^2 - \frac{2}{3}\alpha - \mu = 0 \tag{4.4}$$

This is the indicial equation, which can be solved to determine α in terms of μ:

$$\alpha = \frac{1}{12}\left(1 \pm \sqrt{1+36\mu}\right) \tag{4.5}$$

Defining x to be the positive square root $+\sqrt{1+36\mu}$, we can identify the two leading power behaviours allowed by this equation

$$\begin{aligned}\alpha_1 &= \frac{1}{12}(1-x) = -\frac{c}{24}\\ \alpha_2 &= \frac{1}{12}(1+x) = h-\frac{c}{24}\end{aligned} \tag{4.6}$$

which determines the central charge c of this two-character theory, and the dimension h of the non-identity field(s)

$$\begin{aligned}c &= 2(x-1)\\ h &= \frac{1}{6}x\end{aligned} \tag{4.7}$$

Thus we already have the information that the free parameter μ in the differential equation (4.1) is such as to make $x = \sqrt{1+36\mu}$ rational. Note that in Eq. (4.6) we have identified the two values of the index α in a specific way, based on the assumption that the dimension h is positive. This is necessarily so only in a unitary theory, and one should always examine whether the opposite identification is also possible, giving rise to a non-unitary two-character RCFT. Our analysis has no particular prejudice in favour of unitarity.

Assuming nevertheless for definiteness that Eq. (4.6) holds, we proceed to examine the next order in q in the series solution of the differential equation. Labelling the identity character and its expansion coefficients with the superscript (0), we find

$$\frac{a_1^{(0)}}{a_0^{(0)}} = \frac{10x^2+2x-12}{6-x} \tag{4.8}$$

Now all the a_n's are positive integers, so the right hand side of this equation must be a positive rational number. But, as mentioned earlier, the identity character is non-degenerate, so $a_0^{(0)}$ is in fact equal to unity, with the result that the right hand side is actually a positive integer. This turns out to be a very strong constraint on the allowed values of μ, and leads to a discrete set of allowed values. This comes about as follows.

Denote by m_1 the positive integer to which both sides of Eq. (4.8) are equal. Solving for x, we get

$$x = \frac{1}{20}\left[-(m_1+2) \pm \sqrt{(m_1+2)^2 + 240\,(m_1+2)}\right] \tag{4.9}$$

Since x is rational, the square root on the right hand side must be rational. But m_1 is an integer, so this square root can be rational only if it is an integer. Denoting this integer by k, we have the diophantine equation

$$(m_1+2)^2 + 240\,(m_1+2) = k^2 \tag{4.10}$$

which can be rewritten

$$(m_1+122)^2 - k^2 = (120)^2 \tag{4.11}$$

Factorizing the left hand side, we need to find all positive integers k, m_1 such that

$$(m_1+122-k)\,(m_1+122+k) = (120)^2 \tag{4.12}$$

which is clearly a finite set! In fact, the allowed values for $(m_1+122-k)$ are precisely 2, 4, 8, 16, 32, 6, 18, 10, 50, 30, 90, 12, 36, 20, 100, 60, 24, 72, 40, 48, 80, 96. (The order in which these integers are written here looks less arbitrary if we work in terms of their prime decompositions). The next step is to numerically check whether, for each of these values of m_1 and k, the higher-order coefficients a_2, a_3 etc. turn out to be integers. It looks as if this involves an infinite amount of calculation, since we have no guarantee that all coefficients will be integers after only checking a finite number. But in fact, one finds on doing the calculation that once the first few coefficients are integers, all the remaining turn out to be integers as well, upto about 80 terms in the power series expansion.

Clearly, something of fundamental number-theoretic significance is going on. Actually, for modular forms of a given weight, it can be shown [16] that the integrality of some finite number of expansion coefficients suffices to prove that all the remaining coefficients are integral. To my knowledge, no corresponding theorem has been proved for finite-dimensional representations of the modular group. We conjecture that such a property holds.

Checking the higher-order coefficients in this way rules out a number of possible values in the list above. For those values which survive, it is straightforward to compute the central charge and the conformal dimension of the non-identity field(s). Remarkably, almost all the values obtained this way lead one immediately to identify the corresponding model with a known two-character RCFT. The results are listed in Table 1. We have not listed the values of μ, the parameter in the differential equation, as these are not very illuminating. They can, of course, be deduced easily from c and h via Eq. (4.7). In the table, we have included the actual number of primary fields other than the identity present in the theory, denoted by the integer M. This can be independently calculated after solving the differential equation, and will be discussed below.

All the entries in this table, except for the column labelled M (the number of non-identity primaries) have been obtained by the methods described above. The identification with known two-character theories is straightforward, and it is quite remarkable that a list of these theories has been obtained just from considerations of modular geometry and the integrality requirement on the coefficients in the q-expansion. At this stage, it seems a bit surprising that there is an entry with $c = \frac{38}{5}$, since no such two-character theory is known. It is also puzzling that the one-character E_8 theory has made an appearance. These and other facts will be explained in the next section, where we analyze the consequences of this classification.

5. Reconstruction of RCFT from Modular Geometry

We have found all the values of the parameter μ in the differential equation (4.1) for which the solutions have the right properties to be the characters of an RCFT. It does not follow as yet that these are actually the characters of a consistent theory. On the other hand, we also cannot rule out the possibility that a given set of characters describes more than one distinct theory.

Table 1: Two-character theories

m_1	c	h	M	Identification
1	$\frac{2}{5}\left(-\frac{22}{5}\right)$	$\frac{1}{5}\left(-\frac{1}{5}\right)$	1	non-unitary minimal
3	1	$\frac{1}{4}$	1	$k=1\ SU(2)$
8	2	$\frac{1}{3}$	2	$k=1\ SU(3)$
14	$\frac{14}{5}$	$\frac{2}{5}$	1	$k=1\ G_2$
28	4	$\frac{1}{2}$	3	$k=1\ SO(8)$
52	$\frac{26}{5}$	$\frac{3}{5}$	1	$k=1\ F_4$
78	6	$\frac{2}{3}$	2	$k=1\ E_6$
133	7	$\frac{3}{4}$	1	$k=1\ E_7$
190	$\frac{38}{5}$	$\frac{4}{5}$	?	?
248	8	$\frac{5}{6}$	0	$k=1\ E_8$

To analyze this situation in detail, it is helpful to first make a change of variables which maps moduli space to the complex plane. This is achieved by the transformation

$$\lambda = \frac{\vartheta_2^4(\tau)}{\vartheta_3^4(\tau)} \tag{5.1}$$

which actually maps six copies of the fundamental region in Teichmüller space to the complex plane. In terms of this variable, the modular transformations become

$$\begin{aligned} T: &\quad \tau \to \tau+1 \quad \leftrightarrow \quad \lambda \to \frac{\lambda}{\lambda-1} \\ S: &\quad \tau \to -\frac{1}{\tau} \quad \leftrightarrow \quad \lambda \to 1-\lambda \end{aligned} \tag{5.2}$$

It is easy to check that the differential equation (4.1) is mapped onto

$$\lambda^2(1-\lambda)^2\frac{\partial^2\chi}{\partial\lambda^2} - \frac{2}{3}\lambda(1-\lambda)(2\lambda-1)\frac{\partial\chi}{\partial\lambda} + \mu(\lambda(1-\lambda)-1)\chi = 0 \tag{5.3}$$

This is a Fuchsian differential equation with regular singular points at 0,1 and ∞. It can be checked that it is invariant under modular transformations.

The solution to this equation can be conveniently written down in terms of hypergeometric functions. We have

$$\begin{aligned}\chi_0 &= \left(\frac{\lambda(1-\lambda)}{16}\right)^{(1-x)/6} F\left(\frac{1}{2}-\frac{x}{6}\,,\ \frac{1}{2}-\frac{x}{2}\ \middle|\ 1-\frac{x}{3}\ \middle|\ \lambda\right)\\ \chi_1 &= N\left(\frac{\lambda(1-\lambda)}{16}\right)^{(1+x)/6} F\left(\frac{1}{2}+\frac{x}{6}\,,\ \frac{1}{2}+\frac{x}{2}\ \middle|\ 1+\frac{x}{3}\ \middle|\ \lambda\right)\end{aligned} \tag{5.4}$$

Here N is a normalization constant, and $x = +\sqrt{1+36\mu} = 1+\frac{c}{2}$ as before. We have tentatively identified the upper expression with the identity character, as it is the one with the leading power behaviour in the limit $q \to 0$. The reverse identification is also possible in principle, if one wishes to classify non-unitary theories as well.

The first useful result that one can deduce from this solution is the matrix S_{ij} which implements the modular transformation S (see Eq. (5.2)) on the characters. From standard textbooks on hypergeometric functions, one finds that this matrix (the "monodromy" matrix for these functions) is

$$S = \begin{pmatrix} \dfrac{\Gamma(1-\frac{x}{3})\Gamma(\frac{x}{3})}{\Gamma(\frac{1}{2}-\frac{x}{6})\Gamma(\frac{1}{2}+\frac{x}{6})} & \dfrac{(16)^{x/3}}{N}\dfrac{\Gamma(1-\frac{x}{3})\Gamma(-\frac{x}{3})}{\Gamma(\frac{1}{2}-\frac{x}{6})\Gamma(\frac{1}{2}-\frac{x}{2})} \\ \dfrac{N}{(16)^{x/3}}\dfrac{\Gamma(1+\frac{x}{3})\Gamma(\frac{x}{3})}{\Gamma(\frac{1}{2}+\frac{x}{6})\Gamma(\frac{1}{2}+\frac{x}{2})} & \dfrac{\Gamma(1+\frac{x}{3})\Gamma(-\frac{x}{3})}{\Gamma(\frac{1}{2}+\frac{x}{6})\Gamma(\frac{1}{2}-\frac{x}{6})} \end{pmatrix} \tag{5.5}$$

Thus we have at once written down the modular transformation matrix for all the theories listed in Table 1.

The role of the normalization constant N for the non-identity character needs to be explained. We have defined this so that it corresponds to the degeneracy of the associated primary field. For the identity character, we know that the number of states at the lowest level is 1, and it has been normalized accordingly. There is no such information for the other character, and we simply choose the integer N so that the coefficients in the q-expansion are integers. This does not fix N uniquely, but it allows only a finite number of possibilities, and these can be subjected individually to the consistency checks described below.

As mentioned in Section 2, the diagonal matrix left invariant by S_{ij} determines the modular-invariant partition function of the theory. In the present case this is a 2×2 matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & M \end{pmatrix} \tag{5.6}$$

and the modular-invariant partition function is

$$Z(\tau, \bar{\tau}) = |\chi_0(\tau)|^2 + M|\chi_1(\tau)|^2 \tag{5.7}$$

This means that M must be a positive integer, corresponding to the number of non-identity primaries in the theory. Working this out for each case, one gets the column labelled by M in the table.

The multiplicities of the primaries can be readily understood. The fundamental representations of $SU(3)$ and E_6 are complex, so that the corresponding primary fields appear along with their complex conjugates, which of course have the same character. For $SO(8)$, the well-known property of triality tells us that there are three primaries with the same character: the fundamental, the spinor and the spinor$'$. The case of E_8 is more subtle. We were unable to find a sensible value of N in this case, but it turns out that for any value of N, the matrix S leaves invariant a matrix of the form given in Eq. (5.6), with $M = 0$. It follows that the square of the identity character is itself modular-invariant, and we actually have a one-character theory.

The next step is to find the fusion rules of the primary fields, using Verlinde's theorem [10]. For the theories which have $M = 1$ in the table, this is straightforward, from Eq. (2.5). But if $M > 1$, we need to find the $(1 + M) \times (1 + M)$ matrix S which acts in the space of primaries, which in these cases is larger than the space of characters. A procedure to do this is described in Ref. 13.

A remarkable property of Verlinde's identity is that whereas the entries of the matrix S can very well be fractional, or even irrational, the combinations of these entries which give the fusion-rule coefficients N_{ijk} must be non-negative integers. This is an important consistency check on characters which have been obtained solely from considerations of modular geometry.

The fusion rules have been calculated in Ref. 13 for every theory in Table 1. This calculation has actually been done twice in each case – once for the identification of the identity character made in Eq. (5.4), which is the only valid one for unitary theories, and once for the reverse identification, which could hold if the dimension h is allowed to be negative. It turns out that in every case at most one of these possibilities is consistent in the sense that the N_{ijk} turn out to be non-negative and integral. One case where neither identification works is the $c = \frac{38}{5}$ solution. One identification gives negative fusion-rule coefficients, while the other necessarily leads to a degenerate identity field. We conclude that there is no consistent two-character theory with $c = \frac{38}{5}$, despite the fact that a pair of characters could be found with the right modular and integrality properties. For the other cases, the fusion rules obtained in this way confirm the identification with known two-character RCFT's which was originally made just on the basis of the values of c and h.

After this, one can attempt to study the correlation functions of the theories discovered using modular geometry. This has been discussed in Ref. 13, along with a method to recover the chiral algebra of the theory (or at least the spectrum-generating algebra) from the characters. For reasons of space, these techniques will not be described here, although they are important ingredients in the complete reconstruction of RCFT's from modular geometry. Another question which is not addressed here is the problem of constructing a table analogous to the one presented above, for theories with, for example, $l = 0$ and $n = 3$ (three characters) or theories with $l \neq 0$ and $n = 2$. This turns out to be considerably more complicated, although some progress has been made.

In the next section, we turn instead to the question of solving the differential equations for theories with more than two characters.

6. Contour-Integral Representations for Characters

We have seen that modular-invariant differential equations for characters of RCFT's can be solved, in the two-character case, by mapping moduli space to the complex plane. The solutions are hypergeometric functions, and because their monodromy properties are known, we can find the important matrix S_{ij} which implements the modular transformation $\tau \to -\frac{1}{\tau}$ on the characters. This matrix in turn permits us to obtain the primary field content and the fusion rules of the theory.

One may ask to what extent an analogous procedure can be carried out for theories with more than two characters. This question was first addressed in [13] and studied in detail in [17]. We found that for a large class of RCFT's, the characters can be described in terms of Feigin-Fuchs (FF) contour-integral representations [18–19] on the complex plane. Since hypergeometric functions have contour-integral representations, the FF integrals can be thought of as generalizations of hypergeometric functions suitable for the description of conformal characters. Most important, there exists an explicit procedure [18–19] to determine the monodromy matrices for the functions defined by FF contour integrals.

Consider first the two-character theories which we have discussed in preceding sections. We show that their characters, which were earlier written down as hypergeometric functions, can be expressed as contour integrals. We imagine that moduli space has been mapped onto the complex plane as in the previous section, via the function $\lambda(\tau)$ defined there. Consider now the pair of functions

$$\begin{aligned} \chi_0(\lambda) &= (\lambda(1-\lambda))^{-a-(1/3)} \int_1^\infty dt\,(t\,(t-1)\,(t-\lambda))^a \\ \chi_1(\lambda) &= (\lambda(1-\lambda))^{-a-(1/3)} \int_0^\lambda dt\,(t\,(t-1)\,(t-\lambda))^a \end{aligned} \tag{6.1}$$

These have the following properties, which can be easily checked:

(i) Under modular transformations, whose generators are given in Eq. (5.2), they go into linear combinations of themselves with constant coefficients.

(ii) In particular, they are eigenstates of T with eigenvalues of unit modulus.

(iii) As $\lambda \to 0$ they have the power behaviour:

$$\begin{aligned} \chi_0 &\sim \lambda^{-a-(1/3)} \\ \chi_1 &\sim \lambda^{a+(2/3)} \end{aligned} \tag{6.2}$$

Property (i) implies that χ_0 and χ_1 can be thought of as the linearly independent solutions of a modular-invariant second-order differential equation. Property (ii) tells us that, when re-expressed as functions of q, they have expansions in integer powers of q up to an overall multiplicative factor of the form q^β. Thus χ_0 and χ_1 satisfy some of the necessary conditions to be the characters of a two-character RCFT. If they are indeed consistent characters, we can identify the central charge c and the conformal dimension of the non-identity field(s) h using property (iii):

$$\begin{aligned} -\frac{c}{12} &= -a - \frac{1}{3} \\ -\frac{c}{12} + 2h &= a + \frac{2}{3} \end{aligned} \tag{6.3}$$

It follows that

$$2\frac{c}{24} - h + \frac{1}{6} = 0 \tag{6.4}$$

Now we have already seen that the most general modular-invariant second-order differential equation is Eq. (4.1). Since the functions described above in Eq. (6.1) are solutions of this equation, we can relate the constant a appearing in these functions to the constant μ which specifies the differential equation. One easily finds, by inserting the leading power behaviours into Eq. (4.1), that

$$\mu = a^2 + a + \frac{2}{9} \tag{6.5}$$

From the analysis in the preceding sections, we can conclude that whenever μ has the right value to describe an RCFT, the value of a obtained from the above equation specifies a pair of contour integrals (Eq. (6.1)) which are the characters of this theory. (The parameter x introduced in Section 4 is easily seen to be related to a by $x = 6a+3$).

It is noteworthy that we have re-obtained the solutions of the modular-invariant second order differential equation, this time *without* making use of any information about hypergeometric functions. What is more, by deforming contours [18–19] it is possible to compute explicitly the matrices which describe the linear transformation undergone by these contour integrals under $\lambda \to 1-\lambda$. One obtains, of course, the matrix S of Eq. (5.5).

Note that in this case the number of parameters in the differential equation coincides with the number of parameters in the integrals Eq. (6.1), so that there is a one-to-one correspondence between the possible second-order differential equations and the set of FF integrals Eq. (6.1).

Generalizing to contour-integral representations with two contours, one can find sets of three functions which go into each other under modular transformations, and are therefore candidates for the characters of three- character theories. These functions depend on two independent parameters, while the corresponding modular-invariant third-order differential equation also depends on two parameters. In fact, using the notation and the properties of modular forms discussed in Section 3, we find that the most general modular-invariant third-order differential equation for theories with $l=0$ is

$$\mathcal{D}_\tau^3\chi(\tau)+\mu_1\pi^2E_4(\tau)\mathcal{D}_\tau\chi(\tau)+i\mu_2\pi^3E_6(\tau)\chi(\tau)=0 \tag{6.6}$$

In terms of the variable λ on the complex plane, this becomes

$$\begin{aligned}&\lambda^3(1-\lambda)^3\frac{\partial^3\chi}{\partial\lambda^3}-2\lambda^2(1-\lambda)^2(2\lambda-1)\frac{\partial^2\chi}{\partial\lambda^2}\\&+\lambda(1-\lambda)\{(\tfrac{20}{9}-\mu_1)(1-\lambda(1-\lambda))-2\}\frac{\partial\chi}{\partial\lambda}\\&-\tfrac{1}{3}\mu_2(2\lambda-1)(\lambda+1)(\lambda-2)\chi=0\end{aligned} \tag{6.7}$$

It was found in [13] that the characters for all three-character RCFT's with $l=0$ can be represented in terms of Feigin- Fuchs contour integrals. Instead of writing the explicit forms for the contour integrals in the three-character case, we immediately turn to the description of the general FF contour integrals, from which this special case can be easily deduced.

The general FF integrals appropriate to represent characters turn out to have n_1 contours of one type, labelled by integration variables t_i $(i=1,\cdots,n_1)$, and n_2

contours of another type, labelled by τ_i $(i = 1, \cdots, n_2)$. They are described by two independent parameters a and b, and have the form

$$J \sim (\lambda(1-\lambda))^{\alpha} \int \prod_{i=1}^{n_1} dt_i \prod_{j=1}^{n_2} d\tau_j \prod_{i=1}^{n_1} (t_i(t_i-1)(t_i-\lambda))^{a} \prod_{j=1}^{n_2} (\tau_j(\tau_j-1)(\tau_j-\lambda))^{b}$$
$$\prod_{i<k} (t_i - t_k)^{-2a/b} \prod_{j<l} (\tau_j - \tau_l)^{-2b/a} \prod_{i,j} (t_i - \tau_j)^{-2}$$
$$\equiv (\lambda(1-\lambda))^{\alpha} \mathcal{J}(\lambda) \tag{6.8}$$

where

$$\alpha = \frac{1}{3}\left[-n_1(1+3a) - n_2(1+3b) + \frac{a}{b} n_1(n_1-1) + \frac{b}{a} n_2(n_2-1) + 2n_1 n_2\right] \tag{6.9}$$

It is easy to check that the integrals so defined are invariant, upto a change of integration limits and possible phase factors, under the modular transformations $\lambda \to 1-\lambda$, $\lambda \to \frac{\lambda}{\lambda-1}$. Invariance of the integrand upto a phase under the first of these transformations is manifest. A simple calculation shows that the second transformation also has this property; in fact this determines α in Eq. (6.9). The other exponents in the integrand are fixed by the requirement that there be no simple pole in any $(t_i - \tau_j)$, which permits us to deform the t_i and τ_j contours through each other.

Next we must choose the limits of integration. The integrand has a singularity whenever any of the contours passes through $0, 1, \infty$ or λ. These four points can be connected by two independent contours. These should be chosen so that the transformation $T : \lambda \to \frac{\lambda}{\lambda-1}$ leaves the integration limits unchanged, so that each integral will go into itself upto a phase. Since T interchanges the points 1 and ∞ and leaves 0 fixed, the two independent choices of contour should be 0 to λ and 1 to ∞. Accordingly we define J_{AB} to be the integral with A contours of type t_i between 0 and λ and B

contours of type τ_i between 0 and λ:

$$\begin{aligned} J_{AB} = (\lambda(1-\lambda))^{\alpha} & \int_0^{\lambda} dt_1 \cdots \int_0^{\lambda} dt_A \int_1^{\infty} dt_{A+1} \cdots \int_1^{\infty} dt_{n_1} \\ & \int_0^{\lambda} d\tau_1 \cdots \int_0^{\lambda} d\tau_B \int_1^{\infty} d\tau_{B+1} \cdots \int_1^{\infty} d\tau_{n_2} \\ & \prod_{i=1}^{n_1} (t_i(t_i-1)(t_i-\lambda))^a \prod_{j=1}^{n_2} (\tau_j(\tau_j-1)(\tau_j-\lambda))^b \\ & \prod_{i<k} (t_i-t_k)^{-2a/b} \prod_{j<l} (\tau_j-\tau_l)^{-2b/a} \prod_{i,j} (t_i-\tau_j)^{-2} \\ \equiv (\lambda(1-\lambda))^{\alpha} & \mathcal{J}_{AB}(\lambda) \end{aligned} \tag{6.10}$$

Since we have $0 \le A \le n_1$ and $0 \le B \le n_2$, there are altogether $(n_1+1)(n_2+1)$ independent functions J_{AB} which go into each other under modular transformations.

Consider now the behaviour of each J_{AB} as $\lambda \to 0$. First let us look at the functions $\mathcal{J}_{AB}$ defined in Eq. (6.10) above. To find the $\lambda \to 0$ behaviour, one simply redefines the integration variables for those contours which run from 0 to λ by scaling so that they run from 0 to 1. In this way one gets

$$\mathcal{J}_{AB} \sim \lambda^{\Delta_{AB}} \qquad \text{as} \qquad \lambda \to 0 \tag{6.11}$$

where

$$\Delta_{AB} = A(1+2a) + B(1+2b) - \frac{a}{b}A(A-1) - \frac{b}{a}B(B-1) - 2AB \tag{6.12}$$

It follows that the FF integrals J_{AB} go like

$$J_{AB} \sim \lambda^{\alpha+\Delta_{AB}} \qquad \text{as} \qquad \lambda \to 0 \tag{6.13}$$

where α is defined in Eq. (6.9) and Δ_{AB} in Eq. (6.11). Notice in particular that $\Delta_{00} = 0$. Also note the identity:

$$\sum_{A=0}^{n_1} \sum_{B=0}^{n_2} (\alpha + \Delta_{AB}) = \frac{1}{6} n(n-1) \tag{6.14}$$

where $n = (n_1+1)(n_2+1)$ is the number of integrals J_{AB}. This identity follows from a straightforward computation.

A character χ_i associated with a primary field of conformal weight h_i behaves in the $\lambda \to 0$ limit as

$$\chi_i \sim \lambda^{-\frac{c}{12}+2h_i} \tag{6.15}$$

If we identify the characters with the FF integrals, then we must identify the set of numbers $\{-\frac{c}{12} + 2h_i\}$ with the set $\{\alpha + \Delta_{AB}\}$. Then the identity Eq. (6.14) satisfied by α and the Δ_{AB} translates to the relation

$$l \equiv \frac{nc}{4} - 6\sum h_i + \frac{n(n-1)}{2} = 0 \tag{6.16}$$

Note that the Feigin-Fuchs integrals with any number of contours always depend on only two parameters a and b, besides the integers n_1 and n_2 specifying the numbers of contours. Thus, given a known RCFT with its value of c and spectrum of dimensions h_i, it is not obvious that a set of contour integrals can be chosen such that each one has the right power behaviour near $\lambda = 0$ to be one of the characters. Even if such a set can be found with the right leading behaviours in λ, it does not always follow that they are actually the characters of the theory. This would be so if we could show that the leading behaviour of the characters as $\lambda \to 0$ completely determines the differential equation, since in that case the FF integrals and the characters would be the solutions of the same differential equation.

In the simple cases of two- and three-character theories, the number of constants in the most general modular-invariant differential equation is $n-1$, where n is the number of characters. Since the FF integrals always give $l = 0$, there are precisely $n-1$ independent leading power behaviours in λ. These suffice to determine the parameters in the differential equation, from which it follows that the FF integrals are the characters, upto normalization. To check how far this reasoning works, we need to know the number of independent constants in a general nth order modular-invariant differential equation for an $l = 0$ theory.

It is most convenient to study this problem in terms of the original modular parameter τ. The differential equation when written in monic form has coefficient functions which are modular forms, with no singularities in moduli space. We have already seen the formula for d_k, the dimension of the space of modular forms of a given weight k (Eq. (3.7)). As we saw in Section 3, the nth order differential equation for $l = 0$ can be written, in monic form,

$$\mathcal{D}^n \chi(\tau) + \sum_{k=1}^{k=n} \Phi_k(\tau) \mathcal{D}^{n-k} \chi(\tau) = 0 \tag{6.17}$$

where $\Phi_k(\tau)$ is any modular form of weight $2k$. Then the number of independent parameters N is just

$$N = \sum_{k=1}^{k=n} d_{2k} \tag{6.18}$$

This is equal to $n-1$ only for $n \leq 5$, while for large n it grows quadratically. Thus, for $n \geq 6$, computing the leading behaviour in λ of the FF integrals does not completely determine the differential equation which they satisfy, and so is insufficient to prove that they are the characters. To demonstrate the equivalence of FF integrals and characters for $n \geq 6$, we need to compute a number of terms in the power-series expansion of some FF integral in the variable q, equal to the difference $N - (n-1)$ between the number of parameters in the differential equation and the number of independent leading behaviour of the FF integrals, and show that they match with the expected power series expansion of the character.

This has been carried out in detail in Ref. 17. It is shown there that one can find FF integrals with the right leading behaviour to be the characters of several infinite chains of RCFT's: the A, D and E series $SU(2)_k$ current-algebra theories, the $SU(n)_{k=1}$ current-algebra theories and the A and D series $c < 1$ minimal theories. All these theories have $l = 0$. As one can see from the formulae quoted above, for theories with upto five characters, this matching alone guarantees that we have found the solutions of the corresponding differential equations, which proves that the FF integrals are the characters. In the more general case, it has been checked in Ref. 17 that the first subleading powers in the FF integrals are also consistent with the hypothesis that they are the characters of the corresponding theories. A check of their monodromy properties, and comparison with the known values of the modular transformation matrices in these cases, lends further support to this hypothesis.

If modular-invariant differential equations provide a powerful framework in which all rational conformal field theories can be unified, it seems equally likely that Feigin-Fuchs integrals serve to unify the characters of the RCFT's in a way which permits computation of their essential properties.

7. Conclusions

In this lecture I have described an approach to the classification and reconstruction of rational conformal field theories. The basic premises of this approach are that characters of RCFT's are sets of functions which transform into linear combinations of themselves under modular transformations, and that their power-series expansion in q has integer coefficients. These properties, and the requirement that the identity field be non-degenerate, lead to a systematic classification of RCFT's in terms of two parameters: $l/6$, the number of zeroes of the Wronskian in the interior of moduli space, and n, the number of characters. The integrality requirement on the expansion coefficients leads to diophantine equations. The relevant properties of an RCFT: the primary field content, the fusion rules, the correlators and the chiral algebra, can be reconstructed starting from the characters. Explicit formulae for the characters can be obtained in terms of Feigin-Fuchs contour integrals.

Clearly, the collection of RCFT's forms a rich and beautiful structure, and we have yet to unravel all of its important properties. It remains to be seen what new information one can gain from the classification scheme proposed here, of relevance to string theory and to critical phenomena in two dimensions.

8. Acknowledgements

I am grateful to the organizers of the IIIrd Regional Conference on Mathematical Physics, Islamabad, Pakistan, particularly Faheem Hussain and Asghar Qadir, for their kind invitation to speak at this conference. The experience of visiting Pakistan was heartwarming and memorable, and I wish to express my gratitude to all the wonderful people who made it so. The work reported here was done in collaboration with Samir Mathur, Sudhakar Panda and Ashoke Sen, and I am happy to thank them for a most enjoyable collaboration.

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