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DYNAMICS OF HOROSPHERICAL FLOWS

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Let G be a reductive Lie group; that is, the adjoint representation is completely reducible. A subgroup U of G is said to be *horospherical* if there exists $g \in G$ such that

$$U = \{u \in G \mid g^j u g^{-j} \longrightarrow e \quad \text{as} \quad j \longrightarrow \infty\}$$

where e is the identity element of G. The action of a horospherical subgroup (respectively a maximal horospherical subgroup) on a homogeneous space G/Γ, where Γ is a discrete subgroup, is called a *horospherical flow* (resp. a *maximal horospherical flow*). The classical horocycle flows associated to surfaces of constant negative curvature, studied by G. A. Hedlund and other authors, can be viewed as horospherical flows on homogeneous spaces of $SL(2, \mathbf{R})$.

In the case when Γ is a *co-compact* discrete subgroup of $SL(2, \mathbf{R})$ it was proved by H. Furstenberg that the horocycle flow is uniquely ergodic; i.e. it admits a unique invariant probability measure (cf. [6]). In particular, this implies a result of G. A. Hedlund that the horocycle flow is minimal; i.e. every orbit is dense. Satisfactory generalisations of these assertions for all horospherical flows on *compact* homogeneous spaces are available in literature (cf. [5] and [7]).

The purpose of this note is to announce similar results for maximal horospherical flows on homogeneous spaces G/Γ where Γ is any lattice; i.e. Γ is a discrete subgroup such that G/Γ admits a finite G-invariant measure, but may not necessarily be compact. When G/Γ, as above, is noncompact the horospherical flow fails to be uniquely ergodic. The general task therefore is to obtain a description of all ergodic invariant measures of the horospherical flow. We prove the following.

1. THEOREM. *Let G be a reductive Lie group and Γ be a lattice in G. Let N be a maximal horospherical subgroup of G. Let σ be an N-invariant ergodic probability measure. Then there exists a connected Lie subgroup L of G and an element $x \in G$ such that* (i) *$Lx\Gamma/\Gamma$ is a closed L-orbit in G/Γ and* (ii) *σ is L invariant and $\sigma(G/\Gamma - Lx\Gamma/\Gamma) = 0$.*

It is not difficult to show that in the special case when G/Γ is compact Theorem 1 yields the known results for maximal horospherical flows.

The proof is achieved using a characterisation of the haar measure of a

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semisimple Lie group, proved by the author in [1]. Detailed proofs may be found in [3]. Here we shall describe the candidate for L in a certain (crucial) special case. Let G be a semisimple $\mathbf{R}$-algebraic group defined over $\mathbf{Q}$, admitting no nontrivial compact factors and let Γ be a lattice which is commensurable with $G_{\mathbf{Z}}$; i.e. $\Gamma \cap G_{\mathbf{Z}}$ has finite index in both Γ and $G_{\mathbf{Z}}$. Let P be a minimal $\mathbf{Q}$-parabolic $\mathbf{R}$-subgroup of G and let R be a minimal $\mathbf{R}$-parabolic subgroup of G contained in P. Let N be the unipotent radical of R. Then N is a maximal horospherical subgroup of G. Now let σ be a N-invariant ergodic probability measure on G/Γ. Let π be the measure on G defined by

$$\pi(E) = \int_{G/\Gamma} \sum_{\gamma \in \Gamma} \check{\chi}_E(g\gamma)\, d\sigma(g)$$

where E is any Borel subset of G and $\check{\chi}_E$ is the function defined by $\check{\chi}_E(g) = 1$ or 0 according as $g^{-1} \in E$ or $g^{-1} \notin E$. We recall that by Bruhat decomposition G is a union of finitely many double cosets of the form PgR, $g \in G$. Therefore there exists a double coset Pg_0R such that (i) $\pi(qPg_0R) > 0$ for some $q \in G_{\mathbf{Q}}$ and (ii) $\pi(r(\overline{Pg_0R} - Pg_0R)) = 0$ for all $r \in G_{\mathbf{Q}}$ (bar overhead stands for closure operation). Let $Q = \{g \in G \mid g\overline{Pg_0R} = \overline{Pg_0R}\}$. Then Q is a parabolic subgroup of G. It turns out (a posteriori) that Q is defined over $\mathbf{Q}$. Let V be the smallest normal subgroup of Q containing N and let $L = (\overline{V(Q \cap \Gamma)})^0$ (the connected component of the identity in $\overline{V(Q \cap \Gamma)}$). We prove that assertions (i) and (ii) are satisfied with respect to this subgroup for a suitable x.

The general result is then deduced using Margulis's arithmeticity theorem and other well-known results on lattices.

REMARK. It may be noted that if we start with a parabolic subgroup Q which is defined over $\mathbf{Q}$ and contains a maximal horospherical subgroup N and define L as in the preceding paragraph, then there exist closed L-orbits on G/Γ (as above) which admit a finite L-invariant measure. Further the latter is ergodic as a N-invariant measure. The assertion in Theorem 1 is indeed the converse of this.

REMARK. For arithmetic lattices the assumption in Theorem 1 that σ be a probability measure is redundant. This is because by Theorem 4.1 in [2] finiteness of σ is automatic if we just assume σ to be locally finite.

Applying Theorem 1 we deduce the following results about subsystems of maximal horospherical flows.

2. THEOREM. *Let G, Γ and N be as in Theorem 1. Then any minimal compact (nonempty) N-invariant subset of G/Γ is an orbit of a connected Lie subgroup (containing N).*

3. THEOREM. *Let G be a reductive $\mathbf{R}$-algebraic group defined over $\mathbf{Q}$ and let Γ be a subgroup commensurable with $G_{\mathbf{Z}}$. Let P be a minimal $\mathbf{Q}$-parabolic*

$\mathbf{R}$-subgroup of G and let N be the unipotent radical of a minimal $\mathbf{R}$-parabolic subgroup contained in P. Let V be the smallest normal subgroup of P containing N and let $L = \overline{(V(P \cap \Gamma))}^0$. Then any closed N-invariant subset of G/Γ contains a closed orbit of L.

The proof of Theorem 3 depends on the fact that Γ being an arithmetic lattice, any closed N-invariant nonempty subset of G/Γ contains the support of a N-invariant measure (cf. Proposition 7.1, [2]).

It may be mentioned here that for certain horospherical flows on $SL(n, \mathbf{R})/SL(n, \mathbf{Z})$ the closure of any orbit is the orbit of a connected Lie subgroup of $SL(n, \mathbf{R})$. This is proved in [4] by a different method. A similar result is expected to be true in greater generality.

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