TIFR/TH/93-63
December 1993

Topological Models of Non-critical Strings

Sunil Mukhi

Tata Institute of Fundamental Research
Homi Bhabha Rd, Bombay 400 005, India

I describe the non-critical $c = 1$ string theory in the continuum, from the point of view of topological symmetry. The $c = 1$ string is first discussed in the Liouville formulation, then the topological invariance of non-critical strings is demonstrated, and finally a manifestly topological coset model of compactified $c = 1$ string theory is presented. It is shown that this model reproduces correlation functions of this string theory in the continuum directly at nonzero cosmological constant, without recourse to perturbation theory in the cosmological constant and analytic continuation procedures. The model also gives directly the partition function of the compactified theory at the self-dual radius.

1. $c = 1$ Liouville String Theory

1.1. Introduction

The theory of interest to us will be the bosonic Polyakov string coupled to a (possibly compact) free scalar field $X(z, \bar{z})$ on the world-sheet. The action for this theory, on a Riemann surface Σ, is

$$S^X = \frac{1}{8\pi} \int_\Sigma d^2\xi \sqrt{g} g^{ab} \partial_a X \partial_b X \tag{1.1}$$

In conformal gauge, the holomorphic matter stress-energy tensor is

$$T^X(z) = -\frac{1}{2} \partial X \partial X(z) \tag{1.2}$$

and it has central charge $c_X = 1$.

In the process of going to conformal gauge:

$$g_{ab}(z,\bar{z}) = e^{\phi(z,\bar{z})}\hat{g}_{ab}(z,\bar{z}) \tag{1.3}$$

we encounter the Liouville field $\phi(z,\bar{z})$ which fails to decouple due to an anomaly in the Weyl invariance of the action above. Here $\hat{g}_{ab}(z,\bar{z})$ is a reference metric on Σ. The Liouville mode can be thought of as one more "dimension" of the string background. For self-consistency, it should be a free conformal field of central charge $c_\phi = 25$[1].

The procedure of fixing the gauge also introduces reparametrization ghosts $b(z), c(z)$ of spins $2, -1$ which form a fermionic conformal field theory with central charge $c_{\text{ghost}} = -26$. Thus the total theory has zero central charge, and techniques of conformal field theory can be brought to bear on it. The full system after gauge fixing has stress-energy tensor

$$T(z) = T^X(z) + T^\phi(z) + T^{\text{ghost}}(z) \tag{1.4}$$

where T^X is written above, and

$$\begin{aligned} T^\phi &= -\frac{1}{2}\partial\phi\partial\phi(z) + \sqrt{2}\partial^2\phi \\ T^{\text{ghost}} &= c\partial b(z) - 2b\partial c(z) \end{aligned} \tag{1.5}$$

It is amusing to note that the full matter-Liouville-ghost system has two bosonic and two fermionic fields, which suggests the possibility of a "supersymmetry" in the gauge-fixed theory. The actual situation is more subtle, and we will return to it later. However, one immediate consequence of this observation is that excluding zero modes, the determinants obtained from the path integral on a genus-1 surface (for which the spins of the fields are irrelevant, since we can choose a flat metric) do indeed cancel out.

The unconventional form of the Liouville stress-energy tensor is due to the presence of a "background charge", coming from the following action for the Liouville field:

$$S^\phi = \frac{1}{8\pi}\int d^2\xi\sqrt{\hat{g}}(\hat{g}^{ab}\partial_a\phi\partial_b\phi + 2\sqrt{2}R(\hat{g})\phi) \tag{1.6}$$

where the linear term in the action, proportional to the Ricci curvature scalar on the world-sheet, is responsible for the background charge term in T^ϕ. Unfortunately, this term makes the action ill-defined, since the zero-mode integration

$$\int_{-\infty}^{\infty} d\phi_0 e^{-2\sqrt{2}\phi_0(\frac{1}{8\pi}\int_\Sigma d^2\xi\sqrt{\hat{g}}R(\hat{g}))} \tag{1.7}$$

diverges exponentially. Indeed, we have

$$\frac{1}{4\pi}\int_{\Sigma} d^2\xi\sqrt{\hat{g}}R(\hat{g}) = \chi_{\Sigma} = 2-2h \tag{1.8}$$

where χ_{Σ} is the Euler characteristic of a genus-h Riemann surface. Thus it is clear that the integral diverges as $\phi \to -\infty$ for $\chi = 2$ (the sphere), while it diverges as $\phi \to \infty$ for $\chi = -2, -4\ldots$ (genus greater than one). To stabilise the theory, we add the "cosmological operator"

$$\mu\int_{\Sigma} d^2\xi\sqrt{\hat{g}}e^{\sqrt{2}\phi} \tag{1.9}$$

to the action. With a suitable choice of sign, this stabilises the theory at genus $h \geq 2$.

The coefficient of ϕ in the exponent was determined self-consistently by demanding that it be marginal in the free Liouville CFT. Vertex operators $\exp(k\phi)$ in this theory have conformal dimension $-\frac{1}{2}k(k-2\sqrt{2})$, and setting this equal to 1 fixes $k=\sqrt{2}$. Stabilising the theory in genus 0 or 1 has to be taken care of separately, arising from the fact that the moduli space of unpunctured Riemann surfaces is undefined for these values of the genus. As a result, we will only consider amplitudes of three or more fields in genus 0, and of one or more fields in genus 1, and these turn out to be stabilised if we make suitable choices of the relevant fields, as we will see later.

1.2. Spectrum: States of Ghost Number 1

To solve the $c=1$ string theory, we should first determine the spectrum of physical operators and states[2] [3]. Apparently the situation is quite similar to that in the critical bosonic string, in 26 spacetime dimensions. However, there are two essential differences. One is that because of the background charge, the cosmological perturbation is essential, and $\mu = 0$ is a very singular point in the paramater space of the theory. We will, however, temporarily ignore this and proceed to examine the spectrum at $\mu = 0$ first. This is because the standard CFT techniques available to us to construct the physical states are not straightforward to implement when the cosmological perturbation is turned on. The other difference is that the low dimensionality of the target spacetime in some sense reduces the connectivity of the space of configurations. This leads to many unexpected "new" physical states, compared to the critical string. There is also a (less unexpected) reduction in the number of "old" physical states, as compared with the critical string.

The most systematic procedure to find the spectrum of a string background is to write down the nilpotent BRS charge and study its cohomology. The techniques will indeed be similar to those used in the critical string, and it is only the outcome which will, as we mentioned above, be somewhat different. The chiral BRS operator is

$$Q_B = \oint dz \ [c(T^X + T^\phi) + \frac{1}{2} : cT^{\text{ghost}} :] \tag{1.10}$$

The full spectrum of physical states is the equivariant cohomology of $Q_B + \bar{Q}_B$ relative to $b_0^- = b_0 - \bar{b}_0$.

First we will investigate the (chiral) cohomology of Q_B relative to b_0. Then we will see what happens when we combine left and right movers and "relax" the equivariance condition to be with respect to b_0^- only.

Proceeding by analogy with the critical string, we first consider the vertex operators of ghost number 1:

$$V_{k_X,k_\phi}(z) = c(z)e^{ik_X X(z)}e^{k_\phi \phi(z)} \tag{1.11}$$

where we concentrate on only the chiral part. This is analogous to the tachyone in the critical string. The requirement that this be in the cohomology of Q_B leads to the constraint that the conformal dimension be equal to 1:

$$\Delta(k_X, k_\phi) = \frac{1}{2}k_X^2 - \frac{1}{2}k_\phi(k_\phi - 2\sqrt{2}) = 1 \tag{1.12}$$

which has two classes of solutions

$$k_X = \sqrt{2}s, \qquad k_\phi = \sqrt{2}(1 \mp |s|) \tag{1.13}$$

where s is a real number. Thus one infinite set of physical operators is given by

$$V_s^\pm(z) = c(z)e^{i\sqrt{2}sX(z)}e^{\sqrt{2}(1\mp|s|)} \tag{1.14}$$

Actually, s is an arbitrary real number if X is a non-compact scalar field on the world-sheet. If on the other hand X is compact with radius $R = \frac{1}{\sqrt{2}}$ (the self-dual value) then s is any half-integer in our units.

Now let us examine the analogue of a "graviton" vertex operator (or rather, its chiral part):

$$V_{k_X,\epsilon_X;k_\phi,\epsilon_\phi}(z) = c(z)(\epsilon_X \partial X(z) + \epsilon_\phi \partial\phi(z))e^{ik_X X(z)}e^{k_\phi\phi(z)} \tag{1.15}$$

This time, computing the cohomology of the BRS operator gives us a more non-trivial answer. One finds precisely two *discrete* operators in the physical spectrum, corresponding to $\epsilon_\phi = 0$, $k_X = 0$ and $k_\phi = 0$ or $k_\phi = 2\sqrt{2}$. (Of course, we are only describing representatives of the cohmology, so these solutions can be described equivalently by other operators which differ by a BRS-trivial amount from these).

We conclude that the two (chiral) operators $c\partial X$ and $c\partial X e^{2\sqrt{2}\phi}$ are discrete physical operators of the $c = 1$ string. The "physical" explanation of their existence[4] is that in two spacetime dimensions, the physical-state conditions (implemented by the BRS operator) should leave only transverse degrees of freedom – but there are no transverse directions when the total spacetime dimension is 2. Thus the graviton (or its chiral counterpart) should have no field-theoretic degrees of freedom at all. However, this argument does not rule out a discrete number of degrees of freedom, and that is what is actually left. So these two fields are the "remnants" of a would-be chiral graviton field in two dimensions.

It turns out that this is a general feature of this theory: for every would-be tensor state of the critical string, we will find discrete remanants in the $c = 1$ string theory, occurring for finitely many values of the matter and Liouville momenta. In particular, it can be shown that the polarisation in the Liouville direction can always be chosen to vanish, so these excitations always "point" in the X-direction. The general analysis leads to the result that there are infinitely many (chiral) physical states of ghost number 1, of the form

$$Y^{\pm}_{s,n} = c(z)\mathcal{P}_{s^2-n^2}(\partial X, \partial^2 X, \ldots)e^{i\sqrt{2}nX(z)}e^{\sqrt{2}(1\mp s)\phi(z)} \tag{1.16}$$

where $s = 0, \frac{1}{2}, 1, \ldots$ and $-s \leq n \leq s$ in integer steps. Here, $\mathcal{P}_{s-n}$ is some fixed polynomial of degree $(s^2 - n^2)$ in derivatives of the X field. Note, however, that in the special case where $n = \pm s$, and there is no polynomial in front, the restriction that s should be an integer no longer applies.

The analysis we have performed so far must actually be examined afresh after combining left- and right-movers. If the X field is compact of radius R, we must respect the quantisation condition[5]:

$$(k_X^{(L)}, k_X^{(R)}) = (\frac{M}{R} + 2NR, \frac{M}{R} - 2NR) \tag{1.17}$$

At infinite radius this reduces to $k_X^{(L)} = k_X^{(R)}$, while at the other extreme end where $R = \frac{1}{\sqrt{2}}$ (the self-dual radius) we find that $k_X^{(L)}$ and $k_X^{(R)}$ are essentially independent. (More precisely, in this latter case, they are both quantised independently so that their corresponding label n takes values between $-s$ and s in integer steps.) Thus at the self-dual radius we have the non-chiral physical fields of ghost number 2:

$$Y^{\pm}_{s,n,n'}(z,\bar{z}) = Y^{\pm}_{s,n}(z)\bar{Y}^{\pm}_{s,n'}(\bar{z}) \tag{1.18}$$

while at infinite radius we have the subset of these fields with $n = n'$. For rational values of the radius, the situation is intermediate, while at irrational radius there are no discrete fields at all. Note that the tachyons should not be considered discrete operators, in the following sense: the BRS cohomology permits tachyons of all (continuous) momenta, though the ones which actually exist depend on the compactification radius and they form a discrete set if this radius is finite. In contrast, the remaining discrete operators are forced to have discrete momenta directly by the BRS cohomology, and the ones which actually exist are a further subset of these, depending on the radius.

Note that the fields in the above set are all annihilated by both b_0 and $\bar{b}_0$, and they are in particular in the equivariant cohomology of b_0^-.

From now on we restrict ourselves to the self-dual compactification radius. The physical states we have found lead to moduli of the theory, namely, marginal operators of conformal dimension $(1,1)$ which can be integrated and added to the action to generate a perturbed theory. These are obtained by simply stripping off the ghosts. Writing

$$Y^{\pm}_{s,n,n'}(z,\bar{z}) = c(z)\bar{c}(\bar{z})W^{\pm}_{s,n}(z)\bar{W}^{\pm}_{s,n'}(\bar{z}) \tag{1.19}$$

we find the moduli to be $\int d^2z W^{\pm}_{s,n}(z)\bar{W}^{\pm}_{s,n'}(\bar{z})$.

1.3. Spectrum: States of Other Ghost Numbers

We now return temporarily to the chiral cohomology, and explore one of the most interesting consequences of having discrete states at ghost number 1. These states turn out to be "paired" with equally many discrete states of neighbouring ghost number. To state the result in advance, consider the $Y^{\pm}_{s,n}$ with $|n| \neq |s|$ (thus we exclude the tachyons). Then, the $Y^{+}_{s,n}$ have partners of ghost number 0 and the same matter and Liouville momenta. We label them $\mathcal{O}_{s,n}$. Similarly, the $Y^{-}_{s,n}$ (other than tachyons) have partners of ghost number 2, which we denote $\mathcal{P}_{s,n}$.

This pairing has a relatively simple algebraic explanation, and important physical consequences. First, let us give an example of an analogous phenomenon in the critical string. The operator $\epsilon_\mu c\partial X^\mu$ is in the (chiral) cohomology, being the chiral part of a zero-momentum graviton. The interesting point is that this zero-momentum graviton lies in the cohomology for *any* polarisation tensor ϵ_μ, even though for generic non-zero on-shell momenta, ϵ_μ would be constrained to lie in a 24-dimensional transverse subspace. Thus there are two "extra" states that arise at a special value of the 26-momentum, namely, $k_\mu = 0$. Now supposing we were to perturb this field slightly, by a small parameter δ:

$$\epsilon_\mu c\partial X^\mu \to \epsilon_\mu c\partial X^\mu e^{i\delta\epsilon_\mu X^\mu} \tag{1.20}$$

It is easy to check that for nonzero δ, the perturbed field is no longer in the cohomology. In fact, it becomes BRS exact, and we find

$$[Q_B, e^{i\delta\epsilon_\mu X^\mu}] \sim \delta\epsilon_\mu c\partial X^\mu e^{i\delta\epsilon_\mu X^\mu} \tag{1.21}$$

Thus, the perturbed field has a partner of one lower ghost number, namely $e^{i\delta\epsilon_\mu X^\mu}$, and the two form a doublet under the BRS charge. As long as the perturbation is nozero, neither member of the doublet is in the BRS cohomology (since of course only singlets can be in the cohomology). Now, as we take the perturbing parameter δ to 0, we find that the doublet degenerates into two singlets, one of which is our original field on the left hand side of Eq.(1.20), and the other is just 1 – the identity operator. Thus, we have discovered a new operator, of ghost number 0, in the cohomology of the critical bosonic string, namely the identity operator! Its presence in the cohomology is precisely due to the fact that the operator $\epsilon_\mu c\partial X^\mu$, with which we started, was a "discrete" operator, which is physical only at special momenta.

Since the $c = 1$ string has a plethora of such discrete operators at ghost number 1, we should expect, by the same argument, an equal number of operators at ghost number 0. Actually, this is not quite the case, because on perturbation, a discrete operator can go out of the cohmology in two distinct ways: it can become exact (as in the case above) or it can fail to be closed. In the former case, it forms a doublet with an operator at ghost number 0, which after removing the perturbation becomes a physical state in its own right, while in the latter case it forms a doublet with an operator of ghost number 2 which leads to a new cohmology element there. Indeed, in the $c = 1$ string both possibilities are realised, the former by the Y^+-type discrete operators, and the latter by Y^-. This then gives us the result announced in the first paragraph of this section. Explicit construction of these new operators is possible using precisely the perturbation method described above which proves their existence. But in some cases the new operators have special properties which make their construction even more simple.

Let us now look more closely at the new operators of ghost number 0, the ones we labelled $\mathcal{O}_{s,n}$. Because their ghost number is 0, operator products of two such fields can only produce other fields of the same class, modulo BRS-exact terms. It is convenient to label these fields as $\mathcal{O}_{u,n}$ where $u = s-1$, in which case counting the matter and Liouville momenta (both of which are conserved, at zero cosmological constant) leads to the product rule[6]

$$\mathcal{O}_{u_1,n_1}\mathcal{O}_{u_2,n_2} = \mathcal{O}_{u_1+u_2,n_1+n_2} \tag{1.22}$$

Of course, momentum counting is not sufficient to establish this rule, as one also has to show that the right hand side always comes out with a nonzero coefficient, which moreover can consistently be set to 1 by a choice of normalisation. This can indeed be done, and we conclude that the operators $\mathcal{O}_{u,n}$ of ghost number 0 form a commutative ring, with the multiplication law above. The two generators of this ring (which is still at the level of chiral fields) are $O_{\frac{1}{2},\frac{1}{2}}$ and $O_{\frac{1}{2},-\frac{1}{2}}$. The identity is of course $\mathcal{O}_{0,0} = 1$, which follows trivially from the fact that this field has zero matter momentum, Liouville momentum, ghost number and conformal dimension. Representatives of the nontrivial generators can be chosen to be:

$$O_{\frac{1}{2},\pm\frac{1}{2}} = (cb + \frac{1}{\sqrt{2}}(\partial\phi \pm i\partial X))e^{\frac{1}{\sqrt{2}}(\phi \mp iX)} \tag{1.23}$$

With the information that we have uncovered about the ring structure in this theory (at the chiral level) we can combine left- and right-movers to find out what new fields exist because of this. Clearly, there are elements of the non-chiral "ground ring":

$$\mathcal{O}_{u,n,n'}(z,\bar{z}) = \mathcal{O}_{u,n}\bar{\mathcal{O}}_{u,n'} \tag{1.24}$$

This ring has four generators

$$\begin{aligned} a_1 &= \mathcal{O}_{\frac{1}{2},\frac{1}{2},\frac{1}{2}} \\ a_2 &= \mathcal{O}_{\frac{1}{2},-\frac{1}{2},-\frac{1}{2}} \\ a_3 &= \mathcal{O}_{\frac{1}{2},\frac{1}{2},-\frac{1}{2}} \\ a_4 &= \mathcal{O}_{\frac{1}{2},-\frac{1}{2},\frac{1}{2}} \end{aligned} \tag{1.25}$$

satisfying the single relation

$$a_1 a_2 - a_3 a_4 = 0 \tag{1.26}$$

This relation can be thought of as defining a singular variety in $\mathbf{R}^4$. Thus, the ground ring in $c = 1$ string theory has a simple geometrical interpretation.

The existence of the chiral ring elements $\mathcal{O}_{u,n}$ has another important consequence for the full (non-chiral) spectrum. We can combine a left-moving operator of ghost number 1 with a right-moving operator of ghost number zero, if the Liouville momenta match. Thus, we find operators like

$$Y^+_{s,n}(z)\bar{\mathcal{O}}_{u,n'}(\bar{z}), \qquad \mathcal{O}_{u,n}(z)\bar{Y}^+_{s,n'}(\bar{z}) \tag{1.27}$$

which have ghost number 1. Using descent equations, we find conserved spin-1 currents whose contour integrals give conserved charges on the world-sheet. These are in turn usually interpreted as unbroken symmetry generators in the spacetime string theory. Thus, *it is the existence of infinitely many (chiral) ring elements in this theory that is responsible for the existence of infinitely many unbroken symmetries*[6]. This situation differs sharply from that in the critical string, where there is at most a small, finite-parameter family of unbroken symmetries.

Explicitly, the conserved currents take the form

$$J_{z\,s,n,n'}(z,\bar{z}) = W_{s,n}(z)\mathcal{O}_{u,n'}(\bar{z}) \tag{1.28}$$

with another expression for $J_{\bar{z}}$ which is chosen such that the two-vector current $J_\mu = (J_z, J_{\bar{z}})$ satisfies $\partial_\mu J^\mu = 0$. The corresponding conserved charges are

$$Q_{s,n,n'} = \oint (dz J_z + d\bar{z} J_{\bar{z}}) \tag{1.29}$$

The algebra satisfied by these charges should be the key to unravelling the algebraic structure of $c = 1$ string theory compactified on the self-dual radius. It is straightforward to check[6][7] that

$$[Q_{s_1,n_1,n'_1}, Q_{s_2,n_2,n'_2}] = (s_1 n_2 - s_2 n_1) Q_{s_1+s_2-1,n_1+n_2,n'_1+n'_2} \tag{1.30}$$

which is similar to the wedge subalgebra of W_∞, except for the presence of an extra index n' which plays a passive role (it is conserved but does not appear in the structure constants). From Eq.(1.27) it is evident that there is another set of currents which are the complex conjugates of these, from which we get another infinite set of conserved charges $\tilde{Q}_{s,n,n'}$ which satisfy the same algebra as in Eq.(1.30) above, but with the roles of n and n' interchanged. Thus the full symmetry algebra of the theory incorporates two copies of the algebra in Eq.(1.30).

1.4. Equivariant Cohomology of b_0^-

We briefly return to the question of what happens when we compute the relative cohomology with respect to only b_0^- instead of both b_0 and $\bar{b}_0$, as we have been doing so far. In doing this, we find a whole string of new physical fields that were missed so far. A simple example comes from the operator[8][3]

$$a(z) = c(z)\partial\phi(z) + \sqrt{2}\partial c(z) \tag{1.31}$$

which, because of the ∂c term, is not annihilated by b_0. This operator is just the commutator $[Q_B \phi]$, namely, the BRS variation of the Liouville field, so it is certainly BRS closed. It is not BRS exact, however, in the space of conformal fields, since $\phi(z, \bar{z})$ is not a true conformal field due to its infrared-singular propagator.

Now we can form the combination $(a(z) + \bar{a}(\bar{z}))$, which is not annihilated by b_0 and $\bar{b}_0$ separately, but only by their difference, b_0^-. It turns out that we can multiply this operator by the previously determined cohomology elements of ghost numbers $0, 1, 2, \ldots$ and obtain new cohomology elements of one higher ghost number in each case.

No particular role has been found for these "new" cohomology elements, and it has been suggested that they should indeed be considered BRS trivial, by analogy with well-known arguments in critical string field theory[9].

1.5. Correlators

Having found the spectrum of physical operators in the $c = 1$ string, we would next like to compute correlation functions in this theory. Here we have to face the problem of the cosmological constant. Many correlators formally vanish in the absence of a cosmological term; this is because in that case both Liouville and matter momenta have to be conserved. However, when comparing with the matrix model, we find that the corresponding correlators there do not vanish, and do not conserve the analogue of Liouville momentum, precisely because matrix models are formulated naturally at nonzero μ.

A first guess would be to try and treat the cosmological term as a marginal perturbation of the theory, and to compute in the Hilbert space of the unperturbed theory treating this term perturbatively. This is obviously not going to work in general, as we have already seen that the theory is very singular in the $\mu \to 0$ limit, so that correlators are not generally proportional to positive integer powers of μ. An ingenious trick to deal with this situation[10][11][12] is to formally continue the various parameters of the theory (the central charge of the matter sector, background charges of the fields etc.) to values where the number of cosmological operator insertions becomes a positive integer. Computing the correlator in that situation and then continuing the theory back to the original physical values of the parameters, constitutes a prescription which is surprisingly successful.

Some interesting correlators however, particularly in genus 0, are proportional to positive powers of μ, in which case the computation is simpler, though it still requires the evaluation of complicated multiple integrals and a careful treatment of possible infinities.

To illustrate, suppose we want to compute the four-point function of tachyons in the $+$ sector, namely the operators $Y^+_{s,s}$ and $Y^+_{s,-s}$. We first rename these fields as follows:

$$T_k = \int d^2z e^{\frac{i}{\sqrt{2}}kX} e^{\frac{1}{\sqrt{2}}(2-|k|)} \tag{1.32}$$

where k will take positive and negative integer values. Note that the tachyons have opposite "chirality" (in the 2D spacetime sense) for $k > 0$ and $k < 0$. Now, consider the genus-0 correlator $\langle T_{k_1} T_{k_2} T_{k_3} T_{k_4} \rangle$. Clearly, the matter momentum must be conserved, so there will be an overall factor of $\delta_{\sum_i k_i}$ in the answer. The total Liouville momentum in this correlator determines the power α of μ in the correlator by:

$$\sum_{i=1}^{4} (\frac{1}{\sqrt{2}}(2 - |k_i|)) = 2\sqrt{2} - \sqrt{2}\alpha \tag{1.33}$$

which leads to

$$\alpha = |k_1| + |k_2| + |k_3| - 2 \tag{1.34}$$

Factoring out the power of μ and the momentum-conserving δ-function, one finds through free-field techniques[11]:

$$\langle T_{k_1} T_{k_2} T_{k_3} T_{-k_1-k_2-k_3} \rangle = -\frac{1}{2}[|k_1 + k_2| + |k_1 + k_3| + |k_2 + k_3| - 2] \tag{1.35}$$

which has been obtained directly from matrix models[13]. (Note that in the case of genus-0 correlators there is no radius-dependence.)

Although genus-0 amplitudes and the genus-1 partition function have been calculated using the techniques mentioned above, the computations generally become quite intractable at higher genus. This is really a pity since the answers from matrix models are typically quite simple expressions - but then, that is a measure of the limitations of the standard continuum descriptions of string theory: amplitudes are defined in terms of correlations of local operators, the result of which is to be integrated over moduli space to get physically meaningful numbers. Only a manifestly topological description of string theory could bypass this and produce simple answers for amplitudes in a direct way. We will see in subsequent sections that some success has been achieved in this direction.

2. Topological Invariance in String Theory

2.1. Arbitrary Backgrounds

The conventional bosonic Polyakov string, after gauge-fixing and introduction of ghosts, has a set of chiral fields which look very similar to the generators of the twisted (topological) $N = 2$ superconformal algebra. Recall that this algebra has the four generators $T(z)$, $G^+(z)$, $G^-(z)$ and $J(z)$, of spins $2, 1, 2$ and 1 respectively. Of these, the generators $G^\pm(z)$ are fermionic, while the other two are bosonic. They satisfy the OPE algebra:

$$\begin{aligned}
T(z)T(w) &\sim \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} \\
T(z)G^\pm(w) &\sim \frac{1}{2}\frac{(3\mp 1)G^\pm(z)}{(z-w)^2} + \frac{\partial G^\pm(w)}{(z-w)} \\
T(z)J(w) &\sim \frac{c^T/3}{(z-w)^3} + \frac{J(w)}{(z-w)^2} + \frac{\partial J(w)}{(z-w)} \\
J(z)G^\pm(w) &\sim \pm\frac{G^\pm(w)}{(z-w)} \\
J(z)J(w) &\sim \frac{c^T/3}{(z-w)^2} \\
G^\pm(z)G^\pm(w) &\sim 0 \\
G^+(z)G^-(w) &\sim \frac{c/3}{(z-w)^3} + \frac{J(w)}{(z-w)^2} + \frac{T(w)}{(z-w)}
\end{aligned} \tag{2.1}$$

Here, the constant c^T is the topological central charge of the algebra. Note that the Virasoro subalgebra generated by $T(z)$ has vanishing central charge.

Now, in any bosonic string background, we have the four chiral fields

$$\begin{aligned}
T(z) &= T^X(z) + T^\phi(z) + T^{\text{ghost}}(z) \\
G^+(z) &= c(z)(T^X(z) + T^\phi(z)) + \frac{1}{2} : c(z)T^{\text{ghost}}(z) : \\
G^-(z) &= b(z) \\
J(z) &=: c(z)b(z) :
\end{aligned} \tag{2.2}$$

which have the same spins and statistics as the topological algebra generators in Eq.(2.1) above. Moreover, precisely because we have gauge-fixed and introduced the Liouville and ghost fields, the total stress-energy tensor of this system satisfies a Virasoro algebra with zero central charge, just like the one in the topological algebra. However, unfortunately the generators in Eq.(2.2) form an algebra which does not close, so they cannot (yet) be thought of as topological symmetry generators. Remarkably, it is possible to modify them by total derivative terms in such a way as to make them satisfy precisely the topological algebra[14].

This is possible only if the background admits at least one $U(1)$ current $\tilde{J}(z)$. This current may be anomalous, in which case the Lagrangian of the theory on a curved world-sheet will contain a term proportional to

$$S^{\text{anomalous}} = \frac{Q_{\tilde{J}}}{8\pi} \int d^2\xi \omega_a(\xi) \tilde{J}^a(\xi) \tag{2.3}$$

The coefficient $Q_{\tilde{J}}$ of this term determines the anomaly. It follows that the stress-energy tensor of this $U(1)$ current is

$$T^{\tilde{J}}(z) = -\frac{1}{2}\tilde{J}(z)\tilde{J}(z) + \frac{Q_{\tilde{J}}}{2}\partial\tilde{J}(z) \tag{2.4}$$

Now in any background which admits such a $U(1)$ current, the following modification of the generators in Eq.(2.2) causes them to satisfy the topologically twisted $N = 2$ algebra in Eq.(2.1), without producing any extra terms:

$$\begin{aligned} G^+(z) \rightarrow G^+(z) &+ \frac{1}{2}(-Q_{\tilde{J}} \pm \sqrt{Q_{\tilde{J}}^2 - 8})\partial(c(z)\tilde{J}(z)) \\ &+ \frac{1}{2}(3 + \frac{1}{2}Q_{\tilde{J}}(-Q_{\tilde{J}} \pm \sqrt{Q_{\tilde{J}}^2 - 8}))\partial^2 c(z) \\ J(z) \rightarrow J(z) &- \frac{1}{2}(-Q_{\tilde{J}} \pm \sqrt{Q_{\tilde{J}}^2 - 8}) \end{aligned} \tag{2.5}$$

The coefficients of the extra terms are a little complicated, but they are determined entirely by the anomaly of the $U(1)$ current $\tilde{J}(z)$. That the shifted generators satisfy the topological algebra can be verified by explicit computation. The same computation also reveals that the topological central charge of the algebra, c^T, is

$$c^T = 3(3 + \frac{1}{2}Q_{\tilde{J}}(-Q_{\tilde{J}} \pm \sqrt{Q_{\tilde{J}}^2 - 8})) \tag{2.6}$$

An obvious special case is associated with the value $Q_{\tilde{J}} = 0$, corresponding to a non-anomalous $U(1)$ current. This gives a topological central charge $c^T = 9$, which is known to be a very special value in topological field theories. This can be thought of as the "critical" value of central charge in those theories, for which the partition function in genus h is well-defined without the need for extra insertions. This value is also associated to the Calabi-Yau topological sigma models, which have many beautiful properties both in a mathematical sense and in the context of superstring theory. Indeed, one may ask if the result above indicates a relation between conventional (bosonic) string backgrounds having a non-anomalous current, and Calabi-Yau manifolds. This interesting question has yet to be addressed.

Continuing with the special case of a non-anomalous $U(1)$ current, the simplest case one can imagine is

$$\tilde{J}(z) = \partial X(z) \tag{2.7}$$

where $X(z, \bar{z})$ is a free scalar field with no charge at infinity. Such a field may be interpreted as a flat spacetime direction in the string background. Because of the sign ambiguity in some of the coefficients in Eq.(2.5), there are always at least two topological algebras for a given free scalar field, related by the Z_2 symmetry $X \to -X$, which corresponds to spacetime parity (or time-reversal). Thus the choice of a particular algebra breaks manifest parity (which has intriguing consequences as we will see later) although of course the underlying theory may be parity-invariant.

The most obvious particular example of this is the critical bosonic string which offers 26 free scalar fields from which we can define the current $\tilde{J}(z)$ as in Eq.(2.7). Thus the most general current in that theory is $\tilde{J}(z) = \epsilon_\mu \partial X^\mu$, which points in an arbitrary direction in (25+1) dimensional spacetime. This breaks not only parity but also manifest Lorentz invariance, which may be one reason why this discovery has not so far proved useful in the study of critical string theory.

2.2. $c < 1$ Matter

Let us briefly examine what consequences the above ideas have for the theory of $c < 1$ minimal matter coupled to gravity. It has been known for some time that these theories can be described by coupling certain specially chosen topological matter systems to topological gravity[15].

First, consider the (p, q) minimal models for the special case $(p, q) = (1, k+2)$ where k is some positive integer. These models have a Virasoro central charge given by

$$c_{\text{matter}} = 1 - 6\frac{(k+1)^2}{(k+2)} \tag{2.8}$$

When we couple them to gravity, the corresponding Liouville theory must have a complementary value of central charge:

$$c_L = 26 - c_{\text{matter}} = \frac{(2k+7)(3k+8)}{k+2} \tag{2.9}$$

This in turn fixes the background charge of the Liouville field $\phi(z,\bar{z})$ to be

$$Q_L = \sqrt{\frac{c_L - 1}{3}} = \sqrt{\frac{2}{k+2}}(k+3) \tag{2.10}$$

Now, this class of matter-Liouville systems (along with the reparametrization ghosts) is believed to correspond to "unperturbed" $k+1$-matrix models, while multicritical perturbations of these matrix models are supposed to reproduce the $(p, k+2)$ minimal models coupled to gravity. This identification follows from the matching of critical exponents in the two cases and from some explicit calculations which can be performed in special examples.

But additionally, there is a description of the same systems in terms of topological minimal matter coupled to topological gravity. Here, one starts with the unitary $N=2$ minimal models, labelled by a positive integer k and having central charge $c = 3k/(k+2)$. After twisting to make the theory topological, one finds as usual that the central charge migrates to the topological central charge c^T, which therefore equals $3k/(k+2)$. Now it has been shown that coupling this system to topological gravity (which does not change the topological central charge) is equivalent to the k-matrix model, while suitable marginal perturbations of the topological theory are equivalent to multicritical points of the matrix model.

It follows, indirectly, that the topological description of the $(1, k+2)$ minimal models coupled to gravity has

$$c^T = \frac{3k}{k+2} \tag{2.11}$$

But now we have an independent way to justify the same conclusion. The results of the last section tell us that these models, when coupled to 2D gravity, have a twisted $N=2$ superconformal algebra, whose topological central charge c^T depends only on the background charge associated to the $U(1)$ current which we use to modify the algebra generators. For $c<1$ minimal models there is a unique $U(1)$ current, arising from the Liouville field: $\tilde{J}(z) = \partial\phi(z)$. Now from Eqs.(2.10) and (2.6) we find

$$\begin{aligned} Q_{\tilde{J}} &= \sqrt{\frac{2}{k+2}}(k+3) \qquad \text{or} \quad -2k-3 \\ c^T &= \frac{3k}{k+2} \end{aligned} \tag{2.12}$$

confirming that the topological description of $c<1$ minimal models coupled to gravity has the expected value of c^T, at least for one choice of the sign in Eq.(2.6)[16].

Had we not already known the topological formulation of these models, the result above would have been a guide to look for the right one (although of course identifying the topological central charges is only a necessary and far from sufficient condition to identify two theories). This is indeed the situation for the case of $c = 1$ backgrounds, where a topological description has resisted discovery for a long time, and only recently, following the kind of logic described above, a candidate topological model has been found.

2.3. $c = 1$ Matter

In the $c = 1$ string background, there are two distinct choices of the $U(1)$ current with which to form a topological algebra. The choice $\tilde{J} = \partial X$ gives $c^T = 9$, as we have seen, since the X field has no background charge. The other choice, which is similar to what is done for $c < 1$ models, is to take $\tilde{J} = \partial\phi$. The latter looks more natural as it seems to connect the $c = 1$ background smoothly to the $c < 1$ case, but it does not actually turn out to be very successful. One reason is that the cosmological perturbation renders the Liouville field interacting, so that $\partial\phi$ is no longer holomorphic in this situation. In contrast, choosing $\tilde{J} = \partial X$ (and hence $c^T = 9$) leads to a wealth of interesting results and greatly enhances our understanding of the $c = 1$ string[17], as we will see below.

With this choice, we can first of all ask what is the classification of the physical states with respect to the topological algebra. We will work at the self-dual radius. Consider now the large collection of physical states that were discovered in Section 1. The question is, are they primaries or secondaries of the topological algebra?

It can be shown that the (chiral) primaries of the topological algebra are all the ghost-number 1 vertex operator-type fields of the form:

$$\Phi^{\pm}_{s,n}(z) = c(z)e^{i\sqrt{2}nX(z)}e^{\sqrt{2}(1\mp s)\phi} \tag{2.13}$$

and also the ghost-number 2 fields $\partial c(z)\Phi^{\pm}_{s,n}(z)$. These have conformal dimensions $s^2 - n^2$ as one can easily check. One can think of these as "off-shell tachyons", except for the case $n = \pm s$ for which the tachyons are on-shell.

Now secondaries of the topological algebra above these will have various different ghost numbers since $G^{\pm}(z)$ carry ghost number ± 1. Out of all these states, those of conformal dimension different from zero cannot of course be in the cohomology, by well-known theorems. Those of dimension zero, on the other hand, can be seen to contain in particular the states $Y^{\pm}_{s,n}$ of ghost-number 1, (with $n \neq \pm s$) so that these discrete states are secondaries of the topological algebra. Next, the states of ghost-number 0, $\mathcal{O}_{s,n}$, are also secondaries, but with the added property that they are also primary. Thus these are both primary and secondary of the topological algebra, and they are non-vanishing in the free-field Fock space precisely because of the presence of background charges.

As an example, consider the case $s = 1, n = 0$ with superscript $+$. The primaries are $c(z)$ and $c(z)\partial c(z)$, both of dimension -1. One finds in particular the state $c(z)\partial X(z)$ as a secondary at level one above $c(z)$. This is just $Y^{+}_{1,0}$, the chiral part of the radius-changing operator (a zero-momentum "remnant", in the sense discussed above, of the chiral part of the graviton). Another secondary at level one is the identity operator 1, which is also primary, obviously, and this is the first example of a ground-ring operator, namely $\mathcal{O}_{0,0}$. Other details of this classification are relatively straightforward to fill in.

The main lesson of this is that the wealth of discrete states in the $c = 1$ string at the self-dual radius can be understood as secondaries of the twisted $N = 2$ algebra above the tachyon-like primaries. This is an explanation of the existence of these states, which were originally considered rather mysterious.

3. Topological Formulation of the $c = 1$ String

3.1. The Coset Model

The most important consequence of the topological invariance discussed in the previous section, in the context of $c = 1$ string theory, is that it helps us to find a different formulation of this theory which is manifestly topological, and in which amplitudes and the partition function (even in higher genus) can be explicitly computed in many cases. This means that we have a continuum formulation of this string theory which is apparently as powerful as the very successful matrix-model description.

The topological model for which we are searching cannot be predicted from some general principles, but we do know from the discussion of the previous section that it must possess a topological symmetry with central charge $c^T = 9$. We will look for this model among a general class of $N = 2$ supersymmetric CFT's, the Kazama-Suzuki models[18]. The construction of these models is based on the following facts. Let us start with an $N = 1$ supersymmetric WZW model, based on a group G, and gauge the adjoint action of a subgroup H. Then, if and only if the coset G/H is a Kähler manifold, the model so obtained has $N = 2$ supersymmetry.

Applying this idea to the case where $G = SL(2,R)$ and $H = U(1)$, we find a series of models with central charge $c = 3k/(k-2)$, where k is the level of the $SL(2,R)$ current algebra. After twisting the model so obtained to make it topological, one finds as usual that the central charge of the theory becomes zero, but there is a topological central charge c^T which has the same value as the central charge before twisting. Accordingly, if we want $c^T = 9$ in this class of models, the unique choice is to take level $k = 3$.

Let us now look at the KS model at level 3 in some detail. The stress-energy tensor of the supersymmetric G model is, as usual,

$$T^{(G)}(z) = :J^+(z)J^-(z): + (J^3(z))^2 - \frac{1}{2}(b(z)\partial c(z) + c(z)\partial b(z)) \tag{3.1}$$

where J^+, J^-, J^3 are the spin-1 $SL(2,R)$ currents, b, c are spin-$\frac{1}{2}$ fermions which serve to supersymmetrise the theory, and the coefficient of the first term is unity precisely at $k = 3$.

One can write the stress-energy tensor of the coset model as the difference of the (supersymmetric) G and H stress-energy tensors. However, for our purposes it is more convenient to use an alternative formulation in which to the G stress-energy we *add* a gauge contribution and then pass to a BRS cohomology on this larger space. Thus we have

$$T(z) = T^{(G)}(z) + T^{(\mathrm{gauge})}(z) \tag{3.2}$$

We will write this out explicitly below, but first let us see what the other generators of the $N = 2$ algebra look like. Following Kazama-Suzuki, we find (at $k = 3$)

$$\begin{aligned} G^+(z) &= c(z)J^+(z) \\ G^-(z) &= c(z)J^-(z) \\ J_{N=2}(z) &= 3:c(z)b(z): -\ 2J^3(z) \end{aligned} \tag{3.3}$$

Here we have labelled the $U(1)$ current of the $N = 2$ algebra explicitly to distinguish it from other $U(1)$ currents in the problem.

So far, this is an untwisted $N = 2$ superconformal algebra, with central charge $3k/(k-2)$ as mentioned earlier. Now we render it topological through the twist

$$T(z) \quad \to \qquad T(z) + \frac{1}{2}\partial J_{N=2}(z) \tag{3.4}$$

As a result, the final stress-energy tensor (to be compared with Eq.(3.2)) is

$$T(z) = :J^+(z)J^-(z): + \ (J^3(z))^2 - \partial J^3(z) - 2b(z)\partial c(z) + c(z)\partial b(z) + T^{(\mathrm{gauge})}(z) \tag{3.5}$$

This twist has a rather miraculous consequence. The spins of all the fields in $T^{(G)}$ have changed, and we now have an $SL(2,R)$ current multiplet (J^+, J^3, J^-) of spins (2,1,0) respectively. Moreover, the free fermions (b,c) have changed their spins from $(\frac{1}{2},\frac{1}{2})$ to $(2,-1)$. The currents are now reminiscent of those in the KPZ description of $c=1$ string theory, where they described the gravitational or Liouville sector, while the fermions have become identical to the usual ghost system of bosonic string theory. The central charges are found to be $c=27$ for the twisted currents and of course $c=-26$ for the fermions.

Now we add in the $U(1)$ gauge sector of the theory by making the gauge choice

$$A_z(z) = \partial X(z), \qquad \bar{A}_{\bar{z}}(\bar{z}) = -\bar{\partial}\bar{X}(\bar{z}) \tag{3.6}$$

and defining the free scalar field $X(z,\bar{z}) = X(z) - \bar{X}(\bar{z})$. Fixing the gauge in this way requires the introduction of a pair of fermionic ghosts (B,C) of spins $(1,0)$, hence we get

$$T^{(\mathrm{gauge})}(z) = -\frac{1}{2}\partial X(z)\partial X(z) - B(z)\partial C(z) \tag{3.7}$$

along with the $U(1)$ BRS charge

$$Q_{U(1)} = \int c(z)(J^3(z) - :c(z)b(z): - \frac{i}{\sqrt{2}}\partial X(z)) \quad + \mathrm{c.c} \tag{3.8}$$

So far, everything except the current-algebra sector has been reduced to free fields. We now choose to represent the currents also in terms of free fields, using the Wakimoto representation:

$$\begin{aligned} J^+(z) &= \beta(z)\gamma(z)^2 - \sqrt{2}\gamma(z)\partial\phi(z) + 3\partial\gamma(z) \\ J^3(z) &= \beta(z)\gamma(z) - \frac{1}{\sqrt{2}}\partial\phi(z) \\ J^-(z) &= \beta(z) \end{aligned} \tag{3.9}$$

where (β,γ) are commuting ghosts and ϕ is a free scalar field. Since we know the spins of the currents (after twisting of course), it is easy to deduce that the ghosts have spins $(0,1)$ respectively. This means they contribute a total central charge of $+2$. But the total central charge of the twisted current algebra is 27, so we conclude that the field ϕ has a central charge $c_\phi = 25$, which means it must have a background charge $Q_\phi = 2\sqrt{2}$. Thus, ϕ is clearly identical to the Liouville field in $c=1$ string theory.

To summarise, the full Hilbert space of our topological theory is

$$\begin{array}{lccccccccc} \mathcal{H}: & \mathcal{H}_\phi & \oplus & \mathcal{H}_X & \oplus & \mathcal{H}_{b,c} & \oplus & \mathcal{H}_{B,C} & \oplus & \mathcal{H}_{\beta,\gamma} \\ \text{spins}: & (0) & & (0) & & (2,-1) & & (1,0) & & (0,1) \\ \text{central charge}: & 25 & & 1 & & -26 & & -2 & & 2 \end{array} \tag{3.10}$$

while the physical Hilbert space is obtained from this by taking the quotient with the two BRS charges

$$\begin{aligned} G^+ &= \int cJ^+ = \int c(\beta\gamma^2 - \sqrt{2}\gamma\partial\phi + 3\partial\gamma) \\ Q_{U(1)} &= \int C(\beta\gamma - cb - \partial X^-) \end{aligned} \tag{3.11}$$

(We have defined $X^{\pm} = \frac{1}{\sqrt{2}}(\phi \mp iX)$.)

Note the remarkable fact that the first three Hilbert spaces above are isomorphic to the Hilbert space of the conventional $c = 1$ string quantised in the DDK formalism. The Liouville field arises from the Wakimoto representation for the $SL(2, R)$ current algebra, the $c = 1$ matter scalar field comes from the $U(1)$ gauge field, and the ghosts are just the fermions of the original supersymmetric WZW model, after twisting. The remaining two Hilbert spaces consist of first-order pairs of the same spins and opposite statistics, so it is quite reasonable to expect that they cancel out in some sense. Thus, we have found a very likely candidate for a manifestly topological description of $c = 1$ string theory.

To make this connection completely convincing, one should compute the double cohomology of the BRS charges above on the full Hilbert space. The computation has been described in detail in Ref.[17] and will not be repeated here. The result is a collection of discrete states which are isomorphic to those described in the first few sections above, with tachyons, discrete states of ghost-number 1, ground-ring operators of ghost-number 0 and so on. The only difference between the explicit representatives of the cohomology in the DDK and the KS formulations is that the latter has excitations of the extra β, γ and B, C ghost systems as well. For example:

$$\begin{aligned} Y^+_{1,1} &= \beta^{-1} c e^{\sqrt{2}iX} \\ Y^+_{1,0} &= -c\beta^{-1}\partial\beta + \sqrt{2}ic\partial X \\ Y^+_{1,-1} &= \beta c e^{-\sqrt{2}iX} \end{aligned} \tag{3.12}$$

These three operators, on stripping off the c ghost and integrating to form charges, generate an $SU(2)$ subalgebra of the W_∞ algebra, which is nothing but the enhanced $SU(2)$ symmetry of $c = 1$ CFT at the self-dual radius. Note that on setting $\beta = 1$, these reduce to the corresponding Y^+ operators of the DDK formalism. This property is true of *all* the discrete states of the KS theory above.

3.2. Correlators and Partition Function

In order to show that the $k = 3$ coset model is exactly solvable, we need to investigate the available techniques to solve models of this kind. Let us first recall the situation with the $N = 2$ supersymmetric minimal models labelled by a positive integer k. As we discussed earlier, after coupling to topological gravity, these are equivalent to the $c < 1$ minimal models (unitary and non-unitary) coupled to dynamical 2D gravity.

A powerful technique for the solution of this class of theories comes from their formulation as gauged supersymmetric WZW models, the Kazama-Suzuki (KS) coset models based on $SU(2)_k/U(1)$. It has been shown[19] that algebraic-geometry techniques can be brought to bear on this problem, resulting in expressions for arbitrary correlators in arbitrary genus. These are described as integrals of products of Chern classes of certain bundles over moduli space.

The gravitational primaries in the KS theory coupled to topological gravity are described by $1, g_{11}, g_{11}^2, \ldots, g_{11}^k$ (where g_{ab} is the $SL(2,R)$-valued matrix field of the WZW model). Let us denote the primary g_{11}^r by U_r. Gravitational secondaries are obtained by multiplying these with the usual fields σ_n of pure topological gravity. The selection rules for the correlator $\langle \sigma_{n_1}(U_{r_1}) \ldots \sigma_{n_N}(U_{r_N}) \rangle$ are

$$\sum_{i=1}^{N} \left(\frac{r_i}{k+2} - 1\right) + \sum_{i=1}^{N} n_i = (2g-2)\frac{k+3}{k+2} \tag{3.13}$$

This picture can be suitably modified for our purposes[17]. Since in many ways the $SL(2,R)_k/U(1)$ coset is like the coset of $SU(2)$ at level *minus* k, the model we have described in the previous section should be a "continuation" of the $SU(2)/U(1)$ KS theory with $k = -3$. This continuation may not be rigorously and uniquely defined, but we will see below that sensible results can be extracted by making the most naive continuation with the minimum of assumptions.

To start with, let us restrict to gravitational primaries and set $k = -3$ in the selection rule above. The resulting equation is genus-independent, and remarkably simple:

$$\sum_{i=1}^{N} (r_i + 1) = 0 \tag{3.14}$$

This gives us the first clue about the identification of these fields with the physical fields in the $c = 1$ string (at the self-dual radius). The tachyons in this string theory have discrete momenta, labelled by integers k_i, satisfying momentum conservation in every genus:

$$\sum_{i=1}^{N} k_i = 0 \tag{3.15}$$

Thus, we are tempted to make the identification $k_i = r_i + 1$, as a result of which we would claim that the tachyons in the Lagrangian KS formulation appear as gravitational primaries:

$$T_k = g_{11}^{k-1} \tag{3.16}$$

Note that tachyons with all positive and negative integer momenta are required, so we must allow fields $U_r = g_{11}^r$ for all positive and negative integer values of r. This is not as surprising as it might seem if we think of $SL(2, R$ as $SU(1,1)$, for which the matrix field g_{ab} satisfies

$$|g_{11}|^2 - |g_{12}|^2 = 1 \tag{3.17}$$

so that g_{11} can never be zero.

We now have enough information to compute correlators in the proposed topological description of $c = 1$ string theory and compare with known results from matrix models. Starting with genus 0, the 4-point function of gravitational primaries was explicitly computed for $k > 0$ in Ref.[]. The result is

$$\begin{aligned}\langle U_{r_1} U_{r_2} U_{r_3} U_{r_4} \rangle_{g=0} = \frac{1}{2}&(\min(q_1 + q_2, q_3 + q_4) + \min(q_1 + q_3, q_2 + q_4)\\ &+ \min(q_1 + q_4, q_2 + q_3)) - \frac{k+1}{k+2}\end{aligned} \tag{3.18}$$

where $q_i = r_i/(k + 2)$ are the $U(1)$ charges of the fields. Inserting $k = -3$ and using the identification in Eq.(3.16), we easily find

$$\langle T_{k_1} T_{k_2} T_{k_3} T_{k_4} \rangle_{g=0} = -\frac{1}{2}[|k_1 + k_2| + |k_1 + k_3| + |k_2 + k_3| - 2] \tag{3.19}$$

which is just the same as Eq.(1.35).

Next we consider the case of higher genus. Here, two basic results are known. From Ref.[], we know that the genus-h partition function of the $SU(2)_k/U(1)$ KS model, when continued to $k = -3$, is precisely the (virtual) Euler characteristic $\hat{\chi}_h$ of the moduli space of genus-h Riemann surfaces with no punctures. Now, from the matrix model compactified at the self-dual radius, we have an explicit result for the genus-h partition function (subject to the assumption that the nonsinglet sector can be ignored), and the answer is

$$Z_g = \frac{B_{2h}}{2h(2h-2)} \tag{3.20}$$

where B_{2h} are the Bernoulli numbers. Remarkably, it has been shown[] that this is just the virtual Euler characteristic of the moduli space of genus-h Riemann surfaces. So, our topological formulation of $c = 1$ string theory is powerful enough to reproduce the genus-h partition function, directly in a continuum approach. It is easy to check that genus-h correlation functions of the cosmological operator ($T_0 = g_{11}^{-1}$) are also obtained correctly in our approach. Correlators of arbitrary (discrete) tachyons in higher genus have yet to be computed explicitly.

4. Conclusions

We have seen that non-critical string theories, in both $c < 1$ and $c = 1$ backgrounds, have more effective descriptions as topological field theories on the world-sheet. Moreover, even the critical bosonic string is topological in a very precise sense – although in this case it is not so clear what is the utility of this information.

Until now, it has generally been believed that the Polyakov description of the first-quantised string is the basic one, while other (often more successful) descriptions such as matrix models and topological field theories are very special to certain backgrounds. Perhaps the time has come to revise this opinion and to formulate strings in topological language from the start. This might lead to a new and more powerful description of the configuration space of string theory, and may eventually provide a resolution of the long-standing open problem – to understand background-independent, non-perturbative string theory.

References

[1] F. David, Mod. Phys. Lett. A3 (1988) 1651;
J. Distler and H. Kawai, Nucl. Phys. B321 (1989) 509.

[2] S. Mukherji, S. Mukhi and A. Sen, Phys. Lett. 266B (1991) 337;
P. Bouwknegt, J. McCarthy and K. Pilch, Comm. Math. Phys. 145 (1992) 541;
B. Lian and G. Zuckerman, Phys. Lett. 266B (1991) 21.

[3] E. Witten and B. Zwiebach, Nucl. Phys. B377 (1992) 55.

[4] A.M. Polyakov, Mod. Phys. Lett. A6 (1991) 635.

[5] P. Ginsparg, Nucl. Phys. B295 (1988) 153;
G. Harris, Nucl. Phys. B300 (1988) 588.

[6] E. Witten, Nucl. Phys. B373 (1992) 187.

[7] I. Klebanov and A.M. Polyakov, Mod. Phys. Lett. A6 (1991) 3273.

[8] M. Li, Nucl. Phys. B382 (1992) 242.

[9] S. Mahapatra, S. Mukherji and A. Sengupta, Mod. Phys. Lett. A7 (1992) 3119.

[10] A. Gupta, S. Trivedi and M. Wise, Nucl. Phys. B340 (1990) 475;
M. Goulian and M. Li, Phys. Rev. Lett. 66 (1991) 2051.

[11] P. DiFrancesco and D. Kutasov, Phys. Lett. 261B (1991) 385.

[12] V. Dotsenko, Mod. Phys. Lett. A6 (1991) 3601;
Y. Kitazawa, Phys. Lett. 265B (1991) 262;
D. Ghoshal and S. Mahapatra, Mod. Phys. Lett. A8 (1993) 197;
S. Govindarajan, T. Jayaraman, V. John and P. Majumdar, Phys. Rev. D48 (1993) 839.

[13] G. Moore, Nucl. Phys. B368 (1992) 557;
G. Mandal, A. Sengupta and S. Wadia, Mod. Phys. Lett. A6 (1991) 1465;
K. Demeterfi, A. Jevicki and J. Rodrigues, Nucl. Phys. B362 (1991) 173;
J. Polchinski, Nucl. Phys. B362 (1991) 125.

[14] B. Gato-Rivera and A. Semikhatov, Phys. Lett. 288B (1992) 38.

[15] K. Li, Nucl. Phys. B354 (1991) 711.

[16] M. Bershadsky, W. Lerche, D. Nemeschansky and N. Warner, Nucl. Phys. B401 (1993) 304.

[17] S. Mukhi and C. Vafa, Nucl. Phys. B407 (1993) 667.

[18] Y. Kazama and H. Suzuki, Nucl. Phys. B321 (1989) 232.

[19] E. Witten, Nucl. Phys. B371 (1992) 191.