Living on the edge in a spacetime with multiple warping

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Abstract

The Randall-Sundrum warped braneworld model is generalised to six and higher dimensions such that the warping has a non-trivial dependence on more than one dimension. This naturally leads to a brane-box like configuration alongwith scalar fields with possibly interesting cosmological roles. Also obtained naturally are two towers of 3 branes with mass scales clustered around either of Planck scale and TeV scale. Such a scenario has interesting phenomenological consequences including an explanation for the observed hierarchy in the masses of standard model fermions.

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1 Introduction

Theories with extra spacetime dimensions have drawn considerable attention since the original proposals of Kaluza and Klein. There has been a renewed interest in such theories since the emergence of string theory. Several new ideas in such directions evolved to explore various implications of the presence of extra spatial dimensions in the context of particle phenomenology and cosmology [1–7]. Although these ideas are not necessarily derived from string theory, there have been serious efforts towards establishing links between the two. Two of the most prominent such extra dimensional theories developed in the context of the braneworld models are due to Arkani-Hamed, Dimopolous, Dvali (ADD) [1] on the one hand and Randall and Sundrum (RS) [4] on the other. While the ADD model involves the presence of large compact extra dimensions (the radius(radii) of the extra compact dimension(s) being much larger than the Planck length), the RS model proposes the existence of a warped geometry in $3 + 1$ dimensions in the background of a $4 + 1$ dimensional anti de-Sitter (AdS) bulk. Both the theories claim to solve the so called "naturalness problem" in standard model, which originates from the need of an unnatural fine-tuning of parameters of the theory to stabilize the Higgs mass within the TeV scale against large radiative corrections. Although the presence of large radii in the ADD model indicates the reappearance of the hierarchy in a different guise, the RS scenario is apparently free from such problems. However, in this case the braneworld model itself is not stable and it was first shown by Goldberger and Wise (GW) [8] that by introducing a scalar field in the bulk, the modulus—namely, the brane separation—in RS model can be stabilized without the need of any unnatural fine-tuning. Assumption of a negligibly small scalar back-reaction on the metric in the GW approach prompted further work in this direction where the modifications of the RS metric due to back-reaction of bulk fields have been derived [9]. The stability issues in such cases have been re-examined as also the effects of other bulk fields like gauge field or higher form fields been studied in several works [10–13]. Such warped geometries are expected to have additional consequences in particle phenomenology over and above the hierarchy issue.

As a natural extension to the RS scenario with one extra spatial dimension, several extensions of the RS model to more than one extra dimension have been proposed [14]. In particular some cosmological implications of warped geometry in six dimension have been explored in the context of dynamical compactification of extra dimension [15]. Most of these consider the presence of several independent S_1/Z_2 orbifolded dimensions along with M_4. We, however, propose a more intricate scenario wherein the warped compact dimensions get further warped by a series of successive warping leading to multiple warping of the space-time with various p-branes sitting at the different orbifold fixed points satisfying appropriate boundary conditions. Various lower dimensional branes along with the standard model 3-brane exist at the intersection edges of the higher dimensional branes. Thus the resulting geometry of the D-dimensional space-time is $M^{1,D-1} \rightarrow \{[M^{1,3} \times S^1/Z_2] \times S^1/Z_2\} \times \cdots$, with $(D-4)$ such warped directions. A series of scales are thus generated from each of these successive warpings and we show that such a spacetime with multiple warping leads to interesting phenomenology.

The original RS model corresponded to a 5-dimensional AdS spacetime wherein the extra dimension was S^1/Z_2 orbifolded. Two branes (called the standard model brane and Planck scale brane) were placed at the two orbifold fixed points and appropriate brane tensions at the boundaries of the orbifold were determined in terms of the bulk cosmological constant. In such a model, it was shown that a TeV scale can be generated at the standard model brane without any unnatural fine tuning of parameters. We propose here a model in a $(5 + 1)$-dimensional bulk AdS spacetime where both the extra coordinates are compactified in succession on circles with Z_2 orbifoldings. We show that the six dimensional Einstein's equation can be solved exactly for such a geometry and that the resulting solution for the metric is doubly warped. Although the warping of the metric along one of the compact coordinates resembles exponential warping as found by Randall and Sundrum, the other one turns out to be a hyperbolic warping. Such a solution with doubly orbifolded boundary conditions results in a box-like picture of the bulk, where the walls of the box are $(4 + 1)$-dimensional branes. The

Z_2 orbifoldings along the two compact coordinates puts stringent conditions on the brane tensions. Four $(3 + 1)$-dimensional branes are formed at the four edges of the intersecting 4-branes. We can then identify our $3 + 1$ dimensional standard model brane with one of the edges by requiring the desired TeV scale while the Planck scale brane resides at another edge. We thus live at one edge of the proposed spacetime ith multiple warping. The other two edges correspond to two more $3 + 1$ dimensional branes with some intermediate energy scales. With the aforementioned Z_2 orbifoldings further requiring coordinate-dependent brane tensions, a possible origin for the same is proposed. As could be expected, the mass of a standard model scalar field like the Higgs boson is doubly warped from the Planck scale down to the TeV scale and thereby the fine tuning problem is resolved. The crucial aspect of the double warping scenario turns out to be unequal warping in two directions. We show that while warping in one direction is large, the other is necessarily small.

We then generalise our result to a large number of extra dimensions, where several scales are generated and because of unequal warping, half of these are clustered around Planck scale while the rest are around TeV scale. We then argue that such a clustered scale brane model can offer a possible explanation of the observed mass differences in the standard model fermions.

2   Six dimensional doubly warped spacetime

The spacetime that we are interested in is a doubly compactified six-dimensional one with a Z_2 orbifolding in each of the compact directions. In other words, the manifold under consideration is $M^{1,5} \rightarrow [M^{1,3} \times S^1/Z_2] \times S^1/Z_2$. To set the notation, the non-compact directions would be denoted by x^μ ($\mu = 0..3$) and the orbifolded compact directions by the angular coordinates y and z respectively with R_y and r_z as respective moduli. The corresponding metric is,

$$ds^2 = b^2(z)[a^2(y)\eta_{\mu\nu}dx^\mu dx^\nu + R_y^2 dy^2] + r_z^2 dz^2 \tag{1}$$

where $\eta_{\mu\nu} = diag(-1, 1, 1, 1)$. Since orbifolding, in general, requires a localized concentration of energy, we introduce four 4-branes ($4 + 1$ dimensional objects) at the orbifold fixed "points", namely $y = 0, \pi$ and $z = 0, \pi$.

The total bulk-brane action is thus given by,

$$
\begin{aligned}
S &= S_6 + S_5 + S_4 \\
S_6 &= \int d^4x\, dy\, dz\, \sqrt{-g_6}\ (R_6 - \Lambda) \\
S_5 &= \int d^4x\, dy\, dz\ [V_1\, \delta(y) + V_2\, \delta(y - \pi)] \\
&\quad + \int d^4x\, dy\, dz\ [V_3\, \delta(z) + V_4\, \delta(z - \pi)] \\
S_4 &= \int d^4x\, dy\, dz\, \sqrt{-g_{vis}}[\mathcal{L} - \hat{V}]\ .
\end{aligned}
\tag{2}
$$

Note that, in general, we have, for the brane potential terms $V_{1,2} = V_{1,2}(z)$ whereas $V_{3,4} = V_{3,4}(y)$. The presence of the term S_4 indicates the contributions due to possible 3-branes located at $(y, z) = (0, 0), (0, \pi), (\pi, 0), (\pi, \pi)$.

The full six dimensional Einstein's equation can be written as,

$$
\begin{aligned}
-M^4\,\sqrt{-g_6}\left(R_{MN} - \frac{R}{2}\,g_{MN}\right) &= \Lambda_6\,\sqrt{-g_6}\,g_{MN} \\
&+ \sqrt{-g_5}\,V_1(z)\,g_{\alpha\beta}\,\delta_M^\alpha\,\delta_N^\beta\,\delta(y) + \sqrt{-g_5}\,V_2(z)\,g_{\alpha\beta}\,\delta_M^\alpha\,\delta_N^\beta\,\delta(y - \pi) \\
&+ \sqrt{-\widetilde{g_5}}\,V_3(y)\,\widetilde{g}_{\tilde\alpha\tilde\beta}\,\delta_M^{\tilde\alpha}\,\delta_N^{\tilde\beta}\,\delta(z) + \sqrt{-\widetilde{g_5}}\,V_4(y)\,\widetilde{g}_{\tilde\alpha\tilde\beta}\,\delta_M^{\tilde\alpha}\,\delta_N^{\tilde\beta}\,\delta(z - \pi)
\end{aligned}
\tag{3}
$$

Here M,N are bulk indices, α, β run over the usual four spacetime coordinates (x^μ) and the compact coordinate z while $\tilde{\alpha}, \tilde{\beta}$ run over x^μ and the compact coordinate y. And, finally, $g, \tilde{g}$ are the respective metrics in these (4+1)-dimensional spaces.

On substituting our ansatz for the metric (eqn.1), the yy and zz components of Einstein's equations reduce to a set of two simpler equations, namely

$$
\begin{aligned}
2\,M^4 \left[3\,r_z^2\,a'^2 + 3\,R_y^2\,a^2\,\dot{b}^2 + 2\,R_y^2\,a^2\,b\,\ddot{b}\right] &= -b^2\,a^2\,r_z\,R_y^2\,[r_z\,\Lambda + V_3\,\delta(z) + V_4\,\delta(z-\pi)] \\
2\,M^4 \left[3\,r_z^2\,a'^2 + 5\,R_y^2\,a^2\,\dot{b}^2 + 2\,r_z^2\,a\,a''\right] &= -b\,a^2\,r_z^2\,R_y\,[R_y\,\Lambda\,b + V_1\,\delta(y) + V_2\,\delta(y-\pi)]
\end{aligned}
\tag{4}
$$

where primes denote differentiation w.r.t. y, while dots denote differentiation w.r.t. z. Starting with the bulk part of eqn.(4), and rearranging terms, we have

$$
\frac{a'^2}{a^2} = c^2 = R_y^2 \left[\frac{\dot{b}^2}{r_z^2} + \frac{2\,b\,\ddot{b}}{3\,r_z^2} + \frac{b^2\,\Lambda}{6\,M^4}\right]
\tag{5}
$$

where c is an arbitrary constant. The solution to the above is given by

$$
\begin{aligned}
a(y) &= \exp(-c\,y) & b(z) &= \frac{\cosh(k\,z)}{\cosh(k\,\pi)} \\
c &\equiv \frac{R_y\,k}{r_z\,\cosh(k\,\pi)} & k &\equiv r_z\sqrt{\frac{-\Lambda}{10\,M^4}}.
\end{aligned}
\tag{6}
$$

It can be easily ascertained that eqn.(6) satisfies each of the two eqns.(4) as long as they are restricted to the bulk. As is quite apparent from the form of the solution, the presence of an exponential warping (as in RS model) in the y- direction necessitates a negative value for the bulk cosmological constant Λ, thereby signalling an AdS bulk. Of course, for $c^2 < 0$, an alternative (oscillatory) solution for both $a(y)$ and $b(z)$ are possible. However, this, manifestly, does not lead to the desired warping of the spacetime metric and hence shall not be considered any further. Similarly, we discount solutions of the form $b^2(z) = -(c^2\,r_z^2/k^2\,R_y^2)\,\sinh^2(k\,z)$ as this leads to a bulk metric with a $(4,2)$ signature.

Note that the Z_2 orbifolding in the y-direction, namely $y \equiv -y$, demands that $a(y) = \exp(-c\,|y|)$ whereas the symmetric form of $b(z)$ obviates the need for an analogous requirement. The full metric thus takes the form

$$
ds^2 = \frac{\cosh^2(k\,z)}{\cosh^2(k\,\pi)}\left[\exp\left(-2\,c\,|y|\right)\,\eta_{\mu\nu}\,dx^\mu\,dx^\nu + R_y^2\,dy^2\right] + r_z^2\,dz^2 .
\tag{7}
$$

Next, we focus our attention on the boundary terms so as to determine the brane tensions. Using $a(y) = exp(-c|y|)$, substituting for c from equ. (6) and integrating the second of eqns.(4) over an infinitesimal interval across the two boundaries at $y = 0, y = \pi$ respectively, we obtain

$$
V_1(z) = -V_2(z) = 8M^2\sqrt{\frac{-\Lambda}{10}}\,\operatorname{sech}(k\,z) .
\tag{8}
$$

In other words, the two 4-branes sitting at $y = 0$ and $y = \pi$ have z-dependent tensions, a feature that will return to in the next section. The fact that these tensions are equal and opposite is reminiscent of the original RS-form and is but a consequence of the exponential warping and the fact of these branes sitting at the orbifold fixed "points".

Similarly, starting with the first of eqns.(4), using the solution for $b(z)$ from (6) and integrating over an infinitesimal interval across $z = 0$, we find,

$$
V_3(y) = 0
\tag{9}
$$

This, of course, was to be expected since the smooth behaviour of $b(z)$ as $z \to 0$ obviates the necessity for any localized energy density at $z = 0$. On the other hand, integrating over an infinitesimal interval across $z = \pi$ gives

$$V_4(y) = -\frac{8\,M^4\,k}{r_z}\,\tanh(k\pi)\ , \tag{10}$$

a constant, unlike the case for $V_{1,2}(z)$, but quite similar to the case for the original RS model. This, again, is not unexpected, for $V_{3,4}$ were introduced to account for the orbifolding in the z-direction and with g_{zz} being a constant, the corresponding hypersurfaces should only have a constant energy density. The fact of g_{yy} being a non-trivial function of y, however, made it mandatory that the two hypersurfaces accounting for the y-orbifolding must have a z-dependent energy density.

2.1 Brane identities and the extent of warping

We have thus determined the tensions for all of the 4-branes in the theory. As indicated earlier, the intersection of two 4-branes may be identified with a 3-brane with a tension that, to the leading order, is an algebraic sum of the energy densities contributed by each of the 4-branes. With this identification, the theory, thus, contains four 3-branes located at $(y,z) = (0,0), (0,\pi), (\pi,0), (\pi,\pi)$.

With the 3-brane located at $(y = 0, z = \pi)$ suffering no warping of the metric on it, it is logical that it be identified with the Planck brane. Note that there is no unique assignement of the Standard Model(visible) brane! Each of the other three offers a valid choice depending on the values of the parameters (k, c). The latter, of course, are determined in terms of Λ, r_z and R_y, with the 6-dimensional Planck mass M being essentially the same as the 4-dimensional one because of the relation,

$$M_P^2 \sim \frac{M^4\,r_z\,R_y}{2\,c\,k}\left[1 - e^{-2c\pi}\right]\left[\frac{tanh(k\pi)}{cosh^2(k\pi)} + \frac{tanh^3(k\pi)}{3}\right] \tag{11}$$

Each choice would have its own unique phenomenological consequences. If we adopt the conservative view that there exists no other brane with a natural energy scale lower than ours, we must identify the SM brane with the one at $y = \pi, z = 0$. For such a choice,

$$V_{\text{vis}} = -8M^2\sqrt{\frac{-\Lambda}{10}} \qquad V_{\text{Planck}} = 8M^2\sqrt{\frac{-\Lambda}{10}}\,[\text{sech}(k\,\pi) - \tanh k\,\pi] \tag{12}$$

with the two other 3-branes located at $(0,0)$ and (π,π) having tensions intermediate to the above. Note that whereas the Planck-brane must always have a positive tension (given by eqn.12), it is not mandatory that the SM brane must be a negative tension one. For example, we could have identified the latter with the one at $(0,0)$ with the consequence that now, $V_{\text{vis}} \simeq V_{\text{Planck}}$ but by paying the price of having at least one brane—that corresponding to our present choice—having a lower energy scale.

Before ending this section, we examine the possible mass warping in the scalar sector of the standard model, or, in other words, the status of the naturalness problem in such a scenario. With the action for a free scalar propagating on the visible brane being given by

$$S_H = \int d^4x\,\sqrt{-g_{vis}}\,\left[g_{vis}^{\mu\nu}\,D_\mu H\,D_\nu H - m_0^2\,H^2\right]\ , \tag{13}$$

a Planck scale mass m_0 is warped to

$$m = m_0\,\frac{r_z\,c}{R_y\,k}\,\exp(-\pi\,c) = m_0\,\frac{\exp(-\pi\,c)}{\cosh(k\,\pi)} \tag{14}$$

on the TeV brane which is quite akin to (but not exactly the same as) the RS case. An important point to note is that if we want a substantial warping in the z-direction (from $z = 0$ to $z = \pi$), $k\,\pi$ must be

substantial, i.e. of same order of magnitude as in the usual RS case. But with c being determined by eqn.(6), this immediately means that c must be small, unless there is a large hierarchy between the moduli r_z and R_y. This, in turn, means that we cannot have a large warping in y-direction as well without introducing a new and undesirable hierarchy. Similarly, if we demand a large hierarchy in the y-direction (a situation very close in spirit with RS), we must necessarily live with a relatively small $k \lesssim \mathcal{O}(1)$ and hence little warping in the z-direction.

An interesting consequence that emerges from this is that, of the two branes located at $(y = 0, z = 0)$ and $(y = \pi, z = \pi)$, one must have a natural mass scale close to the Planck scale, while for the other it is close to the TeV scale. The latter statement immediately points to phenomenologically interesting possibilities which shall be addressed later.

3 Origin of the coordinate dependent brane tension

We have noticed in the previous section that two of the (4+1)-dimensional brane tensions namely $V_{1,2}$ are functions of the coordinate z. While such coordinate-dependence might seem counterintuitive at first, it should be realized that the Israel junction conditions only stipulate that there be a concentration of energy-density at the $y = 0, \pi$ hypersurfaces and that these distributions must have the stipulated z-dependence. A particularly simple mechanism for arranging such an energy concentration has its origin in a scalar field confined to the respective branes.

Consider a scalar φ on one 4-brane, say on the brane at $y = y_0$, (where y_0 is either 0 or π) with a potential $\mathcal{V}(\varphi)$. Since the metric on this brane is

$$
\begin{aligned}
ds^2 &= b_0^2 \, \cosh^2(k\,z) \, \eta_{\mu\nu} dx^\mu dx^\nu + r_z^2 \, dz^2 \\
b_0 &\equiv e^{-2\,c\,|y_0|} \, \cosh^{-2}(k\,\pi) \, ,
\end{aligned}
\tag{15}
$$

the action for the scalar is

$$
S_\varphi = \int d^4x \, dz \, \sqrt{-g_5} \, \left[g_5^{AB} \, \partial_A \varphi \, \partial_B \varphi + \mathcal{V}(\varphi) \right]
\tag{16}
$$

with g_5^{AB} being given by eqn.(15). This leads to an equation of motion of the form

$$
r_z^2 \frac{\partial \mathcal{V}}{\partial \varphi} = 8\,k \, \tanh(kz) \, \varphi' + 2\,\varphi''
\tag{17}
$$

where the primes denote differentiation with respect to z. Denoting

$$
\mathcal{V}(\varphi(z)) \equiv \rho(z) \qquad \Longrightarrow \qquad \frac{\partial \mathcal{V}}{\partial \varphi} = \frac{\rho'}{\varphi'} \, ,
\tag{18}
$$

the equation of motion becomes

$$
r_z^2 \, \rho' \, \cosh^8(kz) = \frac{d}{dz} \left[\cosh^8(kz) \, (\varphi')^2 \right] \, .
\tag{19}
$$

Since, for the energy density to give the required brane tension, we must have

$$
\rho(z) + \left(\frac{\varphi'}{r_z} \right)^2 = V_{1,2} \, ,
\tag{20}
$$

as the case may be, we now need to find simultaneous solutions of eqns. (19&20) for each of $V_{1,2}$.

Concentrating first on the 4-brane at $y = \pi$ we find,

$$
\rho(z) = v_0 \left[-\frac{7}{6} \operatorname{sech}(k\,z) + \xi \, \operatorname{sech}^4(k\,z) \right] \, , \qquad v_0 \equiv 8M^2 \sqrt{\frac{-\Lambda}{10}} \, ,
\tag{21}
$$

where ξ is a constant of integration, and

$$\frac{\varphi'^2}{r_z^2} = v_0 \left[\frac{1}{6} \operatorname{sech}(k\,z) - \xi \operatorname{sech}^4(k\,z) \right] . \tag{22}$$

Positivity of the right hand side (over the entire 4-brane) requires

$$\xi \leq \frac{1}{6} . \tag{23}$$

The solution of eqn.(22) involves elliptic integrals. Rather than present the exact, but cumbersome, expressions, we choose to display the profile of $\varphi(z)$ in Fig.1. Since the value of $\varphi(z)$ is not of any physical relevance, we have fixed the constant of integration such that $\varphi(0) = 0$. As is quite apparent, the variation of $\mathcal{V}(\varphi)$ with φ is not a rapid one, and the bulk of the energy density stored in φ is on account of the rapidly varying metric. What is also reassuring is that the dependence of $\rho(z)$ on the parameter ξ is not extreme.
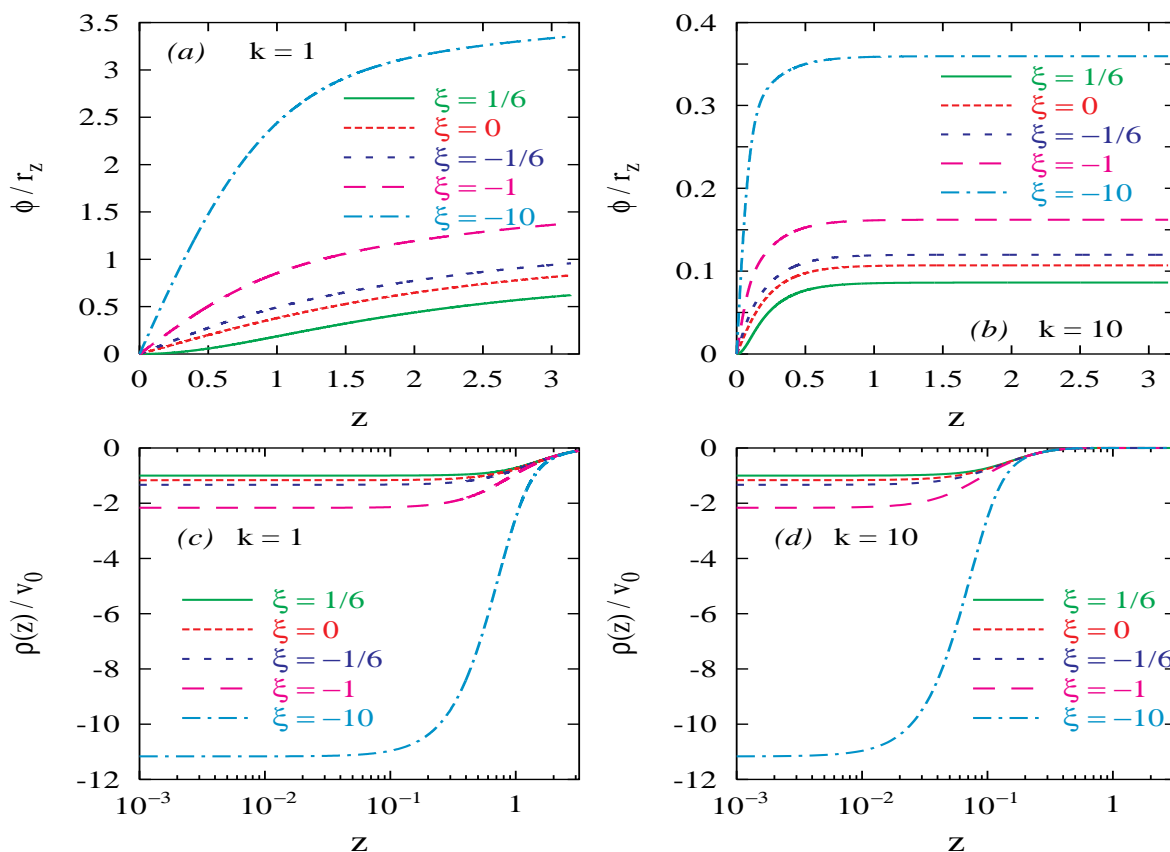

Figure 1: *The profile of the field $\varphi(z)$ (upper panels) on the 4-brane at $y = \pi$ as well as the corresponding potentials (lower panels) $\rho(z)$. The field is defined so that $\varphi(0) = 0$. Left (right) panels correspond to $k = 1\,(10)$.*

It may be observed from eqns.(18, 21 & 22) that although the potential for the scalar field $\mathcal{V}$ and the scalar field ϕ are expressed in terms of the compact coordinate z, it is, in general, extremely difficult to invert the relation and express the scalar potential $\mathcal{V}$ in terms of $\phi(z)$ through some algebraic equation. This, however, can be achieved in certain limits. For example, in the $\xi \to -\infty$ limit, we obtain

$$\beta(\varphi - \varphi_0) = \tanh(kz)$$

where $\beta^2 = -v_0 \, \xi \, r_z^2 / k^2$ and φ_0 is an integration constant. This, of course, is reminiscent of a kink solution. The corresponding scalar potential is

$$\mathcal{V}(\varphi) = -\alpha v_0 [1 - \beta^2 (\varphi - \varphi_0)^2]^2 \ .$$

While this may seem to represent a potential unbounded from below, note that the solution need not be a runaway one. Rather, $\phi = 0$ is a deep local minimum of this solution, and the classical configuration described above stretches from this minimum to a point far short of the summit that ϕ would need to cross to be able to reach the runaway global minimum. It should also be borne in mind that such potentials are not uncommon in effective field theories in general, and, in particular the low energy actions derived from string theory which perhaps is the best candidate for ultraviolet completion of such theories that we are concerned with. It should also be realised that large $|\xi|$ is not the only scenario wherein a closed form solution can be expected, but perhaps is the simplest one. In the opposite limit, namely $\xi \to 0$, eqn.(22) yields

$$\varphi' = A \sqrt{\text{sech}(k \, z)} \ , \qquad A \equiv \sqrt{\frac{v_0 \, r_z^2}{6}} \ . \tag{24}$$

Integrating this, we have, for large $|k \, z|$,

$$exp(-kz) \approx \frac{3 \, k}{4 \, v_0 \, r_z^2} (\varphi - \varphi_0)^2 \ , \quad V(\varphi) \approx -\frac{7}{4 \, r_z^2} (\varphi - \varphi_0)^2$$

where φ_0 is an integration constnt. A much more interesting solution can be obtained by expanding eqn.(24) around $z = 0$, namely

$$\varphi'(z) \approx A \left[1 + \frac{k^2 \, z^2}{4} \right]^{-1/2}$$

to yield

$$\varphi(z) \approx \frac{2 \, A}{k} \, \tan^{-1} \frac{k \, z}{2} \ , \qquad \mathcal{V}(\varphi) \approx \frac{-7 \, v_0}{6} \, \text{sech} \left(2 \, \tan \frac{k \, \phi}{2 \, A} \right) \tag{25}$$

Note that the above potential is a periodic one! In Fig.2, we compare it with the exact numerical solutions presented in Fig.1 for the $\xi = 0$ case. The remarkable agreement bears testimony to the goodness of the approximation in eqn.(25), which is not surprising since it also analytically matches with the approximate solution obtained above for large $k \, z$. For non-zero finite values of ξ, a good approximate solution is admittedly more difficult to obtain, but the above examples illustrate that it may not be impossible to!

While, for the 4-brane at $y = \pi$, the z-dependent brane energy density can easily be accounted for in terms of the scalar $\varphi(z)$, a similar analysis for the brane situated at $y = 0$, leads to a somewhat different conclusion. For this ($y = 0$) brane we have

$$\rho = v_0 \left[\frac{7}{6} \, \text{sech}(k \, z) + \tilde{\xi} \, \text{sech}^4(k \, z) \right] \ , \qquad \frac{\varphi'^2}{r_z^2} = v_0 \left[\frac{-1}{6} \, \text{sech}(k \, z) + \tilde{\xi} \, \text{sech}^4(k \, z) \right] \ . \tag{26}$$

Once again, positivity of φ'^2 requires

$$\tilde{\xi} \geq \frac{1}{6} \, \cosh^3(k \, \pi) \ . \tag{27}$$

and as in the previous case, the large $\tilde{\xi}$ limit yields the potential for the scalar field $\mathcal{V}(\phi)$ as,

$$\mathcal{V}(\varphi) = \tilde{\xi} v_0 [1 - \beta^2 (\varphi - \varphi_0)^2]^2 \tag{28}$$

It may be observed that in this case the potential is not unbounded from below. Proceeding similarly, one can find the form of the scalar potential in small $|\xi|$ regime also. Several comments are in order here:
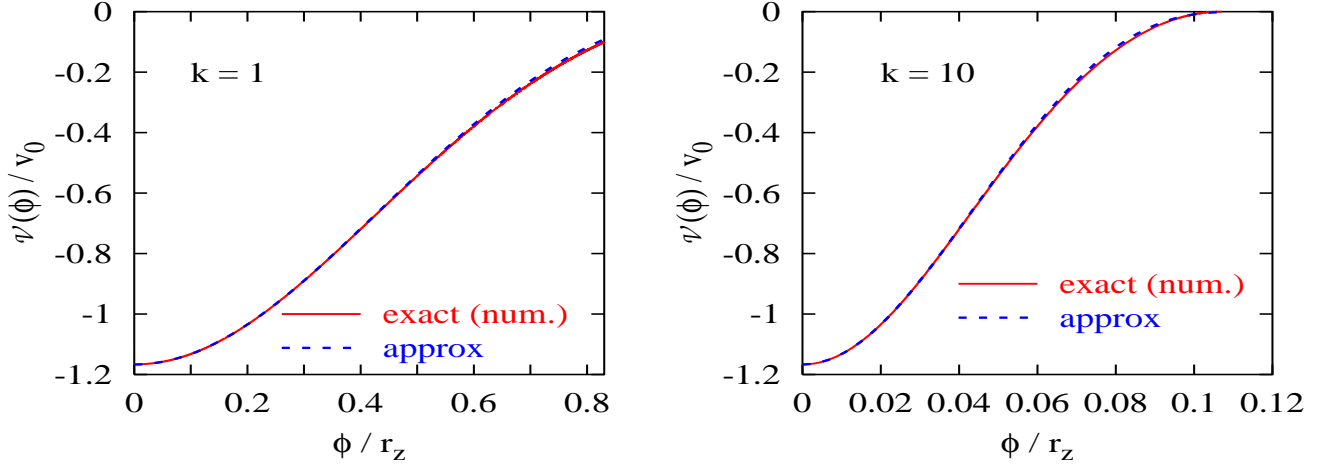
Figure 2: *The potential $\mathcal{V}(\varphi)$ on the 4-brane at $y = \pi$ as function of φ for $\xi = 0$. The solid (red) curve is the exact numerical solution while the dashed (blue) curve is the approximation of eqn.(25). Left (right) panels correspond to $k = 1\,(10)$.*

- If the hierarchy between the Planck scale and the TeV scale is to be explained primarily by the warping in th z-direction, then $\cosh(k\,\pi)$ is large and eqn.(27) implies a very large value for the parameter $\tilde{\xi}$. Although it might be argued that this unnaturalness is just a consequence of the particular parametrization of $\varphi(z)$ on this brane, the large difference between ξ and $\tilde{\xi}$ is indeed disquieting.

- A (drastic) way out of this would be to exchange the field $\varphi(z)$ (on this particular brane) for a phantom scalar field (i.e., one whose kinetic term has the opposite sign). This, obviously, would necessitate $\tilde{\xi} \leq 1/6$ rather than eqn.(27).

  The presence of a phantom field in the theory does not necessarily imply a discernible role for it on the SM 3-brane. However, if we identify the latter with the one located at $(y = 0, z = 0)$—see Sect.2.1—then this raises the interesting possibility of obtaining a dark energy candidate with a non-trivial equation of state.

  However, as is well-known, such a scalar field is not admissible in a fundamental theory. Thus, invoking such a course would necessitate considering the present theory as an effective field theory description of a different theory. Though this, admittedly, is somewhat counterintuitive in a theory purporting to be valid until M_P, yet such an eventuality cannot be ruled out in principle.

- A possible alternative to a phantom-like nature for $\varphi(z)$ would be to postulate a non-minimal coupling of the same to gravity on the brane, thereby effecting a change in both of eqns.(26). This, however, needs further investigation.

- Perhaps the simplest way around eqn.(27) is to appeal to the fact that if we demand that the warping in the y-direction is to account for the Planck scale–TeV scale hierarchy, then c is large and k is small (see Sect.2.1). This, in turn, implies that not too large a value for $\tilde{\xi}$ can still ensure positivity of φ'^2.

While the alternatives listed above present several possible solutions to the problem of a z-dependent energy density concentrated on the 4-brane at $y = 0$, it should be realized that each will have its own unique set of phenomenological consequences (and, in the case of one, require an ultraviolet completion). We postpone any such discussion to a future occasion and turn instead to a brief examination of some outstanding issues.

The braneworld model proposed here faces the usual problem of stability of the moduli r_z and R_y, that one encounters even in the original 5 dimensional Randall-Sundrum model. To stabilize the

single modulus r_c in that model, the most well known mechanism was formulated by Goldberger and Wise where a bulk scalar field is used to stabilize the brane separation. Here, one may carry a similar analysis by incorporating a six dimensional bulk scalar field so that the back-reaction of the scalar field on the background metric is negligibly small. By integrating out the scalar field, an effective potential for the moduli can be generated. The minimization conditions of the potential would give the stabilized value of the moduli. Since the corresponding solutions involve several hypergeometric functions and elliptic integrals, we desist from presenting them here.

Finally, because of the flat nature of the metric on the 3-branes, the induced cosmological constant on the TeV-brane, as in the RS case, vanishes identically.

4 Seven and higher dimensional spacetime with multiple warping

In our task of extending our solutions to even higher dimensions, we start with a seven dimensional spacetime wherein three dimensions are successively warped. In other words, the manifold of interest is $\left[\left\{M^{(1,3)} \times [S^1(/Z_2)]\right\} \times [S^1(/Z_2)]\right] \times [S^1(/Z_2)]$. As in Sec.2, the total bulk-brane action is given by,

$$
\begin{aligned}
S &= S_7 + S_6 + S_5 + S_4 \\
S_7 &= \int d^4x\, dy\, dz\, dw\, \sqrt{-g_7}\ (R_7 - \Lambda_7) \\
S_6 &= \int d^4x\, dy\, dz\, dw\ [V_1\, \delta(w) + V_2\, \delta(w - \pi)] \\
&+ \int d^4x\, dy\, dz\, dw\ [V_3\, \delta(z) + V_4\, \delta(z - \pi)] \\
&+ \int d^4x\, dy\, dz\, dw\ [V_3\, \delta(y) + V_4\, \delta(y - \pi)]
\end{aligned}
\tag{29}
$$

with appropriate actions (S_5) for twelve possible 4-branes at the edges $(z, w) = (0, 0), (0, \pi), (\pi, 0), (\pi, \pi)$, $(z, y) = (0, 0), (0, \pi), (\pi, 0), (\pi, \pi)$ and $(y, w) = (0, 0), (0, \pi), (\pi, 0), (\pi, \pi)$ and eight possible 3-branes at the corners, $(y, z, w) = (0, 0, 0), (0, 0, \pi), (0, \pi, 0), (0, \pi, \pi), (\pi, 0, 0), (\pi, 0, \pi), (\pi, \pi, 0)$ and (π, π, π). As a natural extension to our previous result we make the following metric ansatz:

$$
ds^2 = f^2(w) \left[b^2(z) \left\{ a^2(y)\eta_{\mu\nu}dx^\mu dx^\nu + R_y^2\, dy^2 \right\} + r_z^2\, dz^2 \right] + dw^2
\tag{30}
$$

where $\eta_{\mu\nu} = diag(-1, 1, 1, 1)$. Solving Einstein's equation, one obtains, for the metric coefficients $a(y), b(z)$ and $f(w)$,

$$
a^2(y) = e^{2\,c\,y}
$$

$$
b^2(z) = \begin{cases}
b_1^2(z) &= \dfrac{c^2}{k^2\, R_y^2}\, \cosh^2[k\, r_z\, (z - z_0)] \\[2ex]
b_2^2(z) &= \dfrac{-c^2}{k^2\, R_y^2}\, \sinh^2[k\, r_z\, (z - z_0)]
\end{cases}
$$

$$
f^2(w) = \begin{cases}
f_1^2(w) &= \dfrac{15\, k^2}{\Lambda_7}\, \cosh^2\left[\sqrt{\dfrac{\Lambda_7}{15}}\, \mathcal{R}_w\, (w - w_0)\right] \\[2ex]
f_2(w) &= -\dfrac{15\, k^2}{\Lambda_7}\, \sinh^2\left[\sqrt{\dfrac{\Lambda_7}{15}}\, \mathcal{R}_w\, (w - w_0)\right]
\end{cases} ,
\tag{31}
$$

where we have assumed that $c^2 > 0$.

Note that functions of the form $b(z) \sim e^{\alpha z}$, or $f(w) \sim e^{\beta w}$ are not solutions. As eqn.(31) suggests, there are four materially different solutions in the bulk, $[f_i(w), b_j(z)]$, $i,j = 1,2$. However, it is easy to see that for three of the four combinations, the seven dimensional spacetime is endowed with two timelike and five spacelike direction. Discarding such solutions, we are left with only $f(w) = f_1(w)$, $b(z) = b_1(z)$, or in other words,

$$
\begin{aligned}
ds^2 &= \frac{\cosh^2(\ell\,w)}{\cosh^2(\ell\,\pi)} \left\{ \frac{\cosh^2(k\,z)}{\cosh^2(k\,\pi)} \left[\exp\left(-2\,c\,y\right) \eta_{\mu\nu} dx^\mu dx^\nu + R_y^2\, dy^2 \right] + r_z^2\, dz^2 \right\} + \mathcal{R}_w^2\, dw^2 \\
\ell^2 &\equiv \frac{\Lambda_7\, \mathcal{R}_w^2}{15} \\
k &\equiv \frac{\ell\, r_z}{\mathcal{R}_w\, \cosh(\ell\,\pi)} \\
c &\equiv \frac{\ell\, R_y}{\mathcal{R}_w\, \cosh(k\,\pi)\, \cosh(\ell\,\pi)} = \frac{k\, R_y}{r_z\, \cosh(k\,\pi)}\ .
\end{aligned}
\tag{32}
$$

As before, the factors of $\cosh(\ell\,\pi)$ and $\cosh(k\,\pi)$ in the metric are included to ensure that the natural scale never surpasses unity.

It may be observed that the 5-brane at $w = \pi$ does not have a flat metric (y- and z-dependences). Now, to obtain substantial warping in the w-direction (from $w = \pi$ to $w = 0$), one would need $\ell\pi$ to be substantial (same order of magnitude as the usual Randall-Sundrum case). However, this immediately means that both k and c in eqn.(32) are small (for $r_z, R_y \sim \mathcal{R}_w$). Which, in turn, implies that we cannot have a large warping in either of y- and z-directions. Of course, if we do not demand a very large warping in w [$\cosh(\ell\,\pi) \sim \mathcal{O}(1)$, or, in other words, $\ell\,\pi \sim \mathcal{O}(1)$], then we can have a large warping in z (or y).

The seven-dimensional (triply-warped) theory, then, has a structure very analogous to that of the six-dimensional (doubly-warped) one, not only in the functional dependence of the metric, but also as far as the extent of warping is concerned. As can easily be recognised, the solution can be almost trivially extended to even higher dimensions.

Note that orbifolding demands that we have to have branes situated at the edges of the n-dimensional hypercube, and possibly 3-branes at the corners. Now, if one direction (say z_1) suffers from a large warping, then those in the other directions are necessarily small. This, then, leads to a situation where all the 3-branes at the same $z_1 (= z_1^0)$ coordinate as ours must have a natural scale relatively close to ours (TeV), although still separated from us by the small warpings in the $(n-1)$ directions orthogonal to us. In other words, if we have SM-like fields in each of these 3-branes, the apparent mass-scales (on each brane) would be close to TeV with some splittings. This leads to a phenomenologically interesting possibilities which we discuss in the following section.

5 Some Phenomenological Consequences

5.1 Fermion Masses

We now speculate on some possible phenomenological consequences and constructs. The hierarchy among the masses of the standard model fermions has been a subject of interest for a long time. There have been various efforts in this direction through scenarios like radiative corrections, different grand unification schemes etc. [16]. In a slightly different context of a universal extra dimensional model it has been shown [17] that the requirement of anomaly cancellation in presence of two extra dimensions constrains the number of fermion generations in standard model to three. We now explain how our model of multiple warped geometry can give rise to the observed mass splitting in these standard model fermions. As we have seen in Section 2, in a 6-dimensional doubly warped scenario, the extent

of warping in the two directions are, in general, very different. In fact, if warping were to explain the large hierarchy between the Planck scale and the apparent scale for the electroweak interactions (namely the TeV scale), then the two warpings necessarily have to be very different in magnitude. In other words, we have a situation such that there are two branes close to the Planck scale with two more being at the electroweak scale. And the second TeV-like brane could as well have a natural scale slightly below us as above us. In other words, if we have SM-like fields in each of these 3-branes, the apparent mass-scales (on each brane) would be close to TeV with some splitting between them.

Now, imagine the SM fermions being defined by 5-dimensional fields, restricted to the 4-brane at $z = 0$, which now defines the "bulk" for these fields. If the major warping has occurred in the z-direction, then the natural mass scale of these fields is still $\mathcal{O}(\text{TeV})$. The presence of a y-dependence in the metric obviously leads to a non-trivial bulk wavefunction. Furthermore, since this 4-brane also intersects two other 4-branes at $y = 0$ and $y = \pi$ respectively, on the resultant 3-branes, the fermion fields are allowed have brane-kinetic terms in addition to the bulk kinetic term [18]. The presence of such boundary kinetic terms immediately alters the fermion wavefunction in the bulk (4-brane) as well as on the 3-branes. This, in turn, changes the overlap of the fermion wavefunction with that of a scalar located on the 3-brane and thus the effective Yukawa coupling. Note that the brane kinetic term is the resultant of interactions of the given fermion field with the other fields on the brane. Thus, slightly differing interactions on the distant 3-brane would result in a hierarchy amongst the effective Yuakawa couplings on our 3-brane and hence the fermion masses.

It is easy to see that this feature is repeated in the case of higher-dimensional ($d = 4+n$) constructs (Sections 3, 4). In addition, certain other features may also appear. Note that orbifolding demands that we have to have branes situated at the faces and edges of the n-dimensional hypercube, and possibly 3-branes at the corners. Once again, if one direction (say z_1) suffers from a large warping, then those in the other directions are necessarily small. This, then, leads to a situation where all the 3-branes at the same $z_1(= z_1^0)$ coordinate as ours must have a natural scale relatively close to ours (TeV), although still separated from us by the small warpings in the $(n-1)$ directions orthogonal to us. Now consider the different SM fermion fields to be higher-dimensional ones, but confined to *different* p-branes, which are all situated at $z_1 = z_1^0$ and intersect to give our 3-brane. For each such fermion, the corresponding p-brane defines the bulk. On account of the slightly different warping on each of these p-branes. these fermions will have differing expressions for the wavefunction in the respective bulk and thus on the SM 3-brane, thereby resulting in a hierarchy of Yukawa couplings. Furthermore, the fermions would be associated with naturally differing brane kinetic energies which, in turn, leads to further fine-tuning of the Yukawas. It should be noted that the above is only a plausibility argument in favour of a geometrodynamical origin of fermion Yukawa masses in a multi-warped universe. A realistic structure needs yet to be constructed.

5.2   Graviton tower

A different consequence, not necessarily related to the one discussed above, pertains to the nature of the Kaluza-Klein towers. Assuming, for simplicity, that the SM fields are confined to our 3-brane alone, we are faced with just one relevant field, namely the graviton. Clearly, we have a mutiple, and intertwined, tower in place. To divine the exact nature of the tower including the spacings between the modes and the corresponding eigenfunctions, requires us to solve the graviton equation of motion. This can be done, in the weak-field limit, a la Randall-Sundrum on effecting some changes in variables. Although it is obvious that the exact equations are much more complicated than in the RS case, several qualitative features are easy to appreciate:

- If, as has been argued already, the bulk of the hierarchy problem is addressed by the exponential warping in the y-direction, then k is small, and the z-dependence of the metric is small. This, then, reduces the the situation to essentially a RS $\otimes$ ADD one, with the ADD radius being very small. In other words, the graviton tower, as felt by low-energy experiments, would be almost

identical to the RS case.

- For the opposite case, viz. $k\pi \sim 10$, we again have a similar scenario, with the caveat that the graviton wavefunction (and masses) would be changed somewhat compared to the RS case.

- Especially with the opening of many (upto 6, if we have string theory in mind) extra dimensions, a more interesting possibility opens itself. If we allow for a progression of small ($\lesssim \mathcal{O}(10)$) hierarchies between the moduli, then many intermediate scales become available to us. The multiplicity of towers as well as the intermediate scale may become very relevant in collider phenomenology.

6 Conclusions

In summary, the exact solutions of higher dimensional Einstein's equation for a mutiply warped spacetime with negative cosmological constant has been found. It is shown that the hierarchy problem can be resolved geometrically without invoking any further hierarchy among the various moduli provided the warping is large in one direction and small in the other. Thus, in the case of a six dimensional spacetime, one of the compact dimensions is nearly flat while the other is srongly warped. The resulting geometry is thus similar to a combination RS and ADD scheme of compactifications. We have further shown that such a situation automatically leads to a spectral splitting of scales around the Planck and TeV scale thereby providing a clue to the mass splitting of the standard model fermions. The 4-dimensional brane tension turns out to be dependent on compact coordinates, indicating the existence of an effective scalar field distribution along the branes which is expected to have non-trivial effects on the physics in the bulk. We further speculate that the excitation of bulk fields like scalar, gravity , gauge and higher form tensor fields alongwith their appropriate Kaluza-Klein modes may give rise to interesting phenomenological signatures in our search for extra dimension in the forthcoming collider experiments.

Acknowledgments

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