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# A NOTE ON INVARIANT FINITELY ADDITIVE MEASURES

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ABSTRACT. We show that under certain general conditions any finitely additive measure which is defined for all subsets of a set $X$ and is invariant under the action of a group $G$ acting on $X$ is concentrated on a $G$-invariant subset $Y$ on which the $G$-action factors to that of an amenable group. The result is then applied to prove a conjecture of S. Wagon about finitely additive measures on spheres.

It is well known that if $G$ is an amenable group acting on a set $X$ then there exist plenty of $G$-invariant finitely additive probability measures on $(X, \mathfrak{P}(X))$ where $\mathfrak{P}(X)$ is the class of all subsets of $X$ (cf. [**3**] for details). However, such measures may fail to exist when $G$ is nonamenable. In [**6**] S. Wagon conjectured that if $G$ is a group of isometries of $S^n$, the $n$-dimensional sphere, such that for any $G$-invariant subset $Y$, the group $\{g/Y \mid g \in G\}$ of restrictions of elements of $g$ to $Y$ is nonamenable, then there does not exist any $G$-invariant finitely additive probability measure on $(S^n, \mathfrak{P}(S^n))$.

In this note we establish the above-mentioned conjecture. Further, we formulate a condition on actions of (abstract) groups, involving the isotropy subgroups and fixed point sets, which implies similar assertions in a more general situation (cf. Theorem 1.1). The condition holds for actions of subgroups $G$ of any compact Lie group $\mathfrak{G}$ acting on homogeneous spaces of $\mathfrak{G}$. It also holds for actions of subgroups $G$ of algebraic $\mathbf{R}$-groups $\mathfrak{G}$ acting on homogeneous spaces of $\mathfrak{G}$ by algebraic $\mathbf{R}$-subgroups. Thus, in all these cases we are able to conclude that $G$-invariant finitely additive probability measures (defined for all subsets) are concentrated on invariant sets on which the action factors to that of an amenable quotient of $G$ (cf. §2).

A particular consequence is that if $G$ is a nonamenable subgroup of $\mathrm{GL}(n+1, \mathbf{R})$ acting irreducibly on $\mathbf{R}^{n+1}$, then for the natural $G$-actions on $\mathbf{R}^{n+1} - (0)$, $S^n$ or $\mathbf{P}^n$, there exist no invariant finitely additive probability measures (cf. Corollaries 2.2 and 2.4). We recall that by a theorem of A. Tarski [**5**] this is equivalent to existence of paradoxical decompositions for the action (cf. [**6** and **7**] for motivation and some results in that direction).

In some of the $G$-actions discussed above, e.g. $G$ a group of isometries of $S^n$ or $G$ a subgroup of a compact Lie group $\mathfrak{G}$ acting on a homogeneous space of $\mathfrak{G}$, there exists a natural countably additive $G$-invariant probability measure defined on the class of Borel subsets. We prove that the measure extends to a $G$-invariant finitely additive measure defined on all subsets if and only if $G$ is amenable (cf. Theorem

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1.4 for a general result). For the case of groups of isometries of $S^n$ this was first proved by S. Wagon [**6**].

In [**1**] the author obtained results analogous to those discussed above for the case of subgroups of $\mathrm{GL}(n, \mathbf{Z})$ acting on $\mathbf{T}^n$ as group automorphisms. The present method is analogous but much simpler. In the case of the action on $\mathbf{T}^n$ the condition of Theorem 1.1 is not satisfied and consequently the proofs depend on various specific features of $\mathbf{T}^n$.

**1. Main results.** Let $G$ be a group acting on a set $X$. For $x \in X$ we denote by $G_x$ the isotropy subgroup under the $G$-action; viz. $G_x = \{g \in G \mid gx = x\}$. For any subgroup $H$ of $G$ we denote by $F_H$ the set of fixed points of $H$; that is $F_H = \{x \in X \mid hx = x \text{ for all } h \in H\}$.

For any set $E$ we denote by $\mathfrak{P}(E)$ the class of all subsets of $E$. A (possibly empty) subclass $\mathfrak{C}$ of $\mathfrak{P}(E)$ is said to be of *type* $\mathfrak{L}$ if the following holds: for any family $\{E_\alpha\}_{\alpha \in \Lambda}$, where $\Lambda$ is an indexing set and $E_\alpha \in \mathfrak{C}$ for all $\alpha \in \Lambda$, there exist $k \geq 1$ and $\alpha_1, \alpha_2, \dots, \alpha_k \in \Lambda$ such that $\bigcap_{\alpha \in \Lambda} E_\alpha = \bigcap_{i=1}^k E_{\alpha_i}$.

THEOREM 1.1. *Let $G$ be a group acting on a set $X$. Suppose $\{G_x \mid x \in X\}$ and $\{F_H \mid H$ a subgroup of $G\}$ are of type $\mathfrak{L}$. Let $\mu$ be a $G$-invariant finitely additive probability measure on $(X, \mathfrak{P}(X))$. Then there exists a normal subgroup $Q$ of $G$ such that $G/Q$ is amenable and $\mu(F_Q) = 1$.*

We need the following lemma from [**4**] (cf. Proposition 3.5 of [**4**]).

LEMMA 1.2. *Let $G$ be a group acting on a set $X$. Suppose that for all $x \in X$ the isotropy subgroup $G_x$ is amenable. Suppose also that there exists a $G$-invariant finitely additive probability measure $\mu$ on $(X, \mathfrak{P}(X))$. Then $G$ is amenable.*

We first prove the following.

PROPOSITION 1.3. *Let $G$ be a group acting on a set $X$. Suppose $\{G_x \mid x \in X\}$ is of type $\mathfrak{L}$. Suppose also that there exists a $G$-invariant finitely additive probability measure $\mu$ such that for any nontrivial subgroup $H$ of $G$, $\mu(F_H) < 1$. Then $G$ is amenable.*

PROOF. If possible let $G$ be nonamenable. Let $\mathcal{N}$ be the class of all nonamenable subgroups $S$ of $G$ such that either $S = G$ or $S = \bigcap_{i=1}^k G_{x_i}$, where $k \geq 1$ and $x_1, \dots, x_k \in X$. Let $\mathcal{M}$ be the subset of $\mathcal{N}$ consisting of minimal elements of $\mathcal{N}$; that is $M \in \mathcal{M}$, $N \in \mathcal{N}$, and $N \subset M$ implies $N = M$. Since the class of subgroups $\{G_x \mid x \in X\}$ is of type $\mathfrak{L}$, it follows that $\mathcal{M}$ is nonempty and that in fact any $N \in \mathcal{N}$ contains a minimal element (otherwise there would exist a sequence $x_1, \dots, x_k, \dots,$ in $X$ such that $\bigcap_{i=1}^n G_{x_i}$ is a strictly decreasing sequence).

Let $M \in \mathcal{M}$. Being nonamenable, in particular $M$ is nontrivial. Hence, by hypothesis, $\mu(F_M) < 1$ or, equivalently, $\mu(X - F_M) > 0$. In particular, $X - F_M$ is nonempty. Consider the $M$-action on $X - F_M$. The isotropy subgroup of any $x \in X - F_M$ under the $M$-action is $M \cap G_x$ and it is a proper subgroup of $M$. We note that $M \cap G_x$ is amenable for all $x \in X$; if not, $M \cap G_x \in \mathcal{N}$, which would mean $M$ is not minimal in $\mathcal{N}$. On the other hand, on $X - F_M$ we have an $M$-invariant finitely additive probability measure $\nu$ defined by $\nu(E) = \mu(E)/\mu(X - F_M)$ for all $E \subset X - F_M$. By Lemma 1.2 these observations imply that $M$ is amenable—a contradiction. Hence $G$ must be amenable.

PROOF OF THE THEOREM. Consider the class of subgroups $\mathfrak{C} = \{H \mid \mu(F_H) = 1\}$. Let $F = \bigcap\{F_H \mid H \in \mathfrak{C}\}$. Since $\{F_H \mid H \subset G\}$ is of type $\mathfrak{L}$ there exist $k \geq 1$ subgroups $H_1, \ldots, H_k \in \mathfrak{C}$ such that $F = \bigcap_{i=1}^{k} F_{H_i}$. Then $\mu(F) = 1 - \mu(X - F) \geq 1 - \sum \mu(X - F_{H_i}) = 1$. Let $Q$ be the subgroup generated by $\bigcup\{H \mid H \in \mathfrak{C}\}$. Since $\mu$ is $G$-invariant, $\mathfrak{C}$ is invariant under conjugation and consequently $Q$ is a normal subgroup of $G$. Further, clearly $F_Q = F$ so that $\mu(F_Q) = 1$ and hence $Q$ is the unique maximal element of $\mathfrak{C}$. To complete the proof, we need only show that $G/Q$ is amenable. $F_Q$ is a $G$-invariant set and the action of $G$ on $F_Q$ factors to an action of $G/Q$. The restriction of $\mu$ to $F_Q$ is a $G/Q$-invariant finitely additive probability measure on $F_Q$. It is evident that the $G/Q$-action on $F_Q$ satisfies the conditions of Proposition 1.3 ($G/Q$ and $F_Q$ in the place of $G$ and $X$ respectively). While the isotropy subgroups form a class of type $\mathfrak{L}$ because of the corresponding property for the $G$-action, the set of fixed points of any nontrivial subgroup has $\mu$-measure $< 1$ because of our choice of $Q$ as the unique maximal element of $\mathfrak{C}$ as defined above. Hence, by Proposition 1.3, $G/Q$ is amenable, which proves the theorem.

Let $(X, \mathfrak{M}, m)$ be a measure space; that is, $\mathfrak{M}$ is a $\sigma$-algebra of subsets of $X$, and $m$ is a (countably additive) measure defined on $\mathfrak{M}$. Let $G$ be a group acting on $X$ preserving $\mathfrak{M}$ and $m$; that is, for all $g \in G$ and $E \in \mathfrak{M}$, $gE \in \mathfrak{M}$ and $m(gE) = m(E)$. The $G$-action is said to be *weakly measurable* if for any subgroup $H$ of $G$, $F_H \in \mathfrak{M}$. A weakly measurable action is said to be *essentially factorable* through a normal subgroup $H$ of $G$ if $m(X - F_H) = 0$. A finitely additive measure $\mu$ on $(X, \mathfrak{P}(X))$ is said to extend the measure $m$ on $(X, \mathfrak{M})$ if $\mu(E) = m(E)$ for all $E \in \mathfrak{M}$.

THEOREM 1.4. *Let $G$ be a group acting on a measure space $(X, \mathfrak{M}, m)$, preserving $\mathfrak{M}$ and $m$. Suppose $m(X) = 1$ and the $G$-action is weakly measurable. Suppose also that $\{G_x \mid x \in X\}$ and $\{F_H \mid H$ a subgroup of $G\}$ are of type $\mathfrak{L}$. Then there exists a $G$-invariant finitely additive measure $\mu$ on $(X, \mathfrak{P}(X))$ extending the measure $m$ on $(X, \mathfrak{M})$ if and only if the $G$-action is essentially factorable through a normal subgroup $Q$ such that $G/Q$ is amenable.*

PROOF. Suppose there exists a $G$-invariant finitely additive measure $\mu$ on $(X, \mathfrak{P}(X))$ extending $m$. By Theorem 1.1 there exists a normal subgroup $Q$ of $G$ such that $G/Q$ is amenable and $\mu(F_Q) = 1$. Since $F_Q \in \mathfrak{M}$ and $\mu$ extends $m$, we get $m(F_Q) = 1$, that is, the action is essentially factorable through $Q$.

Conversely, suppose the action is essentially factorable through a normal subgroup $Q$ such that $G/Q$ is amenable. By a well-known result (cf. [**3**, Theorem 5.1]) the measure $m_Q$ on $(F_Q, \mathfrak{M}_Q)$, where $\mathfrak{M}_Q = \{E \cap F_Q \mid E \in \mathfrak{M}\}$, defined by $m_Q(E) = m(E)$ for all $E \in \mathfrak{M}_Q$ extends to a $G/Q$-invariant (under the factor action) finitely additive measure $\mu_Q$ on $(F_Q, \mathfrak{P}(F_Q))$. Put $\mu(E) = \mu_Q(E \cap F_Q)$, for all $E \in \mathfrak{P}(X)$. Then $\mu$ is a $G$-invariant finitely additive measure on $(X, \mathfrak{P}(X))$ extending the measure $m$ on $(X, \mathfrak{M})$.

**2. Examples.** We now apply Theorems 1.1 and 1.4 to various situations.

(i) Let $X = S^n$, the $n$-dimensional sphere (of unit vectors in $\mathbf{R}^{n+1}$, with respect to the usual norm) and let $G$ be a subgroup of $O(n+1)$, the orthogonal group, acting as isometries of $S^n$. For each $x \in S^n$, $G_x = G \cap O(n+1)_x$. Since evidently $\{O(n+1)_x \mid x \in S^n\}$ is evidently of type $\mathfrak{L}$, so is $\{G_x \mid x \in S^n\}$. On the other hand, for each subgroup $H \subset G \subset O(n+1)$, $F_H$ is a compact differentiable submanifold with finitely many connected components. It follows therefore that $\{F_H \mid H$ a subgroup of $G\}$ is of type $\mathfrak{L}$. Hence we get the following.

COROLLARY 2.1. (a) *Let $\mu$ be a $G$-invariant finitely additive probability (or finite) measure on $(S^n, \mathfrak{P}(S^n))$. Then there exists a normal subgroup $Q$ of $G$ such that $G/Q$ is amenable and $\mu(S^n - Y) = 0$, where $Y = F_Q = \{x \in S^n \mid qx = x$ for all $q \in Q\}$. $Y$ is a $G$-invariant set and $\{g/Y \mid g \in G\}$, the group of restrictions of elements of $G$ to $Y$, is an amenable group.*

(b) *There exists a $G$-invariant finitely additive measure $\mu$ on $(S^n, \mathfrak{P}(S^n))$ extending the standard ($O(n+1)$-invariant) measure $m$ on $(S^n, \mathfrak{B})$, where $\mathfrak{B}$ is the $\sigma$-algebra of Borel sets, if and only if $G$ is amenable.*

PROOF. (a) Let $Q$ be as given by Theorem 1.1. Then since $Q$ is normal, $Y = F_Q$ is $G$-invariant. Since $Q$ fixes each $y \in Y$ the group $\{g/Y \mid g \in G\}$ is a quotient of $G/Q$ and therefore amenable (cf. [**2**]).

(b) If such an extension exists, then by Theorem 1.4 there exists a normal subgroup $Q$ such that $G/Q$ is amenable and $m(S^n - F_Q) = 0$. The latter condition implies that $F_Q$ is dense in $S^n$ and, since it is closed, we have $F_Q = S^n$. But identity is the only isometry fixing all points. Hence $Q$ is trivial and consequently $G$ is amenable. Conversely, if $G$ is amenable, then by a well-known result (cf. Theorem 5.1 of [**3**]) there exists a $G$-invariant finitely additive measure on $(S^n, \mathfrak{P}(S^n))$ extending $m$.

Part (b) was proved earlier by Wagon (cf. [**6**]). He also noted part (a) in the particular case $n = 2$ and conjectured its validity for all $n$ (cf. [**6**]); the corollary establishes the conjecture. (We should, however, note the following: In the statement of the conjecture in [**6**, p. 81], it is not explicit whether the measure is meant to be finite. But in the particular case of $S^2$ which is settled in [**6**] and from which the author motivates the conjecture, the measure is implicitly assumed to be finite (cf. Theorems 6 and 7 and Proposition 1 in [**6**]). Whether the conjecture is true for $\sigma$-finite additive measures is not clear, even for $S^2$.)

In [**6**] it is actually shown (cf. Theorem 7 of [**6**]) that for any nonamenable group $G$ of isometries of $S^2$ there does not exist any (finite) $G$-invariant finitely additive measure on $(S^2, \mathfrak{P}(S^2))$. We note that if $G$ is a nonamenable subgroup of $O(3)$ then the natural action of $G$ on $\mathbf{R}^3$ is irreducible (that is, there is no proper nontrivial invariant subspace). The above-mentioned assertion from [**6**] generalizes to the following.

COROLLARY 2.2. *Let $G$ be a nonamenable subgroup of $O(n+1)$ acting irreducibly on $\mathbf{R}^{n+1}$. Then there does not exist any $G$-invariant finitely additive probability measure on $(S^n, \mathfrak{P}(S^n))$.*

PROOF. If possible let $\mu$ be such a finitely additive measure. By Corollary 2.1 there exists a normal subgroup $Q$ of $G$ such that $G/Q$ is amenable and $\mu(F_Q) = 1$. Evidently, $F_Q = S^n \cap W$, where $W = \{x \in \mathbf{R}^{n+1} \mid qx = x$ for all $q \in Q\}$. Since $Q$ is normal $W$ is a $G$-invariant subspace of $\mathbf{R}^{n+1}$. By irreducibility either $W = (0)$ or $\mathbf{R}^{n+1}$. The former is impossible since $F_Q$ would then be empty, while actually $\mu(F_Q) > 0$. On the other hand, if $W = \mathbf{R}^{n+1}$ then $F_Q = S^n$, which means $Q$ must be the trivial subgroup; but in that case $G$ must be amenable—a contradiction to the hypothesis. Hence there cannot exist such a finitely additive invariant measure.

(ii) The above examples generalize in a natural way to homogeneous spaces of compact Lie groups. Let $\mathfrak{G}$ be a compact Lie group and $\mathfrak{S}$ be a closed subgroup of $\mathfrak{G}$. Let $X = \mathfrak{G}/\mathfrak{S}$ and let $G$ be a subgroup of $\mathfrak{G}$ acting on $X$ on the left. Then arguments as above yield the following.

COROLLARY 2.3. (a) *Let $\mu$ be a $G$-invariant finitely additive probability measure on $(X, \mathfrak{P}(X))$. Then there exists a normal subgroup $Q$ of $G$ such that $G/Q$ is amenable and $\mu(X - F_Q) = 0$.*

(b) *There exists a $G$-invariant finitely additive measure $\mu$ on $(X, \mathfrak{P}(X))$ extending the $\mathfrak{G}$-invariant (countably additive) probability measure on $(X, \mathfrak{B})$, where $\mathfrak{B}$ is the $\sigma$-algebra of Borel sets, if and only if $G$ is amenable.*

(iii) Similar arguments may be made for homogeneous spaces of algebraic groups: Let $\mathfrak{G}$ be an algebraic $\mathbf{R}$-group (the group of $\mathbf{R}$-elements of an algebraic group defined over $\mathbf{R}$), e.g. $\mathrm{GL}(n, \mathbf{R})$. Let $\mathfrak{S}$ be an algebraic $\mathbf{R}$-subgroup of $\mathfrak{G}$. Let $X = \mathfrak{G}/\mathfrak{S}$ and let $G$ be a subgroup of $\mathfrak{G}$ acting on $X$ on the left. Then for each $x \in X$, $G_x = G \cap \mathfrak{G}_x$. $\{\mathfrak{G}_x \mid x \in X\}$ is a family of algebraic $\mathbf{R}$-subgroups and therefore it is of type $\mathfrak{L}$. Hence so is $\{G_x \mid x \in X\}$. On the other hand, for any subgroup $H$, $F_H$ is the set of $\mathbf{R}$-elements of an algebraic variety. Therefore $\{F_H \mid H$ a subgroup of $G\}$ is of type $\mathfrak{L}$. It follows that Corollary 2.2(a) holds verbatim for $G$ and $X$ as above. In general, $(X, \mathfrak{B})$, where $\mathfrak{B}$ is the Borel $\sigma$-algebra, may not carry any *finite* $G$-invariant (countably additive) measure. If it does then Corollary 2.2(b) also holds verbatim for those $G$ and $X$. We note that since compact Lie groups are algebraic $\mathbf{R}$-groups these results generalise Corollary 2.3. Similarly the following generalises Corollary 2.1.

COROLLARY 2.4. *Let $G$ be a subgroup of* $\mathrm{GL}(n, \mathbf{R})$. *Let $X$ be either $\mathbf{R}^n - (0)$ or $S^{n-1}$ or $\mathbf{P}^{n-1}$ ($(n-1)$-dimensional projective space) and consider the natural $G$-action on $X$. Let $\mu$ be a $G$-invariant finitely additive probability measure on $(X, \mathfrak{P}(X))$. Then there exists a normal subgroup $Q$ of $G$ such that $G/Q$ is amenable and $\mu(F_Q) = 1$. In particular, if $G$ is nonamenable, and the action of $G$ on $\mathbf{R}^n$ is irreducible (e.g. if $G = \mathrm{SL}(n, \mathbf{Z})$) then there does not exist any $G$-invariant finitely additive probability (or finite) measure on $(X, \mathfrak{P}(X))$.*

In the case of $X = \mathbf{R}^n - (0)$ or $\mathbf{P}^{n-1}$ the isotropy subgroups for the (transitive) $\mathrm{GL}(n, \mathbf{R})$-action are algebraic $\mathbf{R}$-subgroups and the result follows from the above remarks. For $S^{n-1}$ it may be deduced either from the result for $\mathbf{P}^{n-1}$ or directly from Theorem 1.1. If the action of $G$ on $\mathbf{R}^n$ is irreducible, then as in the proof of Corollary 2.2 we see that for any normal subgroup $H$ of $G$, $F_H$ must be either empty or $X$ (for any $X$ as above). Hence for such a $G$ there cannot exist any invariant finitely additive measure on $(X, \mathfrak{P}(X))$ unless it is amenable.

In [**1**] we proved the analogous assertion for the actions of subgroups of $\mathrm{GL}(n, \mathbf{Z})$ acting on $\mathbf{T}^n$, the $n$-dimensional torus, as the group of automorphisms. In that case, however, Theorems 1.1 and 1.4 do not apply since $G_x$ for $x \in \mathbf{T}^n$ cannot in general be expressed as $G \cap \mathfrak{G}_x$ for some algebraic group $\mathfrak{G}$; in fact $\{G_x \mid x \in \mathbf{T}^n\}$ is not of type $\mathfrak{L}$. The proof there uses, together with the ideas underlying the proof of Theorem 1.1, certain specific features of the torus.

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