IJHS, 12.1 (1977) 1-18

THE METHOD OF INTEGRAL SOLUTION OF INDETERMINATE EQUATIONS OF THE TYPE : BY = AX ± C IN ANCIENT AND MEDIEVAL INDIA*

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(*Received 10 November 1975*)

Āryabhaṭa I (476 A.D.) and other Indian scholars have given a general method of integral solution of indeterminate equations of the type : $by = ax \pm c$, which involves the knowledge of continued fraction : $p_n \cdot q_{n-1} - q_n \cdot p_{n-1} = \pm 1$ according as n is even or odd, where $\frac{p_n}{q_n}$ and $\frac{p_{n-1}}{q_{n-1}}$ are the n-th and $(n-1)$-th successive approximations of a/b. The paper presents a detailed discussion of the knowledge used by Āryabhaṭa I in his method of solution and of contributions made by the scholars like Bhāskara I, Āryabhaṭa II, Bhāskara II and other medieval scholars towards further simplification and modifications of the rule of Āryabhaṭa I. The original Sanskrit passages from the text of *Karaṇapaddhati* (1596 A.D. ?) with English translations have also been presented in the paper to show to what extent the perfection of the technique has been attained by the author of *Karaṇapaddhati* following traditional lines and the knowledge found its way to other cultural areas.

1. Introduction

Āryabhaṭa I (476 A.D.) and other Indian scholars have given a general method of integral solution of the indeterminate equations of the type by $= ax \pm c$, which involves knowledge of continued fraction. The continued fraction is a process of converting a fraction into a continued division and seems to have arisen in connection with the problem of finding the approximate square-root of numbers that are not perfect squares. Various methods of finding the approximate values of $\sqrt{A}$, where A is an integer, by excess or in defect, were known in different cultural areas of ancient and medieval periods. The earliest important step in the theory of continued fraction is known to have been made by Euclid[1] (*c.*300 B.C.) who applied the process of finding the greatest common divisor of two lines to the greatest common divisor of two numbers. No further improvement of the knowledge has been made by the Greeks except by the Indians who have made a systematic application of the knowledge from the fifth century A.D. onwards. In the sixteenth century

*Read in the Summer School on History of Science organised under the auspices of the Indian National Science Academy in September 2-11, New Delhi, 1974.

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two European scholars Bombelli (b. 1530) and Cataldi (1548-1626) have made a most systematic use of the knowledge of continued fraction, though it is not yet known how they have discovered it[2]. The knowledge of continued fraction based on a method of finding the greatest common divisor of the prime numbers a and b may be placed as under :

$$
\begin{array}{l}
b)\; a \;\; (a_1 \\
\quad \underline{ba_1} \\
\quad r_1)\; b \;\; (a_2 \\
\qquad \underline{r_1a_2} \\
\qquad r_2)\; r_1 \;\; (a_3 \\
\qquad\quad \underline{r_2a_3} \\
\qquad\quad r_3)\; r_2 \;\; (a_4 \\
\qquad\qquad \underline{r_3a_4} \\
\qquad\qquad\quad r_4 \\
\qquad\qquad\quad \cdot \\
\qquad\qquad\quad \cdot \\
\qquad\qquad\quad \cdot \\
\qquad\qquad\quad r_{n-2})\; r_{n-3} \;\; (a_{n-1} \\
\qquad\qquad\qquad\quad \underline{r_{n-2}a_{n-1}} \\
\qquad\qquad\qquad\quad r_{n-1})\; r_{n-2} \;\; (a_n \\
\qquad\qquad\qquad\qquad\quad \underline{r_{n-1}a_n} \\
\qquad\qquad\qquad\qquad\qquad r_n
\end{array}
$$

In modern symbols,

$$\frac{a}{b} = a_1 + \frac{1}{a_2+}\;\frac{1}{a_3+}\;\frac{1}{a_4+}\;\cdots \quad \text{and so on.}$$

Here

$$\left.\begin{aligned}
a &= ba_1 + r_1 \\
b &= r_1a_2 + r_2 \\
r_1 &= r_2a_3 + r_3 \\
r_2 &= r_3a_4 + r_4
\end{aligned}\right\} \qquad \ldots \quad (1)$$

and so on.

When $r_n = 0$

$$\frac{a}{b} = a_1 + \frac{1}{a_2+}\;\frac{1}{a_3+}\;\frac{1}{a_4+}\;\cdots\;\frac{1}{a_{n-1}+}\;\frac{1}{a_n}$$

If $\frac{p_1}{q_1}, \frac{p_2}{q_2}, \frac{p_3}{q_3}, \ldots \frac{p_n}{q_n}$ be the successive approximations of $\frac{a}{b}$, then

$$\frac{p_1}{q_1} = a_1$$

$$\frac{p_2}{q_2} = a_1 + \frac{1}{a_2} = \frac{a_1 a_2 + 1}{a_2}$$

$$\frac{p_3}{q_3} = a_1 + \frac{1}{a_2 +}\ \frac{1}{a_3} = \frac{a_1(a_2 a_3 + 1) + a_3}{a_2 a_3 + 1}$$

$$\frac{p_4}{q_4} = a_1 + \frac{1}{a_2 +}\ \frac{1}{a_3 +}\ \frac{1}{a_4} = \frac{a_1[a_2(a_3 a_4 + 1) + a_4] + a_2 a_4 + 1}{a_2(a_3 a_4 + 1) + a_4}$$

and so on. ... (2)

Evidently $p_n q_{n-1} - q_n p_{n-1} = \pm 1$ ... (3)

according as n is even or odd.

The results (1), (2) and (3) were known to Āryabhaṭa I (476 A.D.) and other Indian scholars. Datta and a host of other writers[3] have given English translation with modern interpretations of the rule of Āryabhaṭa I and of other scholars upto the time of Bhāskara II (*b.* 1150) and presented interpretation of the results in modern form giving little importance to the knowledge of continued fraction. Here an attempt has been made, how the results (1), (2) and (3) of the continued fraction have been established and used in ancient and medieval India to obtain the solution of $y = \frac{ax+c}{b}$ or $x = \frac{by+c}{a}$ and of some such problems and found their way to other cultural areas.

2. Āryabhata I (b.476 A.D.)

The *Āryabhaṭīya* of Āryabhaṭa I which deals with the solution of indeterminate problems is also known as *Aśmakatantra*. He has been mostly referred as an *Āśmakīya* by Bhāskara I, which indicates that he belonged to *Aśmaka* tribe and lived in the region between the rivers Mahismati and Godavari.[4] He had his education in the school of Kusumpura, which has been identified with two places one near Patna and the other near Madras in Salempur district.[5] Since Āryabhaṭa I's commentators like Bhāskara I, Parameśvara, Rāmakṛṣṇa, Nīlakaṇṭha and others had associations with southern region of India and his Mss. are mostly available in the South, it is presumed he belonged to the South. According to Purāṇic division, the region below Vindhya was regarded as South. This suggests that Āryabhaṭa I probably had his education in the Kusumpura school in Madras and established his own school in the Mahiṣmati and Godāvarī region.

(A) *Solution of* $y = \frac{ax+c}{b}$ *in positive integers when a and b are prime to each other.*[6]

To get a solution for y, the rule as given by Āryabhaṭa I in his *Āryabhaṭīya*[7] prescribes that a and b are to be mutually divided, then the remainder r_{n-1} obtained after some stages (vide H.C.F. process shown before) is multiplied by a number t, known as *mati*, such that the product when added to the *kṣepa* quantity c(c may be positive or negative), is exactly divisible by the subsequent remainder r_{n-2}. Symbolically it may be written as,

$$\frac{r_{n-1}\,t \pm c}{r_{n-2}} = s \text{ or } n \text{ (say)} \qquad \dots \quad (4)$$

Āryabhaṭa I did not explain when c is to be added or subtracted. Bhāskara I (c.600 A.D.) in his *Āryabhaṭīya-bhāṣya* has written, *sameṣu kṣiptaṃviṣameṣu śodhyam*,[8] i.e. "add (the *kṣepa* quantity c) when n (total number of quotients of mutual division) is even and subtract when n is odd. Obviously two cases arise.

Case I : When n is even (i.e. the number of partial quotients used by *Āryabhaṭa I is always one less i.e. odd, vide Table I below*)

Let $n = 4$, then from equation (4), we can write,

$$\frac{r_3 t + c}{r_2} = s, \text{ when } t = \frac{r_2 s - c}{r_3}$$

Evidently the process of division (vide H.C.F. process) stops at the residue r_3 after the partial quotients a_1, a_2, a_3 are obtained. According to the rule given by Āryabhaṭa I, the partial quotients a_1, a_2, a_3, with *mati* t, and final quotient s [calculated from eqn. 4] are to be placed on below the other in Column I of the Table I and the operation in the 2nd, 3rd and 4th column are performed by the Āryabhaṭa I's rule, *adhaupari guṇitamantyāyug*, *i.e.* "(the numbers) below and above are multiplied and the last is added to it" to obtain the final results u and l as follows :

TABLE I

1st column	2nd column	3rd column	4th column
a_1	a_1	a_1	$a_1 l + m = u$ (say)
a_2	a_2	$a_2 m + t = l$ (say)	l
a_3	$a_3 t + s = m$ (say)	m	
t	t		
s			

Now let us calculate the value of m, l and u and investigate their nature.

Here, $\quad m = a_3 t + s$

$$= \frac{a_3(r_2 s - c)}{r_3} + s$$

$$= \frac{s(r_2 a_3 + r_3) - a_3 c}{r_3}$$

$$= \frac{sr_1 - a_3c}{r_3} \text{ [by equation (1)]}$$

Again, $l = a_2m + t$

$$= \frac{a_2(sr_1 - a_3c)}{r_3} + \frac{r_2s - c}{r_3}$$

$$= \frac{s(a_2r_1 + r_2) - c(a_2a_3 + 1)}{r_3} \text{ [by equation (1)]}$$

$$= \frac{sb - c(a_2a_3 + 1)}{r_3}$$

and $u = a_1l + m$

$$= a_1 \frac{sl - c(a_2a_3 + 1)}{r_3} + \frac{sr_1 - a_3c}{r_3}$$

$$= \frac{s(a_1b + r_1) - c[a_1(a_2a_3 + 1) + a_3]}{r_3}$$

$$= \frac{sa - c[a_1(a_2a_3 + 1) + a_3]}{r_3}$$

Now $$\frac{u}{l} = \frac{sa - c[a_1(a_2a_3 + 1) + a_3]}{sl - c(a_2a_3 + 1)}$$

$$= \frac{sa - cp_3}{sb - cq_3}$$

If there are n number of partial quotients,,

$$\frac{p_n}{q_n} = \frac{a}{b} \text{ and } \frac{u}{l} = \frac{sa - cp_{n-1}}{sb - cq_{n-1}}$$

Evidently,

$$p_n.l - q_n.u$$
$$= p_n(sb - cq_{n-1}) - q_n(sa - cp_{n-1})$$
$$= -c(p_nq_{n-1} - q_np_{n-1})[\because p_n = a,\ q_n = b]$$
$$= -c \text{ (by equation (3), when } n \text{ is even)} \qquad \ldots (5)$$

Case II. When n is odd (i.e. the number of partial *quotients is even, vide Table II(below)*.

Let $n = 5$, then from equation (4), we get $\frac{r_4t - c}{r_3} = n$

when $$t = \frac{r_3q_2 + c}{r_4}$$

Now we place below the partial quotients a_1, a_2, a_3, a_4, *mati* t, then final quotient n, and obtain l and u following the process as explained before.

TABLE II

1st column	2nd column	3rd column	4th column	5th column
a_1	a_1	a_1	a_1	$a_1 l + y = u$ (say)
a_2	a_2	a_2	$a_2 y + x = l$ (say)	l
a_3	a_3	$a_3 x + t = y$ (say)	y	
a_4	$a_4 t + n = x$ (say)	x		
t	t			
n				

If we calculate x, y, l and u proceeding as before, we find

$$\frac{u}{l} = \frac{na+cp_4}{nb+cq_4}$$

When there are n number of partial quotients, we can write,

$$\frac{p_n}{q_n} = \frac{a}{b} \text{ and } \frac{u}{l} = \frac{na+cp_{n-1}}{nb+cq_{n-1}}$$

Evidently

$$p_n.l - q_n.u$$

$$= p_n(nb+cq_{n-1}) - q_n(na+cp_{n-1})$$

$$= c(p_n q_{n-} \quad\quad \because p_n = a,\ q_n = b]$$

$$= -c\ [\quad\quad \text{on (3) when } n \text{ is odd}] \qquad \ldots \quad (6)$$

The equation (5) and (6) are identical. This was purposely done by Āryabhaṭa I, thereby establishing equation (3) for he always wrote

$$p_n.l - q_n.u = -c \text{ or } q_n.u = p_n.l + c \qquad \ldots \quad (7)$$

when u and l were calculated from the Table I and II according as n is even or odd.

From (7) we have

$$bu = al + c \ (\because p_n = a,\ q_n = b)$$

Comparing with the equation $by = ax + c$, obviously $y = u$ and $x = l$ gives the solution. The two numbers u and l were known as (final) *upari* and *adha*, *dvāveva rāśi*, *phala* and *guṇa* etc. by the Indian scholars.

If $u = am + y_1$, and $l = bm + x_1$

then $y = y_1$ (mod. a) and $x = x_1$ (mod. b) are the general solution of $y = \dfrac{ax+c}{b}$.

(B) *Solution of* $x = \dfrac{by+c}{a}$ *in positive integers when a and b are prime to each other*

The method of solution is exactly similar, only select

$$\frac{r_{n-1}t \pm c}{r_{n-2}} = s \text{ or } n$$

according as the number of partial quotients in the Table is odd or even

Then $\qquad x = u = bm + x_1$

$\therefore \qquad x = x_1 \,(\text{mod.b})$

and $\qquad y = l = am + y_1 = y_1 \,(\text{mod. } a)$

(C) *Illustration* :

$$45x + 7 = 29y$$

Following Āryabhaṭa I s rule, let us divide 45 by 29 following a H.C.F. process.

now,

```
29)45(1
   29
   ---
   16)29(1
      16
      ---
      13)16(1
         13
         --
          3)13(4
            12
            --
             1)3(3
               3
               -
               0
```

Hence $\qquad \dfrac{45}{29} = 1 + \dfrac{1}{1+}\;\dfrac{1}{1+}\;\dfrac{1}{4+}\;\dfrac{1}{3}$

Case I. If the process of division is stopped at the remainder 3, the number of partial quotients, is odd : then we write,

$$\frac{3 \times 2 + 7}{13} = 1$$

Here *mati* = 2, final quotient = 1

Arranging the quotients 1, 1, 1, *mati* 2, and final quotient 1 in Column I of Table III and the numbers l and u are obtained as follows.

TABLE III

I	II	III	IV
1	1	1	$1\times5+3=8=u$ (say)
1	1	$1\times3+2=5$	$5=l$ (say).
1	$1.2+1=3$	3	
2 (*mati*)	2		
1			

Evidently $y=u=8=45\times0+8=8$ (mod. 45)

and $x=5$ (mod. 29).

Case II. When the process of division continues upto the remainder 1, *and the number of partial quotients are even*

Then
$$\frac{1\times10-7}{3}=1.$$

Here *mati* $=10$, final quotient $=1$.

Placing the quotients 1, 1, 1, 4, *mati* 10, and final quotient 1, one below the other and proceeding the operation as before, we get, $u=143$, and $l=92$.

$$y=u=143=45\times3+8=8\ (\text{mod. } 45).$$

$$x=l=92=5\ (\text{mod. } 29).$$

This shows that the method of solution based on continued fraction was rightly understood by Āryabhaṭa I (476 A.D.).

3. Bhaskara I (600 A.D.)

Bhāskara I was not a direct disciple of Āryabhaṭa I. He imbibed his knowledge of astronomy from his father[9] who has been an astronomer of the school of Āryabhaṭa I. In his commentary on the *Āryabhaṭiya* he has adduced six mathematical problems on remainder and twenty-four astronomical problems[10] to explain the rule of Āryabhaṭa I. The method of indeterminate analysis used here is known as *kuṭṭākāra* or simply *kuṭṭa* by Bhāskara I. In English equivalent it is known as 'pulverizer', since the values of the coefficients (a, b) become smaller and smaller. Bhāskara I has classified the pulverizer into two groups, viz. residual pulverizer (*sāgra kuṭṭākāra*) and non-residual pulverizer (*niragra kuṭṭākāra*). Such classifications is not available in the *Brāhmasphuṭa-siddhānta* of Brahmagupta (628 A.D.), a contemporary of Bhāskara I, but same classification is found to have been mentioned by Govinda (800-850), Paramośvara (1430) and other medieval scholars. The terms *sama-* and *viṣama-kuṭṭākara* have also been used by Mahāvīra (850) and the author of *Yuktibhāṣā* (c. 1500-1600).

(A) *Solution of* $y = \frac{ax-c}{b}$

Bhāskara I has applied the method of Āryabhaṭa I to solve astronomical problems like calculation of *ahargaṇa,* complete revolutions performed by a planet, expressed mainly in the form

$$y = \frac{ax-c}{b} \qquad \ldots \quad (8)$$

where a = *bhājya,* dividend, or revolution number of planets; b = *hāra,* divisor or civil days in a yuga; c = *agra,* residue of the revolution of the planets; x = *guṇakāra* or *ahargaṇa,* and y = *phala,* or complete revolutions performed by a planet. He has given a rule[11] indicating the method of solution of equation (8). Now it may be seen that the equation (8) of Bhāskara I and Āryabhaṭa I's equation $x = \frac{by+c}{a}$ are identical with only difference that the former is expressed in terms of y and the latter in terms of x. In the method of solution of Bhāskara I the number b is divided by a the reason of which is obviously understood if one is acquainted with the method of solution of Āryabhaṭa I. Bhāskara I has preferred in his calculations always an even number of partial quotients, perhaps to avoid repetition of results, obtained from odd number of quotients. Obviously for solution, the Case II of Āryabhaṭa I's rule (B) is applicable here.

$$\left.\begin{array}{ll} \text{Hence} & x = u = x_1 \pmod{a} \\ \text{and} & y = l = y_1 \pmod{b} \end{array}\right\}$$

is the general solution of $y = \frac{ax-c}{b}$

(b) *Further Simplification of the Rule*[12]

The quantities, a, b, c of the equation $y = \frac{ax-c}{b}$ in astronomical problems are generally very large and its solution is a laborious affair. To avoid this, Bhāskara I has supplied the least integral values of a, b, x, and y for planets satisfying the pulverizer $\frac{ax-1}{b} = y$ and then applied following procedures to obtain the solution of

$$y = \frac{ax-c}{b}$$

(i) If $x' = r$, $y' = s$ be a solution of $\frac{ax'-1}{b} = y'$, then $x = cr$ and $y = cs$ is a solution of

$$\frac{ax-c}{b} = y \qquad \ldots \quad (9)$$

Evidently, $$\frac{ax'-1}{b}=y'$$

or, $$\frac{ar-1}{b}=s$$

or, $$\frac{a(cr)-c}{b}=(cs)$$

or, $$\frac{ax-c}{b}=y.$$

(ii) If $x=r$, $y=s$ be a solution of $\frac{a'x-c}{b}=y$

then $x=r$ and $y=mr+s$ is a solution of $\frac{ax-c}{b}=y$

where $$a=mb+a'(a<b). \qquad \ldots (10)$$

Evidently, $\frac{a'x-c}{b}=y$

or, $a'r-c=bs$

or, $(mb+a')r-c=b(mr+s)$

or, $ar-c=b(mr+s)$

$x=r_1$ and $y=mr+s$ is s solution of $\frac{ax-c}{b}=y.$

(iii) If $x=r$, $y=s$ be the minimum solution of $y=\frac{ax-c}{b}$

then other solutions of the same pulverizer are :

$$\left.\begin{array}{l} x=bm+r \\ y=am+s \end{array}\right\} \qquad \ldots (11)$$

4. Scholars from seventh to sixteenth century

The method was subsequently discussed by Brahmagupta (*c.* 628), Pṛthudaka (*c.* 850) and Śrīpati (1039), Govindasvāmī (*c.* 850), Mahāvīra (850) with no improvement over Āryabhaṭa I and Bhāskara I's rule[13]. It was elaborated with further simplification[14] by Āryabhaṭa II (950), who continued the mutual division till the remainder in the process of division becomes 1. Then the table (*valli*) was made with quotients a_1, a_2, a_3 and with 1, attached at the end. Evidently it gives :

$$\frac{u}{l}=\frac{a_1(a_2a_3+1)+a_3}{a_2a_3+1}=\frac{p_3}{q_3}$$

If there be n number of partial quotients,

$$\frac{p_n}{q_n} = \frac{a}{b} \quad \text{and} \quad \frac{u}{l} = \frac{p_{n-1}}{q_{n-1}}$$

Evidently, $p_n.l - q_n.u = \pm 1$ according as the number of partial quoients are odd and even

$\therefore \quad q_n u = p_n.l \mp 1$, according as number of partial quotients are odd and even

$= p_n l \pm 1$ according as number of partial quotients are even and odd.

or $\qquad bu = al \pm 1 \qquad$,, $\quad$,, $\quad$,,

or, $\qquad by = ax \pm 1 \qquad$,, $\quad$,, $\quad$,,

This gives the solution of $by = ax \pm 1$ ... (12)

where $\qquad y = u$, and $x = l$.

If (x_1, y_1) be the least solution of (12), then (cx_1, cy_1) is the solution of $by = ax \pm c$.

Obviously two cases arise :

Case I. When the number of quotients are even, the method gives the solution of $by = ax + c$.

Case II. When the number of quotients are odd, the method gives the solution of $by = ax - c$. Then $x = (b - x_1) + bt$, and $y = (a - y_1) + at$ is the general solution of $by = ax + c$.

Bhāskara II[15] (*b.* 1150) has simplified further the method of Āryabhaṭa II by continuing the mutual division till the remainder becomes unity and prepared the table (*valli*) with partial quotients, along with c and o. This shows that Bhāskara II directly calculated $bu = al \pm c$

$$\text{where} \qquad \frac{u}{l} = \frac{cp_{n-1}}{cq_{n-1}}.$$

The remaining operations are similar to that of Āryabhaṭa II. Exactly the same method of Bhāskara II has been given in different words by Nārāyaṇa[16] (1356 A.D.)and Kamalākara[17] (1658 A.D.). The South Indian scholar Parameśvara[18] (1430) in his commentary on the *Āryabhaṭiya* has cited two examples with explanation the rule of Āryabhaṭa I and several others in his *Siddhāntadīpikā*, a commentary on the *Mahābhāskarīya* of Bhāskara I, and considered solution for even number of quotients only. Nilakaṇṭha[19] (1500), however furnished a detailed account of the rule of Āryabhaṭa I in his *Āryabhaṭīyabhāṣya* with quotations from the works of Bhāskara I, Govindasvāmi, Bhāskara II and Parameśvara without any further contribution to the source.

5. Treatment in the *Karaṇapaddhati*[20] (16th century ?) and *Yuktibhāṣā*[21] (1500-1600)

The *Karaṇapaddhati* furnishes interesting information on the calculation of approximations of *mahāhāra* : *mahāguṇa* (i.e. $b : a$) and has applied the both upward (below-top) and downward (top-below) techniques to calculate the successive approximations. The relevant verse[22] runs as follows :

anyonyaṃ vibhajenmahāguṇahārau yāvadvibhakte'lpata
tāvallabdhaphalāni rūpamapi ca nyasyedadho'dhaḥ kramāt
prakṣipyāntyamupāntimena guṇite svordhve tadantyaṃ tyajed
bhuyo'pyeṣa vidhirbhaved guṇahārau syātām tadordhvasthitau ||

English Translation :

"The *mahāguṇa* (*a*) and *hāra* (b) should be simultaneously divided till it becomes negligible (i.e. zero). The respective quotients and a number one are placed one below the other. Then multiply the last but one (*upānta*) by the number placed above it (*upānta ūrddha*) and the last is added to it, and then this last number is left. This is the process of obtaining the *guṇa* and *hāra* placed-in two *urdha* positions."

This method upward (below-top) technique is similar to that of Bhāskara I with the difference that the mutual division is repeated upto the remainder zero, like that of our modern method.

For example,

$$\text{let } \frac{b}{a} = \frac{355}{113} = 3+\cfrac{1}{7+\cfrac{1}{16}}$$

The rule gives a process of calculating the final convergent $\frac{p_n}{q_n}$. Here for $n = 3$, $\frac{p_3}{q_3} = \frac{b}{a} = \frac{355}{113}$. This has been obtained from the quotients 3, 7, 16 and 1 placed one below another following the process discussed before.

The *Karaṇapaddhati* has given an alternative (top-below) process of this rule for calculating the successive approximations of circumference : diameter, i.e. $\frac{p_1}{q_1}, \frac{p_2}{q_2}, \frac{p_3}{q_3}$ ... etc., in the following rule.[23]

anyonyāhṛtabhājyahārakaphalaṃ sarvaṃ tvadho'dho
nyasedekatrādyaphalena hīnamaparatraikam dvayaścapari /
kurjād valyupasaṃhṛtiṃ hyuparitaḥ pūrvapraṇāśaṃ vinā
tyājyaṃ tatprathamordhvagaṃ hāraguṇāśśiṣṭāśca vā svecchayā //

L lish Translation :

"The *hāra* amd *bhājya* (*guṇa*) are simultaneously divided and the results are all placed systematically one below the other in one place (first place). Put the partial quotients without the first quotient in another place (second place) and place one over both the places. Perform the *vallī* operations from the top and leave the number of the first place which was not destroyed in the operation. The *hāra*, *guṇa* and the remaining results (are obtained) as desired.

This is undoubtedly a reverse (top-below) technique of calculating $\frac{p_1}{q_1}, \frac{p_2}{q_2}, \frac{p_3}{q_3}$...etc., followed before. Here the *vallī* operation is performed starting from top like that of our modern method of calculation as follows :

1st place	*hāra*	2md place	*guṇa*
1			
a_1	$a_1(=p_1)$	1	$1(=q_1)$
a_2	$a_1a_2+1(=p_2)$	a_2	$a_2(=q_2)$
a_3	$a_3\{a_1a_2+1\}+a_1\,(=p_3)$	a_3	$a_2a_3+1(=q_3)$
a_4	$a_4[a_3\{a_1a_2+1\}+a_1](=p_4)$	a_4	$a_4\{a_2a_3+1\}+a_2(=q_4)$

Hence the successive approximations of circumference (*hāra*) diameter (*guṇa*), i.e. $\frac{p_1}{q_1}, \frac{p_2}{q_2}, \frac{p_3}{q_3}$ ... etc. are :

$$\frac{p_1}{q_1}=\frac{a_1}{1},\ \frac{p_2}{q_2}=\frac{a_1a_2+1}{a_2},\ \frac{p_3}{q_3}=\frac{a_3\{a_1a_2+1\}+a_1}{a_2a_3+1} \text{ and so on.}$$

The author of *Yuktibhāṣā* (vide appendix of the edited text) has used the same technique to calculate $\frac{p_1}{q_1}, \frac{p_2}{q_2}, \frac{p_3}{q_3}\ldots$.

This shows that Indian scholars had a more or less distinct idea about the application of continued fraction and used the tool $p_nq_{n-1}-q_np_{n-1}=\pm1$ for the solution of $by=ax\pm c$, according as n is even or odd.

6. Transmission

The Greek scholar Nicomachus of Geresa[24] (i.e. 100 A.D.) recorded an example of remainders. The Chinese classic, *Sun Tzu Suan Ching*[25] (4th century A.D.) have also a similar example on the problem of remainders. Both these problems were supplied with answers without any method of solution.

The Chinese scholar I-Hsing (687-727 A.D.) has made use of indeterminate analysis in his *Ta Yen Li Shu* (Book of the Ta Yen Calendar) for the solution of astronomical problems. The indeterminate analysis derived the name *Ta Yen* from 'Great Extension Number" based on a process of continued division.[26] According to George Sarton, *Ta Yen* was similar to the Hindu method of *kuṭṭaka* (pulverizer or multiplier).[27] Needham opined, "the argument of Matthiessen that they were different does not carry conviction".[28] Five centuries later Chhin Chiu-Shao (*c.* 1244) gave a full explanation of the subject in his *Shu Shu Chiu Chang*. His second and third problems appearing in the first and third chapters respectively with calculations approach very nearly to the method of I-Hsing, though the terminology was different. I-Hsing came to India in 673 A.D., became a Tantric-Buddhist monk and learnt Sanskrit. Hence it is quite possible that this Tantric--Buddhist astronomer acquired the technique of solving indeterminate problems from Indian works or scholars and it is through his effort, the knowledge was carried to China.

The Arabic scholar Abū Kāmil[29] (*c.*850-930) has given integral solutions of some indeterminate equations in his *Kitāb al-ṭarā'if fi'l ḥisāb* (Book of Rare Things in the Art of Calculations) but his solutions were obtained by trial. For this reason, Levey did not believe that the Indian Knowledge of indeterminate analysis had passed to the Arabs by the time of Abū-Kāmil.[30].

The problem of remainders has been treated by Ibn al-Haitam (*c.* 1000), and Leonardo Pisano (*c.* 1202) in his *Liber Abaci*.[31]

Regiomontanus (1436-76) got acquainted with the work of L. Pisano in Italy and proposed in a letter a problem on remainders similar to that of *Āryabhaṭa* I. A German Ms. of the fifteenth century has proved a general rule corresponding to the Chinese *Ta Yen* rule[32]. The other Latin European scholar took also active interest in the translations of many earlier Sanskrit texts through Arab and Chinese intermediaries. In view of this it is yet to be seen whether they have received any idea of the knowledge of continued fraction and its application in the general method of integral solution of $by = ax \pm c$ from the source.

NOTES AND REFERENCES

1 *Elements*, VII, 2 *ff* 3; compare with X, 3, 4.

2 Smith, D. E. *A Source Book in Mathematics*, 1, pp. 80-84, Dover Edition, 1959.

3 Datta, B. "Elder Āryabhaṭa's rule for the Solution of indeterminate equation of the first degrees", *Bull. Calcutta math. Soc*, 24, pp. 19-36, 1932; vide also his *History of Hindu Mathematics*, 2, pp. 87-99, Lahore, 1938; Rodet, L., "Lecons de Calcul d'Āryabhaṭa", *J. Asiatique*, 13, Series 7, pp 430-34, 1879; Kaye, G. R. "Notes in Indian Mathematics", *J. Asiatic Soc. Bengal*, No 4, pp. 111-41, 1908; Mazumdar, N. K. "Āryabhaṭa's rule in relation to indetermine equations of the first degree", *Bull. Calcutta math. Soc.*, 3, pp. 11-19, 1911-12; Sengupta, P. C. *Āryabhaṭiya*, translated into English—*Journal of the Department of Letters*, 16, pp. 27-30, 1927; Ganguly, S. K. "The Source of Indian Solutions of the so-called Pellian Equation", *Bull. Calcutta math Soc.*, 19, pp. 151-76, 1928; Clark, W. E. *Āryabhaṭiya*, tr. into English, Chicago, 1930; Sen, S. N. "Āryabhaṭa's Mathematics", *Bull. natn. Inst. Sci. India* 21, pp. 297-319, 1963; Volodarski, A. I. *Ārybhhaṭa*, Moscow 1977. *Āryabhaṭiya* of Āryabhaṭa critically edited with English translation 3 vols, by K. S. Shukla and K. V. Sarma INSA, New Delhi, 1976.

[4] According to Shukla, Āryabhaṭa I belonged to the *Aśmaka* country (*Mahābhāskarīya*, Eng tr p 2) According to Nīlakaṇṭha's comm (on *Āryabhaṭīya*, ii, 1, T S S No 101, p 1) he was *aśmakajanapadajāta*, i e born in that country. K V Sarma has not included Āryabhaṭa I in the list of Kerala astronomers (vide his *A Hist of KeralaSchool of Hindu Astronomy*, pp 41-81, Hoshiarpur, 1972) The *Aśmaka* country has been identified with Kerala by some scholars without any sound basis.

[5] *kusumpure'bhyarcitaṃ jñānam* (*Āryabhaṭīya, Gaṇita, v* 1)
The village *kusumpura* perhaps originated from the name of the tribe *Kusumas* that lived in *Dakṣiṇāpatha* (*Mārkaṇḍeya Purāṇa*, ch 57, verses 45-46), a territory identified with modern district of Madras, Chingleput, North and South Arcot, Salem and South-east portion of Mysore with Kanci (Conjeveram), as capital by Pargiter (*Mārkaṇḍeya purāṇa*, Eng Tr Calcutta, 1904) Near about 7th century A D the power of this tribe attained its zenith

[6] Āryabhaṭa I's rule was intended to find a solution for N such that $N = r_1 \pmod{a} = r_2 \pmod{b} = \ldots$ etc. The rule consequently is directed to solve an equation of the type: $by = ax \pm c$ (where $r_1 - r_2 = c$) according as $r_1 \gtrless r_2$.

[7] *adhikāgrabhāgahāraṃ chindyādūnāgrabhāgahāreṇa |*
śeṣaparasparabhaktaṃ matiguṇamagrāntare kṣiptam ||
adhauparigunitamantyayugūnāgracchedabhājite śeṣam |
adhikāgracchedaguṇaṃ dvicchedāgramadhikāgrayutam ||

(*Āryabhaṭīya, Gaṇita*, Verses 32-33).

Datta's English translation has been taken into consideration in preference to others, since it is based on the earliest commentary of Bhāskara I (600 A.D.).

[8] Datta, B. *Bull. Calcutta math. Soc.*, 24, p. 35, 1932.

[9] *Mahābhāskarīya*, edited by T. S. Kuppanna Sastry, Madras Govt. Oriental Series, Introduction, p. xiii, Madras, 1957.

[10] Shukla, K. S. "Hindu Mathematics in the Seventh Century as found in Bhāskara I's commentary on the *Āryabhaṭīya*", *Gaṇita*, 23, No. 1, pp. 57-79, 1972.

[11] *Mahābhāskarīya*, i. 41-44; vide also English translation of K. S. Shukla, pp. 29-30, Lucknow, 1960.

[12] *Mahābhāskarīya*, i, 45-46 i; i. 47; i. 50.

[13] Datta, B. and Singh, A. *History of Hindu Mathematics*, 2, pp. 101-110, Motilal Banarsidas, Lahore, 1938; vide also *Laghubhāskarīya*, pp. 103-14, Eng. Tr. by K. S. Shukla, Lucknow University, 1963.

[14] *Mahāsiddhānta*, Ch. XVIII, 1-14, vide Sudhakara Dvivedi's edition, Benares, 1910, vide also Datta & Singh, *History of Hindu Mathematics*, 2, pp. 104-109.

[15] *Bījagaṇita, Kuṭṭakavivaraṇa*, verses 50-57, 67.

[16] *Gaṇitakaumudī*, edited by Padmakara Dvivedi, Prince of Wales Saraswati Bhavana Tests No. 57, pt. II, pp. 213-14, Benares, 1942.

[17] *Siddhāntattvaviveka, Mahāpraśnādhikāra*, verses 179-190.

[18] Parameśvara's comm. on the *Āryabhaṭīyabhāṣya*, ed. by H. Kern, pp. 47-41, London, 1874; quotation from *Siddhāntadīpikā* (vide *Mahābhāskarīya*, ed. by T. S. Kuppanna Sastri, p. 55, Madras Govt. Oriental Series, No. 130, Madras, 1957).

[19] *Āryabhaṭīyabhāṣya*, TSS (Trivandrum Sanskrit Series) 101, pp. 161-180.

[20] Edited in the *Trivandrum Sanskrit Series* (*TSS*) No. 126. There is a great deal of difference of opinion as to the time of the *Karaṇapaddhati* (vide Bag, A. K., *Karaṇapaddhati* and its probable date of the text, *Indian J. Hist. Sci.*, 1, No. 2, pp. 98-106, 1966). The verse *vidvān stunnavala* ... quoted by Nīlakaṇṭha (*TSS*. 101, p. 118) to be of Mādhava appears in full detail in the *Karaṇapaddhati* (ch. 6) without any reference to Mādhava. The text

contains no reference to Nilkaṇṭha's *Tantrasaṃgraha* and *Yuktibhāṣā*. This suggests that the *Karaṇapaddhati* was written at a time more or less contemporaneous with that of *Tantrasaṃgraha* and *Yuktibhāṣā*.

[21] Edited with notes by Ramavarma (Maru) Tampuran and A. R. Akhilesvara Iyer, Mangoladayam Press, Trichur, 1948; as regards date, vide Sarma, K. V., "Jyeṣṭhadeva and his identification as the author of *Yuktibhāṣa* ?", *Adyar Library Bulletin*, 22, pp. 35-30, 1958.

[22] *Karaṇapaddhati*, ii. 5 (*TSS*. 126).

[23] *Karaṇapaddhati*, ii. 6.

[24] Dickson, L. E. *History of the Theory of Numbers*, 2, p. 58, New York, 1934.

[25] Wang, Ling, "The Date of the Sun Tzu Suan Ching and the Chinese Remainder Problems", *Proceedings of the Xth International Congress of Hist. of Science*, 1, pp. 489-96, Hermann, Paris, 1964.

[26] Needhan, J. "*Science and Civilization in China*," 3, pp. 119-20.

[27] Sarton, George, *Introduction to the Hist. of Science*, 2, Pt. 2, p. 626, Baltimore, 1931; reprinted 1950.

[28] Needham, J. *Ibid*., 3, p. 122, f.n.; vide also Mikami, Y. *The Development of Mathematics in China and Japan*, p. 58, Leipzig, 1913.

[29] Translated into German by H. Suter, "Das Buch der Seltenheiten der Rechenkunst von abū-kāmil as-Miṣrī", *Bibliotheca Mathematica*, 11, pp. 100-120, 1910-11.

[30] Levey, Martin, *The Algebra of Abū Kāmil*, p. 8, The University of Wisconsin Press, 1966.

[31] Dickson, L. N. *Ibid*., 2, p. 59, New York, 1934; Needham, J., *Ibid*., 3, p. 122.

[32] Dickson, L. E. *Ibid*., 6, p. 60.