W_∞ Identities From Topological 2D String Theory *

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String theory in a ground state corresponding to a two-dimensional target space-time has been solved to all orders in perturbation theory using random-matrix models[1]. In view of increasing evidence that matrix models are closely related to topological field theories in general and to intersection theory on moduli space in particular, one may ask if these theories can be re-formulated, and solved to all orders in string perturbation theory, from a topological viewpoint. For the two-dimensional spacetime with the geometry of a cylinder, where the compact direction has the special self-dual value of the radius, the answer is in the affirmative[2, 3, 4, 5, 6].

The most useful explicit solution to this problem is provided by a particular topological Landau-Ginzburg model with singular superpotential $1/X$, coupled to topological gravity. This theory is solved using results from intersection theory[7, 8] and from the analysis of matter Landau-Ginzburg systems[9, 10], and consistency requirements. Higher-genus amplitudes decompose as a sum of contributions from the bulk and the boundary of moduli space, and can be shown to generate the W_∞ algebra. Since this algebra of constraints on the partition function generates all the correlation functions of the discrete tachyon modes in this theory, it constitutes an explicit solution of the model.

This solution is manifestly equivalent to that found by solving the random-matrix formulation, since there is a 1-1 correspondence between the tachyon field modes in the two theories and the constraint algebra specifying the partition function is the same[11]. Moreover, in the topological framework it can be shown that the genus-g partition function is proportional to the Bernoulli numbers, another well-known matrix-model result[12]. All the above is at nonzero cosmological constant.

In view of the fact that a topological symmetry algebra underlies all string backgrounds, the above result is likely to provide a direction to understand general topological features of string theory itself, independent of the background, while on the other hand matrix models may not be sufficiently powerful to make background-independent statements about string theory.

The analysis of the Landau-Ginzburg topological theory with superpotential $1/X$ reveals that the BRS-invariant observables are

$$T_k = X^{k-1}$$

with $k \in \mathbb{Z}$. Their genus-g correlators $\langle T_{k_1} T_{k_2} \cdots T_{k_N} \rangle_g$ satisfy the following conservation law, independent of genus:

$$\sum_{i=1}^{N} k_i = 0$$

which suggests that they are identified with discrete tachyons of momentum k.

The solution of the theory proceeds via the observation that multipoint correlators on the sphere require the presence of contact terms coming from colliding operators, to satisfy the general symmetry requirements on the structure constants of the operator algebra. These contact terms lead to flow equations on the phase space of the model perturbed by positive and negative tachyons, whose solution is incorporated in the

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statement that the perturbed superpotential is

$$\hat{W}(t) = -X^{-1} + \sum_{k>0} t_k X^{k-1} - \frac{1}{\mu^2} \sum_{k>0} k \frac{\partial}{\partial t_k} X^{-k-1}$$

acting on the partition function $Z(t, \bar{t})$, where t_k are the couplings to the positive tachyons.

In terms of this "quantized" superpotential, the correlation function of negative tachyons in the phase space of positive tachyons is obtained by starting with the three-point correlation function in the perturbed pure matter LG theory on the sphere:

$$\langle T_{k_1} T_{k_2} T_{k_3} \rangle_{g=0}(t) = \oint \frac{T_{k_1}(t) T_{k_2}(t) T_{k_3}(t)}{W'(t)}$$

and integrating. (Here, W is the superpotential in Eq. restricted by dropping the last term, hence corresponds to the theory in the "small" phase space of positive tachyon perturbations.) This leads to

$$\frac{\partial}{\partial \bar{t}_n} Z = \frac{1}{n(n+1)} \oint (-W)^{n+1} Z$$

which is the matrix model result in Ref.[11] restricted to genus zero and the small phase space. The big phase space result is obtained by the the substitution $W \to \hat{W}$, according to the quantization hypothesis.

In higher genus, the corresponding correlation functions are obtained using two pieces of information. One is that if $\mathcal{O}$ is any operator in a matter LG theory, then genus-g and genus-0 correlators are related as follows[9, 7]:

$$\langle \mathcal{O} \rangle_g = \langle (W'')^g \mathcal{O} \rangle_0$$

where the second derivative of the superpotential, W'', can be thought of as the "handle operator". The other information is that the negative tachyons in the present theory have various "picture-changed" representatives corresponding to forms of different dimension on moduli space. The picture-changing operation introduces the observables of pure topological gravity, whose correlators are known from intersection theory on the moduli space of Riemann surfaces.

The result for the negative tachyon 1-point function in genus g (in the small phase space) is a combinatorial formula involving a sum over partitions of g, of the form $1^{\alpha_1} 2^{\alpha_2} \cdots g^{\alpha_g}$ with $\sum_{l=1}^{g} l\alpha_l = g$. Let $p = \sum_{l=1}^{g} \alpha_l$ be the total number of elements in the partition, then

$$\begin{aligned}
\langle\langle \mathcal{T}_{-n} \rangle\rangle_g &= \sum_{\substack{\alpha_1,\ldots,\alpha_g \\ \sum \alpha_l = p,\ \sum l\alpha_l = g}} \langle\langle \mathcal{T}_{-n} \rangle\rangle_g^{(\alpha_1,\ldots,\alpha_g)} \\
\langle\langle \mathcal{T}_{-n} \rangle\rangle_g^{(\alpha_1,\ldots,\alpha_g)} &= \frac{1}{2^{2g}} \frac{\prod_{j=1}^{2g-2+p}(j-n)}{\prod_{l=1}^{g} \alpha_l!\,((2l+1)!)^{\alpha_l}} \oint \frac{\prod_{l=1}^{g} (\partial^{2l} W)^{\alpha_l}}{W'} \left(\frac{(-W)^{n-2g+2-p}}{-(n-2g+2-p)} \right)'
\end{aligned}$$

(Here, ∂ represents $\partial/\partial X$.)

Some of the terms in this combinatorial sum arise as "bulk" terms on moduli space, while the rest must be added self-consistently as boundary contributions corresponding to contacts between handles of the genus-g Riemann surface.

The formula above can be shown by direct combinatorial arguments to be precisely the small-phase-space reduction of the W_∞ identity on the partition function that is obtained from matrix-models. A suitable limit of the above expression also gives, explicitly, the genus-g partition function of this theory as

$$Z_g = \frac{B_{2g}}{2g(2g-2)}$$

where B_{2g} are the Bernoulli numbers.

For the big phase space, the quantization of the superpotential in Eq. suggests that one must replace W by $\hat{W}$ and view the W_∞ identities as differential operators acting on the partition function which depends on $t, \bar{t}$. Carrying out this procedure carefully (which will not be described here) completes the self-consistent solution of self-dual two-dimensional string theory from the topological formulation. It would be interesting to examine whether this solution tells us something about the compactification of moduli space relevant to this string theory, and conversely whether a more detailed study of moduli space can provide derivations in place of some of the self-consistency assumptions.

References

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