LATTICE DOUBLE PACKINGS IN THE PLANE

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It is proved that the lattice double packing density of a symmetrical convex domain is twice its lattice packing density. An example is given to show that this result cannot be extended to the class of symmetrical star domains.

§ 1. Let $R_n$ be the $n$-dimensional Euclidean space. Let $S$ be an open set in $R_n$ with volume $V(S)$. A lattice $\Lambda$ in $R_n$ is said to be a $k$-fold packing for $S$ if no point of the space $R_n$ is contained in more than $k$ translates of $S$ by points of $\Lambda$. In particular if $k = 1$, $\Lambda$ is a packing lattice for $S$ in the usual sense. The density of the best $k$-fold lattice packing of $S$ is defined by

$$\delta_k(S) = \sup \frac{V(S)}{d(\Lambda)}$$

where $d(\Lambda)$ is the determinant of $\Lambda$ and the supremum is taken over all lattices $\Lambda$ which provide a $k$-fold packing for $S$. For $k = 1$, $\delta_1(S) = \delta(S)$ is the usual density of the best lattice packing of $S$. It is easy to show that

$$\delta_k(S) \geq k\delta(S). \quad \ldots \quad \ldots \quad (1)$$

If $S$ is a circle with centre $O$, Heppes (1959) proved that equality holds in (1) if and only if $k \leq 4$. Blundon (1963) obtained $\delta_5(S)$ and $\delta_6(S)$ for the circle. For a sphere in three-dimensional space $\delta_3(S)$ was determined by Few and Kangasabapathy (1969).

In § 2 we extend the result of Heppes to all symmetrical convex domains in $R_n$ for lattice double packings. In § 3 we give an example to show that this result cannot be extended to the class of symmetrical star domains.

§ 2. Let $S$ be an open set in $R_2$. A lattice $\Lambda$ is said to be admissible for $S$ ($S$-admissible) if $\Lambda$ has no point other than the origin in $S$. The critical determinant $\Delta(S)$ of $S$ is defined as the infimum of the determinants of all admissible lattices for $S$. It is well known that $\Lambda$ is a packing lattice for $S$ if and only if $\Lambda$ is admissible for the difference set $D(S)$ of $S$. It follows that

$$\delta(S) = \frac{a(S)}{\Delta(D(S))}.$$ 

where $a(S)$ is the area of $S$. If $K$ is a convex domain with centre $O$, then $D(K) = 2K$, hence

$$\delta(K) = \frac{a(K)}{4\Delta(K)}.$$
We prove the following theorem:

**Theorem 1**—Let $K$ be a symmetrical convex domain with centre $O$. Then $\delta_s(K) = 2\delta(K)$.

**Proof:** In view of (1) it remains to prove that $\delta_2(K) < 2\delta(K)$. Since $\delta(K) = \frac{a(K)}{4\Delta(K)}$, it suffices to prove that if $A$ is any lattice which provides a lattice double packing for $K$, then $d(A) > 2\Delta(K)$.

Let $A$ be a double packing lattice for $K$. Let $f(X)$ be the gauge function of $K$ so that $K$ is given by $f(X) < 1$.

For any point $A$ other than $O$, we must have $f(A) > 1$; for if $f(A) < 1$, then $O$ belongs to the three sets $K$, $K + A$ and $K - A$.

If $f(A) > 2$ for all points of $A$ other than the origin, then $A$ is admissible for $2K$ which implies that

$$d(A) > \Delta(2K) = 4\Delta(K)$$

and the assertion follows.

Suppose that there is an $A \in A$, $A \neq O$ such that $f(A) < 2$. Among all such lattice points we choose $A$ such that $f(A)$ is least. Thus $1 < f(A) < 2$ and $1 < f(A) < f(P)$ for all $P \in A$, $P \neq O$. Then $A$ is clearly a primitive point of $A$. In case $O$, $\pm A$ are the only points of $A$ in $2K$, then the lattice $A'$ generated by $2A$ and $B$, where $A$, $B$ is a basis of $A$, is admissible for $2K$.

Therefore,

$$d(A) = \frac{1}{2}d(A') > \frac{1}{2}\Delta(2K) = 2\Delta(K)$$

and the assertion follows in this case.

Suppose that there exists a point $B$ of $A$ other than $O$, $\pm A$ in $2K$. Then $f(B) < 2$. Since $f(nA) = |n|f(A) > |n| > 2$ if $n \neq 0$, $\pm 1$ is an integer; $B$ is linearly independent of $A$. We claim that $A$, $B$ generate $A$. Let $A$, $C$ be a basis of $A$. Then $B = mA + nC$ for some integers $m$, $n$; $n \neq 0$. On replacing $C$ by $-C$ if necessary we can suppose that $n > 1$. We have

$$1 < f(A) < f(B) = f(mA + nC) < 2.$$ 

so that

$$f\left(\frac{m}{n}A + C\right) < \frac{2}{n}.$$ 

Choose an integer $k$ such that $\left|\frac{m}{n} - k\right| < \frac{1}{n}$.

Therefore,

$$f(A) < f(kA + C) = f\left(\frac{m}{n}A + C + \left(k - \frac{m}{n}\right)A\right)$$

$$< f\left(\frac{m}{n}A + C\right) + \left|\frac{m}{n} - k\right|f(A)$$

$$< \frac{2}{n} + \frac{1}{n}f(A)$$
i.e. \( f(A) < \frac{4}{n} \), which is a contradiction to \( f(A) > 1 \) if \( n > 4 \).

If \( n = 3 \), we have \( \left| k - \frac{m}{3} \right| < \frac{1}{2} \), so that
\[
f(A) < f(kA + C) < \frac{3}{2} + \frac{1}{2} f(A)
\]
i.e. \( f(A) < 1 \) which is a contradiction.

If \( n = 2 \), then \( f\left(\frac{m}{2}A + C\right) < 1 \). This is clearly not possible if \( m \) is an even integer, for then \( \frac{m}{2}A + C \) is a non-zero lattice point. If \( m \) is an odd integer \( 2k - 1 \) say, then we have
\[
f(kA + C - \frac{1}{2}A) < 1.
\]

Also we have \( f\left(\frac{A}{2}\right) < 1 \) and \( f\left(-\frac{A}{2}\right) < 1 \). Hence \( A/2 \) belongs to the three sets \( K, K + A \) and \( K + kA + C \), a contradiction. Thus we must have \( n = 1 \) and hence \( A, B \) generate \( A \).

Both \( B + A \) and \( B - A \) cannot lie in \( 2K \) for then \( \frac{1}{2}(A+B) \) belongs to the three sets \( K, K + A \) and \( K + B \). Replacing \( A \) by \( -A \) if necessary we can suppose that \( f(B + A) > 2 \). Also we cannot have \( f(B - 2A) < 2 \), for then \( f\left(\frac{B}{2} - A\right) < 1 \) which implies that \( B/2 \) lies in the three sets \( K, K + B \) and \( K + A \).

Thus we have a basis \( A, B \) of \( A \) satisfying

(i) \( f(A) < f(C) \) for \( C \in A, C \neq 0 \).

(ii) \( 1 < f(A) < f(B) < 2 \).

(iii) \( f(B + A) > 2, \ f(B - 2A) > 2 \).

Since \( f(A) < 2 \), the sets \( K \) and \( K + A \) overlap. Let \( P \) be a point common to the boundaries of \( K \) and \( K + A \) on the same side of \( OA \) as \( B \). Then \( P - A \) is common to the boundaries of \( K - A \) and \( K \). There are points arbitrarily near \( P \) which are already double covered by \( K \) and \( K + A \). Since \( K \) is open, \( P \) cannot belong to any \( K + C \) for \( C \in A \). This implies that \( K + P \) does not contain any lattice point. Similarly \( K + P - A \) is free of lattice points. The points \( O, A, 2P, 2P - A \) lie on the boundary of \( K + P \) and the points \( O, -A, 2P - A, 2P - 2A \) lie on the boundary of \( K + P - A \). It is then clear that the set \( \bigcup_{m=-\infty}^{\infty} (K + P + mA) \) covers the entire strip between \( OA \) and the line \( l \) through \( 2P \) parallel to \( OA \). Hence the point \( B \) lies on or above the line \( l \).

The line \( l \) meets the boundary of \( 2K \) in the points \( 2P \) and \( 2P - 2A \), so that it meets \( 2K \) in a segment of length \( |2A| \). Since \( B \) is above this line and \( 2K \) is convex, the line through \( B \) parallel to \( OA \) cuts \( 2K \) in a segment \( CD \) of length \( < |2A| \).
Let $A'$ be the lattice generated by $2A$ and $C$. Then $d(A') = 2d(A)$. So it suffices to prove that $d(A') > 4\Delta(K)$. This follows from the following lemma:

**Lemma**—The lattice $A'$ generated by $2A$ and $C$ is admissible for $2K$.

**Proof:** Let if possible $2mA + nC$ be a point of $A'$ other than 0 in $2K$. Without loss of generality we can suppose $n > 0$. Since $2mA \notin 2K$ for $|m| > 1$, we cannot have $n = 0$. If $n = 1$, then $2mA + C \notin 2K$ for this point does not belong to the open segment $CD$ in which the line through $C$ parallel to $OA$ meets $2K$.

If $n > 2$, then $2mA + nC \in 2K$ implies

$$f\left(\frac{2m}{n}A + C\right) < 1.$$ 

Let $C = B + rA$, so that $f\left(\frac{2m}{n}A + B\right) < 1$.

Hence we shall arrive at a contradiction if we can prove that $f(B + \lambda A) > 1$ for all real $\lambda$.

Suppose $f(B + \lambda A) < 1$ for some real $\lambda$. Then clearly we must have $-2 < \lambda < 1$, since $f(B - 2A) > 2$, $f(B) < 2$ and $f(B + A) > 2$. Also $\lambda$ is not an integer for then $B + \lambda A$ is a non-zero point of $A$, so that $f(B + \lambda A) > 1$. We have three cases:

(i) $0 < \lambda < 1$.

(ii) $-2 < \lambda < -1$.

(iii) $-1 < \lambda < 0$.

We now show that in all cases we get a contradiction.

**Case (i):** $0 < \lambda < 1$

Since $f(B + \lambda A) < 1$, we have

$$f(A) = f(B + \lambda A - \lambda A) < f(B + \lambda A) + \lambda f(A) < 1 + \lambda f(A)$$

i.e.

$$(1 - \lambda)f(A) < 1. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

Also

$$2 < f(B + A) = f(B + \lambda A + (1 - \lambda)A) < f(B + \lambda A) + (1 - \lambda)f(A) < 1 + (1 - \lambda)f(A)$$

i.e. $(1 - \lambda)f(A) > 1$, which contradicts (2).

**Case (ii):** $-2 < \lambda < -1$

In this case we have

$$f(A) < f(B - A) = f(B + \lambda A - (1 + \lambda)A) < f(B + \lambda A) - (1 + \lambda)f(A) \quad \text{(since } 1 + \lambda < 0)$$

or

$$(2 + \lambda)f(A) < 1. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3)$$
Also,
\[ 2 < f(B - 2A) = f(B + \lambda A - (2 + \lambda)A) \]
\[ < f(B + \lambda A) + (2 + \lambda)f(A) \]
\[ < 1 + (2 + \lambda)f(A) \]
i.e. \((2 + \lambda)f(A) > 1\), which contradicts (3).

**Case (iii):** \(-1 < \lambda < 0\)

In this case we have
\[ f(A) < f(B) = f(B + \lambda A - \lambda A) \]
\[ < f(B + \lambda A) - \lambda f(A) \]
\[ < 1 - \lambda f(A) \]
i.e. \((1 + \lambda)f(A) < 1\), or
\[ f(A + \lambda A) < 1 \] ... ... ... ... (4)

Also,
\[ f(A) < f(B - A) < f(B + \lambda A - (1 + \lambda)A) \]
\[ < f(B + \lambda A) + (1 + \lambda)f(A) < 1 + (1 + \lambda)f(A) \]
i.e. \(-\lambda f(A) < 1\), or since \(\lambda < 0\),
\[ f(-\lambda A) < 1. \] ... ... ... ... (5)

Also,
\[ f(B + \lambda A) < 1. \] ... ... ... ... (6)

The inequalities (4), (5) and (6) imply that the point \(-\lambda A\) belongs to the three sets \(K + A\), \(K\) and \(K + B\), a contradiction.

Thus in all cases we arrive at a contradiction and the Lemma follows.

This completes the proof of Theorem 1.

§ 3. In this section we give an example of a symmetric star domain \(S\) for which \(\delta_2(s) > 2\delta(s)\).

Let
\[ P_1 = (3, 0), \quad P_2 = (15, 3), \quad P_3 = (15, 15), \quad P_4 = (3, 15) \]
and
\[ P_5 = (0, 3). \]

Let \(S\) be the region bounded by the line segments \(P_1 P_2, P_2 P_3, P_3 P_4, P_4 P_5\) and their reflections in the axes and the origin (see Fig. 1). Then \(S\) is a symmetrical star domain with centre \(O\). Let \(a(S)\) denote its area. Then we prove the following Theorem:

**Theorem 2**—Let \(S\) be the star domain described above. Then
\[ \delta(S) = \frac{a(S)}{900} \]
and
\[ \delta_2(S) > \frac{a(S)}{443} > 2\delta(S). \]
Proof: Let $D(S)$ denote the difference set of $S$. As remarked in § 2, we have

$$\delta(S) = \frac{a(S)}{\Delta(D(S))}$$

where $\Delta(D(S))$ denotes the critical determinant of $D(S)$. Thus the first assertion will follow if we can show that $D(S)$ is the square

$$T: \max(|x|, |y|) < 30$$

for $\Delta(T) = \frac{1}{4}$ area of $T = 900$.

Since $S$ is a star domain, so is $D(S)$. Also $D(S) \subset T$, since $S \subset \frac{1}{4}T$. The assertion will follow if we can prove that every point of the boundary of $T$ is in $D(S)$. Due to symmetries of $S$ it is enough to prove that the points $(30, \lambda) \in D(S)$ for $0 \leq \lambda \leq 30$. Since $S$ is symmetrical about the origin, $2S \subset D(S)$. Therefore $(30, \lambda) \in D(S)$ for $6 \leq \lambda \leq 30$. For $0 \leq \lambda \leq 6$, we have

$$(30, \lambda) = (15, 9+\lambda)-(15, 9) \in D(S).$$

Hence $D(S) = T$.

To prove the second assertion we claim that the lattice $\Lambda$ generated by $A = (15, 7), B = (-14, 23)$ is a double packing lattice for $S$. For this it suffices to prove that no point of the triangle $OAB$ is covered more than twice by the translates of $S$ through points of $\Lambda$. It is easy to verify that the only sets which intersect $OAB$ are the translates of $S$ through the points $O, A, B, -A$ and $A+B$. In Fig. 2, the parts of the triangle $OAB$ which are covered exactly twice are shaded (by means of lines) and the portion which is single covered is dotted and the remaining portion is not covered at all. Hence $\Lambda$ is a double packing lattice for $S$, so that

$$\delta_d(S) > \frac{a(S)}{d(\Lambda)} = \frac{a(S)}{443} > 2\delta(S).$$

This proves Theorem 2.
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REFERENCES


Professor L. Fejes Tóth asked whether the result of Theorem 1 is true for non-symmetrical convex domain. The answer to the question is in the negative as can be easily seen in the case of a triangle \( K \). For \( \delta(K) = 2/3 \) and it is easy to verify that a lattice \( A \) which provides the best lattice covering for \( K \) in fact provides a lattice double packing for \( K \) also, so that \( \delta_2(K) > 3/2 > 2\delta(K) \).