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CONVECTION UNDER TERRESTRIAL AND ASTROPHYSICAL CONDITIONS

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Convection and turbulence are two concepts which are fundamental to many aspects of meteorology, geophysics, and astrophysics. The two concepts are not synonymous: convection implies that a pattern of motion prevails and that its prevalence provides a mechanism for the transport of heat, energy, and momentum; turbulence implies a certain random element in the prevalent motions. At the same time, the two concepts are not mutually exclusive: we may envisage a random element superposed on a convective pattern of the average motions. Nevertheless, it is convenient to keep the two concepts distinct.

Now convection is often, if not always, the result of thermal instability, and the manner in which convection results from thermal instability is best illustrated by considering a layer of liquid heated from below. Then, on account of thermal expansion, the liquid above, being colder, is denser than the liquid below, which is hotter: this is clearly an unstable state of affairs, and if the density gradient is sufficiently adverse (i.e., if the arrangement is sufficiently top-heavy), motions will ensue which will be in the sense of restoring a stable density gradient. This is essentially the manner in which convection originates, even on a large scale as in our atmosphere. There is, however, one difference: in a compressible medium like air, it is not necessary that an adverse density gradient be established for thermal instability; it suffices if the density gradient exceeds the adiabatic. But a theorem of Jeffreys enables us to apply to a compressible medium results derived (either by observation or by theory) for an incompressible fluid: the theorem of Jeffreys is to the effect that an adverse density gradient in an incompressible fluid is formally equivalent to a density gradient in excess of the adiabatic in a compressible fluid.

The manner of the onset of convection in a fluid heated below has been the subject of experimental and theoretical investigation since Bénard's pioneering experiments in 1900 and 1901. Bénard found that if a thin layer of liquid free at its upper surface is heated uniformly at the lower surface, then the layer rapidly resolves itself into a number of hexagonal cells if the temperature gradient exceeds a certain critical value. The walls of these cells are vertical and the motions in the cells are upward at the center and downward at the periphery. By observations made on minute suspended particles, Bénard was able to show that the horizontal components of the motions in the cells are radially outward from the centers of the hexagons and that the downward vertical motion is a maximum along the edges common to the three adjacent walls. Further, when the liquid is subjected to shear, the vertical cells are replaced by horizontal strips. Bénard's experiments have been repeated in air by A. Graham and K. Chandra. In these experiments the motions were made visible by the introduction of smoke and the same phenomenon of hexagonal cells and longitudinal strips or rolls was observed.

The most careful repetitions of Bénard's experiments under quantitative conditions are those of Schmidt and Milverton and Schmidt and Saunders. These latter investigations have established that what decides the stability of a layer of fluid heated below is the numerical value of the nondimensional quantity:

$$R = \frac{g\alpha|\beta|}{\kappa\nu} d^4,$$

where g denotes the value of gravity, α, κ, and ν are the coefficients of volume expansion, thermometric conductivity, and kinematic viscosity, respectively, d is the depth of the layer considered, and $\beta = -|dT/dz|$ is the adverse temperature gradient which is maintained. We call R the Rayleigh number after Rayleigh, who first isolated this quantity as a criterion for thermal stability. Experiments show that instability sets in when R exceeds a certain determinate critical value. That higher temperature gradients can be maintained in a liquid of higher viscosity and/or higher thermal conductivity before instability sets in is physically understandable.

The experiments of Schmidt and Milverton establish that the critical value of R is in the neighborhood of 1750 when the liquid is confined between two rigid conducting boundaries; they further show that as the critical value of R is surpassed the pattern of motions is cellular, as in Bénard's experiments.

Before outlining the theory of the onset of convection by thermal instability, we may briefly refer to the fact that there are many observations which point to the occurrence, on a large scale and under natural conditions, of cellular convection similar to that observed in the laboratory. Brunt and Walker have pointed out, for example, the similarity of various cloud forms to the patterns observed with the aid of smoke in a confined layer of air under experimental conditions. The most beautiful confirmation of the occurrence of cellular convection in nature is, however, a result of the work of Woodcock and Wyman. Plate III shows a peculiar banded appearance of the surface of the sea which is commonly observed in the Gulf of Panama. Woodcock and Wyman present strong evidence for believing that these bands are caused by the presence of a system of longitudinal roll vortices of the type observed in the laboratory under shear conditions. Again in a different connection it has been suggested by Low that the polygonal arrangement of stones on Erdman's Tundra near Spitzbergen is caused by cellular convection. More particularly, the assumption is that during thaws the top surface of melted ice is at 4° (at which temperature water has its maximum density) while the bottom surface is still at 0° ; and the adverse density gradient which thus prevails induces cellular convection, which in turn transports the stones to the peripheries of the cells and arranges them in the observed hexagonal pattern. Wasiutynski has gone much further and has attempted to interpret markings on the moon and Mars in terms of Bénard convection cells.

The explanation of the origin of cellular convection when the temperature gradient exceeds a certain critical value was first given by Lord Rayleigh in 1916. The theory has since been developed by Jeffreys and by Pellew and Southwell and is along the following lines: An arbitrary constant temperature gradient is compatible with the equation of heat conduction. But whether a stationary state compatible with the equation of heat conduction is stable or not, can be decided only by considering whether a small displacement from the steady state is damped or amplified with time. In general, small deviations from the steady state may be assumed to vary like $e^{\omega t}$, and the stability (or otherwise) of the initial state to perturbations of the character assumed will depend on whether the real part of ω is positive or negative. If the real part of ω is positive, the initial state is unstable; if the real part of ω is negative, the initial state is stable for the assumed perturbation. For conservative systems, ω can be shown to be real always; accordingly, we pass from stability to instability as ω passes through zero. Since ω is equivalent to $\partial/\partial t$, it follows that the equations governing the state in marginal stability will not involve the time explicitly. This is known as the principle of the exchange of the stabilities. The system we are dealing with is not conservative, however: it includes viscosity, which is a dissipative mechanism; and, in general, nonconservative systems need not satisfy the principle of the exchange of the stabilities. And if a system does not satisfy the principle, the marginal state is one which is not independent of time: it will be a system varying periodically with time, and instability when it sets in will be in the form of oscillations of increasing amplitude, i.e., by overstability in the sense of Eddington. In other words, when instability sets in it could arise either through a stationary pattern of motions (i.e., convection) or through overstability.

Returning to the problem of the Bénard cells, we can show that the principle of the exchange of stabilities is valid and that the equations governing marginal stability are the standard equations of motion and heat conduction in which $\partial/\partial t$ has been set equal to zero. Methods of solving these latter equations together with the appropriate boundary conditions have been developed by Jeffreys, Low, Christopherson, and Pellew and Southwell. The results of these investigations are summarized in Table I:

TABLE I

	R_c	L/d
Two free boundaries	657.5	1.89
One free boundary, one rigid boundary	1100.7	1.56
Two rigid boundaries	1707.8	1.34

The critical Rayleigh numbers for three different sets of boundary conditions and the ratio L/d of the side of the hexagonal cells to the depth is also given for each case.

As we have already pointed out, the experiments of Schmidt and Milverton on a layer of liquid confined on both sides gave a value of about 1750 for the critical Rayleigh number; this is in agreement with the predicted value 1708 within the limits of experimental error.

Now the occurrence of magnetic fields in a number of geophysical and astrophysical problems in which convection is presumed to play a role makes it of interest to examine the effect of an external magnetic field on the onset of convection by thermal instability in a fluid which is also an electrical conductor. In a general way it is clear that the magnetic field will have an inhibiting effect which will be greater, the greater the magnetic field (H) and the greater the electrical conductivity (σ): for when the field is strong (or the conductivity high) the lines of force tend to be glued to the material and this will make motions at right angles to H increasingly difficult; and this will in turn tend to prevent the closing in of the stream lines required for convection. Moreover, when cellular convection does set in, we should expect the cells to be elongated in the direction of the magnetic field, the elongation being greater, the greater the magnetic field. We should also expect that in the limit of infinite electrical conductivity, when the lines of force are permanently glued to the material, convection by thermal instability will become impossible. A detailed theoretical treatment of the problem confirms these anticipations.

Restricting ourselves to the case in which the impressed magnetic field and gravity act in the same direction, we find that the critical Rayleigh number for the onset of instability depends on the strength of the magnetic field and the electrical conductivity, σ, through the nondimensional quantity,

$$Q = \frac{H^2 \sigma}{\rho \nu} d^2 .$$

On the assumption that the principle of the exchange of stabilities is valid, a revision of the classical Rayleigh-Jeffreys theory leads to the results summarized in Table II.

In order to see how effective the magnetic field will be in inhibiting the onset of convection, we shall consider the practical case of mercury at room temperature. We have $\sigma = 1.1 \times 10^{-5}$,

$\rho\nu = 1.7 \times 10^{-2}$, and $Q = 6 \times 10^{-4} H^2 d^2$, where H is measured in gauss. For

$$d = 1 \text{ cm. and } H = 10^3 \text{ gauss}, Q = 600,$$

and the critical Rayleigh numbers for the three cases considered are:

$$\begin{array}{lll} R_c = 9942 & R_c/R_o = 15.12 & \text{case (i)} \\ R_c = 11660 & R_c/R_o = 6.80 & \text{case (ii)} \\ R_c = 4018 & R_c/R_o = 3.65 & \text{case (iii)} \end{array}$$

It would appear from these values that the predicted effect should be easily detectable in the laboratory.

TABLE II

	R_c		
Q	Two Free Boundaries	Two Rigid Boundaries	One Free Boundary, One Rigid Boundary
0	657.5	1708.	1101.
100	2654.	3757.	1699.
500	8579.	10110.	3586.
1000	15210.	17100.	5613.
10,000	119800.	124500.	35040.

As we have already pointed out, the foregoing results depend on the validity of the principle of the exchange of stabilities. An examination of this principle shows that a sufficient condition for its validity is

$$K < \frac{1}{4\pi\sigma} = \eta.$$

This inequality is satisfied under most terrestrial conditions. Thus for mercury at room temperature, $\eta = 7.5 \times 10^3$ cm^2/sec while $K = 4.7 \times 10^{-2}$ cm^2/sec. On the other hand, under astrophysical conditions $K >> \eta$. This is a result of the fact that under astrophysical conditions the thermal conductivity is extremely high because the transport of heat by radiation is a very efficient process. And if $K >> \eta$, then it can be shown that so long as

$$Q < \frac{27}{4}\pi^4 \frac{\eta}{\nu} = Q^*,$$

instability will set in by cellular convection; but for $Q > Q^*$ instability will set in as overstability. This result can be stated in a different way: We know from Alfvén's work that, in a magnetic field, waves can be propagated with a velocity

$$V_m = \frac{H}{(4\pi\rho)^{\frac{1}{2}}} \, ;$$

this is the velocity of the so-called magneto-hydrodynamic wave; and the condition $Q \leqq Q^*$ for instability to set in as cellular convection is equivalent to the requirement that the velocity of the magneto-hydrodynamic wave be less than

$$V^*_m = \frac{\sqrt{27}}{2} \frac{\pi}{d} \eta .$$

When $V_m > V^*_m$, instability will set in as overstability when the Rayleigh number reaches the value $27\pi^4 \eta / 4\nu$. Further, it appears that the frequency of oscillation at marginal stability is essentially determined by the time required for the magneto-hydrodynamic wave to travel the thickness of the atmosphere. The fact that under astrophysical conditions thermal instability does not lead to convection in the usual sense but to overstability must have important bearings on a number of astrophysical phenomena.

The new possibilities raised by the foregoing discussion of thermal convection in the presence of a magnetic field suggest that we may profitably re-examine a variety of related problems in the stability of fluid motions by including electromagnetic effects; and in this new field of mathematical analysis we may expect rapid advances.