A COUNTER-EXAMPLE IN GEOMETRY OF NUMBERS

by V. C. Dumir *and* R. J. Hans-Gill, *Department of Mathematics, Panjab University, Chandigarh 14*

(Communicated by R. P. Bambah, F.N.A.)

(*Received 25 January 1971*)

An example of a symmetrical star domain S is given such that it does not contain any star domain which is minimal for coverings in the class of symmetrical star domains with covering constant $C(S)$. In the process it is also shown that the analogue of Farey's result for convex domains does not hold in the class of symmetrical star domains.

Introduction

§ 1. Let R_n be the n-dimensional Euclidean space. Let K be a closed set in R_n. A lattice Λ in R_n is said to be a covering lattice for K if every point of R_n lies in some translate of K through a point of Λ. The covering constant $C(K)$ of K is defined as the least upper bound of the determinants of all covering lattices of K.

Let $\mathfrak{C}$ be a class of sets in R_n. A set $S \in \mathfrak{C}$ is said to be $\mathfrak{C}$-minimal (for coverings) if $T \in \mathfrak{C}$, $T \subsetneq S$ implies $C(T) < C(S)$. An interesting question is whether in a given class $\mathfrak{C}$ every set $S \in \mathfrak{C}$ contains a $\mathfrak{C}$-minimal set with equal covering constant.

A set S in R_n is said to be a star set if

(i) $S \neq \phi$

and

(ii) if $X \in S$, then $\lambda X \in S$ for $0 \leqslant \lambda \leqslant 1$.

A star set S which has origin in the interior is called a star body if every ray through the origin meets the boundary in atmost one point.

We shall use the following notation:

$\mathfrak{T}$ = class of bounded closed star sets

$\mathfrak{T}'$ = class of bounded O-symmetrical closed star sets

$\mathfrak{S}$ = class of bounded closed star bodies

$\mathfrak{S}'$ = class of bounded O-symmetrical closed star bodies

$\mathfrak{K}$ = class of closed convex bodies containing O in the interior

$\mathfrak{K}'$ = class of O-symmetrical closed convex bodies.

Hans (1967) proved that the answer to the above question is in the affirmative in the classes $\mathfrak{T}$ and $\mathfrak{K}$. The proofs show that analogous result also holds in the class $\mathcal{T}'$ and $\mathcal{K}'$. Here we shall prove that in the classes $\mathcal{S}$ and $\mathcal{S}'$ the answer to the above question is in the negative. In § 2 we show that the bounded O-symmetrical star domain S (shown in Fig. 1) does not contain any $\mathcal{S}$-minimal or $\mathcal{S}'$-minimal star domain with the same covering constant. In § 3 we determine a $\mathcal{T}$-minimal set contained in S with covering constant equal to $C(S)$.

The analogous concept of irreducibility has been studied by various authors, e.g. Mahler (1946, 1947), Rogers (1947, 1952), Woods (1959) and Ollernshaw (1953). In particular, Mahler (1947) asked whether every bounded member of $\mathcal{S}$ contains an $\mathcal{S}$-irreducible member with the same critical determinant. Rogers (1952) constructed an example to show that the answer to Mahler's question is in the negative. However, by slightly modifying the definition of the critical determinant of a set he proved that the result is true in the class $\mathcal{T}'$. An alternative proof of this was given by Hans (1967).

The star domain S of Fig. 1 also provides a counter-example to a generalization of Farey's result about the covering constant of a convex domain in R_2. Farey (1950) proved that every convex domain K contains a space-filling convex domain which has the same covering constant as that of K (it is the largest symmetrical 'hexagon' contained in K). In § 2 we show that this result does not generalize to O-symmetrical star domains by showing that this star domain S does not contain a space-filling star domain with covering constant equal to $C(S)$.

The covering constant and all maximal covering lattices of S were determined by Dumir and Hans-Gill (1972*a*). This domain S also served as a counter-example for some results in the lattice double covering and lattice double packing (Dumir and Hans-Gill 1972*b*) by star domains.

§ 2. Let $P_1 = (3,\ 0)$, $P_2 = (15,\ 3)$, $P_3 = (15,\ 15)$, $P_4 = (3,\ 15)$, $P_5 = (0,\ 3)$. Let S be the set bounded by the broken line segment P_1, P_2, P_3, P_4, P_5 and its reflections in the origin and the coordinate axes. This is the region shown in Fig. 1. S is a star body symmetrical about the axes and the lines $y = \pm x$.

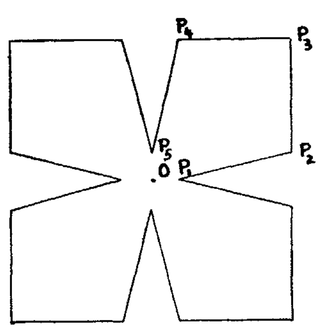

Fig. 1

Theorem A—The covering constant of S is

$$C(S) = 491{\cdot}04.$$

The maximal covering lattices of S are of the type $\Omega\Gamma$ where Ω is an automorphism of S and Γ is the lattice generated by

$$A = \left(-\frac{66}{5}, \frac{66}{5}\right), \quad B = \left(12, \frac{126}{5}\right).$$

This is theorem 3 of Dumir and Hans-Gill (1972*a*).

Here we prove:

Theorem 1—The star domain S does not contain a space-filling star domain with same covering constant as that of S.

PROOF: Let T be any closed star domain contained in S with $C(T) = C(S)$. We shall prove that T cannot be space-filling. Any maximal covering lattice of T is also a maximal covering lattice of S. Hence by Theorem A it is of the type $\Omega\Gamma$ where Ω is an automorphism of S and Γ is the lattice generated by $A = \left(-\frac{66}{5}, \frac{66}{5}\right)$ and $B = \left(12, \frac{126}{5}\right)$. If $(T, \Omega\Gamma)$ is a space-filling, then $(\Omega^{-1}T, \Gamma)$ will be a space-filling. On replacing T by $\Omega^{-1}T$ we can suppose that Γ is a maximal covering lattice of T. We claim that this cannot be space-filling.

Consider the covering of the plane by the sets $S+C$, $C \in \Gamma$. Let $P = \left(-\frac{51}{5}, \frac{66}{5}\right)$, $Q = (-3, 15)$ and $R = \left(-3, \frac{57}{5}\right)$. It is easy to verify that the open triangle PQR is covered by S and no other set $S+C$, $C \in \Gamma$, $C \neq 0$ (*see* Fig. 2). Since $T \subset S$, the sets $T+C$, $C \in \Gamma$, $C \neq 0$ do not meet the

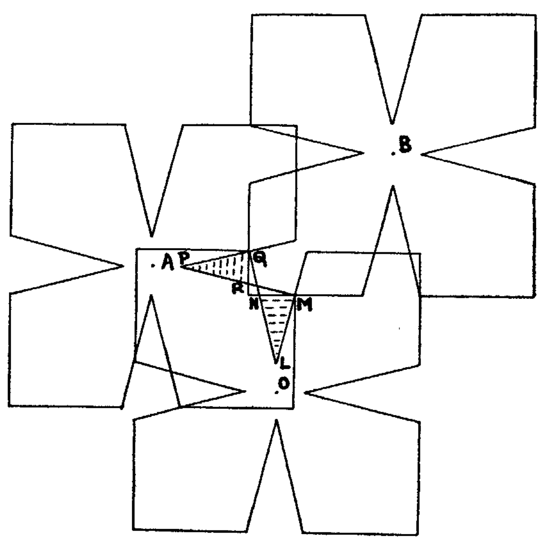

FIG. 2

interior of the triangle PQR. Since Γ is a covering lattice for T, triangle PQR must be contained in T. Hence the triangle OPQ must be contained in T, since T is a star domain.

Let $L = (0,\ 3)$, $M = \left(\frac{9}{5}, \frac{51}{5}\right)$ and $N = \left(-\frac{9}{5}, \frac{51}{5}\right)$. The triangle LMN is contained in $S+A$ and no other translate of S meets its interior (*see* Fig. 2). Therefore triangle LMN must be contained in $T+A$. Since $T+A$ is star with respect to A, the triangle ALM must be contained in $T+A$.

The triangles OPQ and ALM evidently overlap. Hence T and $T+A$ overlap. Therefore T is not a space-filling domain.

Theorem 2—The star domain S does not contain an $\mathcal{S}$-minimal or $\mathcal{S}'$-minimal domain with equal covering constant.

PROOF: It was shown by Hans (1965) that any bounded $\mathcal{S}$-minimal or $\mathcal{S}'$-minimal domain is space-filling. Hence the result follows from Theorem 1.

§ 3. In this section we determine a $\mathfrak{T}$-minimal set contained in S which has covering constant equal to $C(S)$.

Definition: Let $T \in \mathfrak{T}$. A point P on the boundary of T is said to be a $\mathfrak{T}$-minimal point of T if $T' \in \mathfrak{T}$, $T' \subset T$, $P \notin T'$ implies that $C(T') < C(T)$.

The following two Lemmas are easy to prove.

Lemma 1—Let $T \in \mathfrak{T}$. Then T is $\mathfrak{T}$-minimal if and only if every point on the boundary of T is a $\mathfrak{T}$-minimal point of T.

Lemma 2—Let $T \in \mathfrak{T}$ and P a point on the boundary of T. Then P is a $\mathfrak{T}$-minimal point of T if for every maximal covering lattice of T either P lies on the boundary of $\bigcup_{\substack{A \in \Lambda \\ A \neq 0}} (T+A)$ or P lies on a segment OQ where Q is a $\mathfrak{T}$-minimal point of T.

Let

$$Q_1 = (3,\ 0),\ Q_2 = \left(\frac{51}{5}, \frac{9}{5}\right),\ Q_3 = \left(15, \frac{51}{5}\right),\ Q_4 = \left(\frac{351}{25}, \frac{351}{25}\right),\ Q_5 = \left(\frac{51}{5}, 15\right),$$

$$Q_6 = \left(\frac{9}{5}, \frac{51}{5}\right),\ Q_7 = (0,\ 3),\ Q_9 = \left(-\frac{51}{25}, \frac{279}{25}\right),\ Q_{10} = (-3,\ 15),$$

$$Q_{11} = \left(-\frac{51}{5}, \frac{66}{5}\right),\ Q_{14} = \left(-\frac{66}{5}, \frac{51}{5}\right),\ Q_{15} = (-15,\ 3),\ Q_{16} = \left(-\frac{279}{25}, \frac{51}{25}\right),$$

$$Q_{18} = (0,\ -3) = -Q_1.$$

Let Q_8 be the point of intersection of the lines OQ_9 and AQ_7 where $A = \left(-\frac{66}{5}, \frac{66}{5}\right)$; Q_{12} be the point of intersection of the lines OP_{11} and AQ_7. Let Q_{13} and Q_{17} be the reflections of Q_{12} and Q_8 respectively in the line $y = -x$.

Let T be the domain bounded by the broken line segment $Q_1, Q_2 \ldots Q_{18}$ and its reflection in the origin (*see* Fig. 3). Clearly T is a star set with centre O.

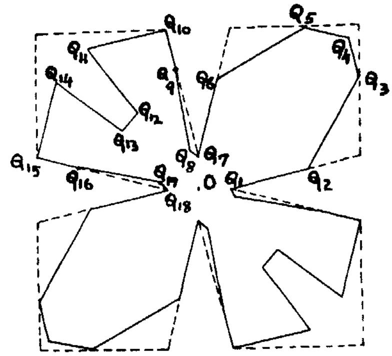

FIG. 3

Theorem 3—Let T be the closed star subset of S described above. Then $C(T) = C(S)$ and T is a $\mathfrak{J}$-minimal set for coverings.

PROOF: It is easy to verify that the lattice Γ is a covering lattice for T, for the $\triangle\, AOB$ is covered by the sets T, $T+A$ and $T+B$ (*see* Fig. 4). Therefore $C(T) \geqslant d(\Gamma) = C(S)$. Since $T \subset S$, $C(T) \leqslant C(S)$. Hence $C(T) = C(S)$.

In order to show that T is $\mathfrak{J}$-minimal, by Lemma 1 it is enough to prove that every point of the boundary of T is $\mathfrak{J}$-minimal for T. Since T is symmetric in O it is enough to prove this for boundary points of T in the upper half plane. Any maximal covering lattice of T is also a maximal covering lattice of S so that by Theorem 1, it is of the type $\Omega\Gamma'$ where Ω is a suitable automorphism of S. The group of automorphisms of S is generated by reflections in the origin, in the X-axis and the line $y = x$; whereas the group of automorphisms of T is generated by reflections in the origin and the line $y = x$. It is easy to see that $\Omega\Gamma$ is a covering lattice of T if and only if Ω is an automorphism of T. Thus it suffices to prove that the conditions of Lemma 2 are satisfied for every point P on the boundary of T for the covering (T, Γ). Because of the symmetry of T and in O, it is enough to verify it for points in the upper half plane.

It is easily seen from Fig. 4 that conditions of Lemma 2 are satisfied. The broken line segments $Q_1\, Q_2 \ldots Q_8$, $Q_9\, Q_{10}\, Q_{11}$, $Q_{12}\, Q_{13}$, $Q_{14}\, Q_{15}\, Q_{16}$ and $Q_{17}\, Q_{18}$ are on the boundary of $\bigcup_{\substack{A \in \Gamma \\ A \neq 0}} (T+A)$. The remaining segments of the boundary in the upper half plane, namely $Q_8\, Q_9$, $Q_{11}\, Q_{12}$, $Q_{13}\, Q_{14}$ and $Q_{16}\, Q_{17}$, are such that they are parts of the rays through the origin such that their end points P_9, P_{11}, P_{14} and P_{16} respectively are $\mathfrak{J}$-minimal.

Hence T is a $\mathfrak{J}$-minimal set contained in S such that $C(T) = C(S)$.

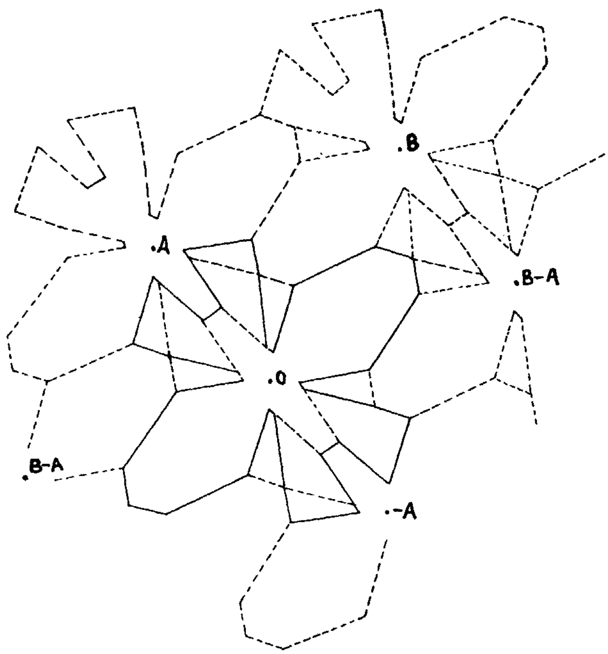

FIG. 4

ACKNOWLEDGEMENT

The authors wish to express their deep sense of gratitude to Professor R. P. Bambah for making some valuable suggestions in the preparation of this note.

REFERENCES

Dumir, V. C., and Hans-Gill, R. J. (1972*a*). Lattice double coverings in the plane. *Indian J. pure appl. Math.*, 3, 466–480.

——— (1972*b*). Lattice double packings in the plane. *Indian J. pure appl. Math.*, 3, 481–487.

Farey, I. (1950). Sur la densite des reseaux de domaines convexes. *Bull. Soc. Math. Fr.*, 78, 152–161.

Hans, R. J. (1965). On Dissertation, Ohio State University, Columbus, Ohio, U.S.A.

——— (1967). Extremal packing and covering sets. *Monatschefte für mathematik*, 71, 203–213.

Mahler, K. (1946). On lattice points in *n*-dimensional star bodies—II (Reducibility theorems) *Indag. math.*, 8, 200–212, 299–311, 343–351.

——— (1947). On irreducible convex domains. *Indag. math.*, 9, 3–12.

Ollernshaw, K. (1953). Irreducible convex bodies. *Q. Jl. Math.*, 4, 293–302.

Rogers, C. A. (1947). A note on irreducible star bodies. *Indag. math.*, 9, 379–383.

——— (1952). The reduction of star sets. *Phil. Trans. R. Soc.*, A 245, 59–93.

Woods, A. C. (1959). On the irreducibility of convex bodies. *Can. J. Math.*, 11, 256–261.