A technique for estimating maximum harvesting effort in a stochastic fishery model

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Exploitation of biological resources and the harvest of population species are commonly practiced in fisheries, forestry and wild life management. Estimation of maximum harvesting effort has a great impact on the economics of fisheries and other bio-resources. The present paper deals with the problem of a bioeconomic fishery model under environmental variability. A technique for finding the maximum harvesting effort in fluctuating environment has been developed in a two-species competitive system, which shows that under realistic environmental variability the maximum harvesting effort is less than what is estimated in the deterministic model. This method also enables us to find out the safe regions in the parametric space for which the chance of extinction of the species is minimized. A real life fishery problem has been considered to obtain the inaccessible parameters of the system in a systematic way. Such studies may help resource managers to get an idea for controlling the system.

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1. Introduction

A quantitative and qualitative understanding of the interaction of different species is crucial for the management of fisheries. Harvesting has generally a strong impact on the population dynamics of a harvested species. The severity of this impact depends on the nature of the implemented harvesting strategy which, in turn, may range from the rapid depletion to the complete preservation of a population. The study of population dynamics with harvesting is a subject of mathematical bioeconomics, and is related to the optimal management of renewable resources (Clark 1990). The exploitation of biological resources and the harvest of population species are commonly practiced in fisheries, forestry, and wild life management. Problems related to the exploitation of multispecies systems are not only interesting but also difficult as there are theoretical as well as practical difficulties in the determination of an optimal policy for the harvesting of a multispecies fishery. First, the difficulty lies in the construction of realistic model of a multispecies system which leads to an analytically tractable optimal control problem. Secondly, it is difficult to carry out dynamic optimization in a problem involving more than two-state variables and many parameters. Thirdly, quantitatively valid estimates of interaction coefficients are available for very few multispecies communities. Moreover fish and wildlife populations often fluctuate unpredictably in numbers as uncertainty is prevalent in resource economics. For the immense interest of both theoretical and field ecologists, we have considered this important ecological problem in a more realistic sense by introducing environmental variability. Another important aspect regarding this type of problem is the knowledge of system parameters which are not possible to measure through experimental observations or field studies. These parameters are known as inaccessible parameters and the prior knowledge of these parameters will help resource managers to monitor the system and chalk out suitable harvesting policy to prevent exploitation of the resources. For this we discuss the problem with the help of mathematical model adopting techniques adopted for proper treatment.

Keywords. Colour noise; harvested competitive system; maximum harvesting; parametric safe zone; solution of stochastic differential equation; spectral density; Tchebycheff's inequality

Finally, we represent the outcomes of our mathematical analysis under different biological scenarios and real life applications to provide an idea to the resource managers for controlling the system parameters which, in turn, save the species from extinction.

1.1 *Brief review*

The problem of inter species competition was considered by Gause (1935) for two species obeying the law of logistic growth. Clark (1990) considered harvesting of a single species in an ecologically competing two fish population model. Modifying Clark's model Chaudhuri (1986, 1988) studied combined harvesting and considered the perspectives of bioeconomics and dynamic optimization of a two-species fishery. Brauer and Soudack (1981) and Dai and Tang (1998) studied constant rate of harvesting in a predator-prey system to allow simultaneous harvesting of both the species. They showed how to approximate the region of asymptotic stability in biological terms, in the initial states which lead to coexistence of the two-species and their global dynamics by efficient computer simulation. McNamara *et al* (1995), Alvarez (1998), Alvarez and Shepp (1998) discussed optimal harvesting under stochastic fluctuations. However, they neglected the competition between harvesters, and assumed that the population is under total and exclusive control.

To formulate the harvesting strategy, the knowledge of maximum value of efforts is essential. Bhattacharya and Begum (1996) gave light in this direction but their study was based on a deterministic situation. In traditional economic studies of rational harvesting planning, the objective of the harvestor is to find a plan that maximizes the present expected value of the cumulative yield. Clark (1990) gave some indications for finding optimal harvesting in a stochastic two-species system. However, he overlooked some of the most important consequences of fluctuations in resource stocks. For example, the stock-recruit function, which needs to be estimated from fishery data, may not be known precisely. Consequently, major uncertainty as to sustainable yields may prevail. Furthermore, in many fisheries the exact magnitude of the current stock is unknown at the time that quotas are specified. The optimal harvesting policy in a competitive system – under environmental variability – is thus still a challenging, important and open problem in ecology.

1.2 *Basic aim of our study*

Resource studies are subject to many stochastic effects. The growth, mortality and reproductive rates of fish, insect and animal population vary in a random manner. Fires, disease, insects attack, environmental fluctuations may harm or destroy forests and fisheries, and these phenomena are usually unpredictable. Random changes in resource stocks have two very important bioeconomic consequences – risk discounting and model as well as parameter uncertainty. The main objectives of fisheries management – or the evaluation of the results of management – are usually based on the abundance of harvested stocks. For the fishery managers to chalk out rational harvesting policy, optimal fishing effort and mortality, total available catches (catchability coefficients), species interaction terms are some of the important parameters of the system that need to be estimated properly. In a more realistic situation, such as, the system with external environmental randomness or under intrinsic stochastic variability, it is practically impossible to give proper estimation of all the system parameters. However, very few of the parameters affecting the system can be estimated through experimental study. Very few techniques have been developed to estimate the inaccessible parameters of the system. Thus a suitable technique is required for estimation of inaccessible parameters in a system under stochastic fluctuations. This may play an important role for controlling the system or to prevent the species from extinction.

Therefore, we have proposed a technique to estimate the inaccessible parameters of the system of a harvested two-species competition model with logistic growth. This modes which is further modified by introducing stochastic fluctuations on both the populations as an example. In a real environment, these parametric estimations may offer a more valid estimate of harvesting effort hitherto prior described ones from purely deterministic case. We have also shown the efficiency of this technique in other real life situations.

2. The mathematical model and a method for finding ecologically stable region

Let us first consider a deterministic harvested two-species competition population growth model:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = rx\,(1-\frac{x}{K}) - axy - q_1 Ex$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = sy\,(1-\frac{y}{L}) - bxy - q_2 Ey, \qquad (1)$$

where r, s, a, b, K, L are all positive constants. Here r, s represent the natural growth rate and K, L environmental carrying capacity of the two-species x and y respectively. Both species follow logistic growth pattern. The interaction terms $-axy$ and $-bxy$ indicate that the two-species compete for the use of common resource. We assume that both the species are subjected to a combined harvesting effort (E) and q_1, q_2 are the catchability coefficients of the two-species.

The possible steady states of (1) are $P_0:(0, 0)$, $P_1:({}^K/_r\,(r-q_1E),\ 0)$, P_2: $(0,\ {}^L/_s\,(s-q_2E))$, $P_*:(x^*,\ y^*)$, where $x^* = K[aL\,(s-q_2E)-s(r-q_1E)]/(abKL-rs)$, and $y^* = L[bK\,(r-q_1E)-r(s-q_2E)]/(abKL-rs)$.

The existence and local stability properties of the equilibria have been elaborately discussed by Chaudhuri (1986). If $rs > abKL$, the dynamical behaviour of the system is stable. If on the other hand $rs < abKL$, the system is unstable around the positive interior equilibrium (P_*).

The conventional theory of harvested populations is based on the assumption that environmental and biological parameters remain constant. But, environmental randomness is an inherent property of the harvested system (May 1975); and, environmental stochasticity on the reproduction factors play an important role on the dynamics of harvested population (Dimentberg 1988). It may be noted that the ecological effects for terrestrial and aquatic systems will depend on the character of the physical frequency distributions, and the general qualitative response of these systems could be inherently different. For terrestrial system, the environmental variability is large at both short time periods and long time periods and could be expected to develop internal mechanisms to the system which would cope with short term variability and minimize the effects of long term variations. Hence, analysis of the system with white noise gives better results. Gaussian white noise is extremely useful to model rapidly fluctuating phenomena. However, as can be seen by studying their spectra, the physical frequency distribution remains almost constant through out the entire span of the spectra; and that these are white to a good approximation. However, for natural aquatic systems, less robust internal processes are needed to handle the smaller amplitude variability at short time periods commensurate with the life span of the species. Hence the physical frequency distributions of the spectra are rapid, perpetual and highly irregular. Therefore, for such processes it is useful to describe the main random quantity as colour noise. Uhlenbeck and Ornstein (1954) process is the appropriate choice to model a coloured noise environment in most of the applications (see, Horsthemke and Lefever 1983). Here we have assumed the Stratonovich interpretation of the stochastic differential equations (Hoel *et al* 1993), which conserves the ordinary rule of calculus and in this case the stochastic differential equations can be considered as an ensemble of ordinary differential equations.

Introducing the environmental stochasticity in the form of colour noise on the growth of both species we shall investigate the dynamical behaviour of the system (1). The behaviour of this system in a random environment is considered within the frame work of the following model:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x\,(r + h_1(t) - \frac{rx}{K}) - axy - q_1Ex$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = y\,(s + h_2(t) - \frac{sy}{L}) - bxy - q_2Ey, \qquad (2)$$

where the perturbed terms $h_i(t)$ ($i = 1, 2$) are uncorrelated colour noise and follow the Ornstein-Uhlenbeck process.

The mathematical expectation and correlation function of the process $h_i(t)$ ($i = 1, 2$) are given by:

$$\langle h_1(t)\rangle = 0,\ \langle h_1(t_1)h_1(t_2)\rangle = ed_0 \exp(-d_0\,|\,t_1 - t_2\,|) \qquad (3)$$

and

$$\langle h_2(t)\rangle = 0,\ \langle h_2(t_1)h_2(t_2)\rangle = e'd'_0 \exp(-d'_0\,|\,t_1 - t_2\,|) \qquad (4)$$

where e, e', d_0, $d'_0 > 0$ are respectively the intensities and the correlation times of the noise and $\langle\cdot\rangle$ represents average over the ensemble of the stochastic process. The $h_1(t)$ and $h_2(t)$ are the solutions of the stochastic differential equation (Uhlenbeck and Ornstein 1954) and are given by:

$$\frac{\mathrm{d}h_1}{\mathrm{d}t} = -d_0h_1 + d_0\sqrt{2e}\,\frac{\mathrm{d}w_1}{\mathrm{d}t}, \qquad (5)$$

and

$$\frac{\mathrm{d}h_2}{\mathrm{d}t} = -d_0'h_2 + d_0'\sqrt{2e'}\,\frac{\mathrm{d}w_2}{\mathrm{d}t}, \qquad (6)$$

where

$$x_i(t) = \frac{\mathrm{d}w_i}{\mathrm{d}t}\ (i = 1, 2)$$

denotes the standard zero mean Gaussian white noise characterized by:

$$\langle x_1(t)\rangle = 0,\ \langle x_1(t_1)\,x_1(t_2)\rangle = d\,(t_1 - t_2) \qquad (7)$$

and

$$\langle x_2(t)\rangle = 0,\ \langle x_2(t_1)\,x_2(t_2)\rangle = d\,(t_1 - t_2) \qquad (8)$$

with $d\,(t)$ the Dirac delta function which is defined as

$d\,(t) = 1$, for $t_1 = t_2$
$= 0$, otherwise.

Substituting $X = \log x$ and $Y = \log y$ in equation (2) and using the transformation $u = X - X^*$, $v = Y - Y^*$, respectively, we obtain the linearized system. We have determined the spectral density functions of the stochastic functions in the linearized system and have obtained the mean square deviations ($D_u(t)$ and $D_v(t)$) of both the populations following the conventional technique which has been elaborately discussed in Sarkar *et al* (2001) and also given in figure 1.

An important concept in stochastic population dynamics is the concept of stability. Once a stochastic system of population is caught within the attraction domain of a stable equilibrium point of the corresponding deterministic system, it will remain there for a long time, attracted by the equilibrium point. Stochastic fluctuations give rise to the deviation from the equilibrium point. Large departures may occur, which can lead to escape from the do-

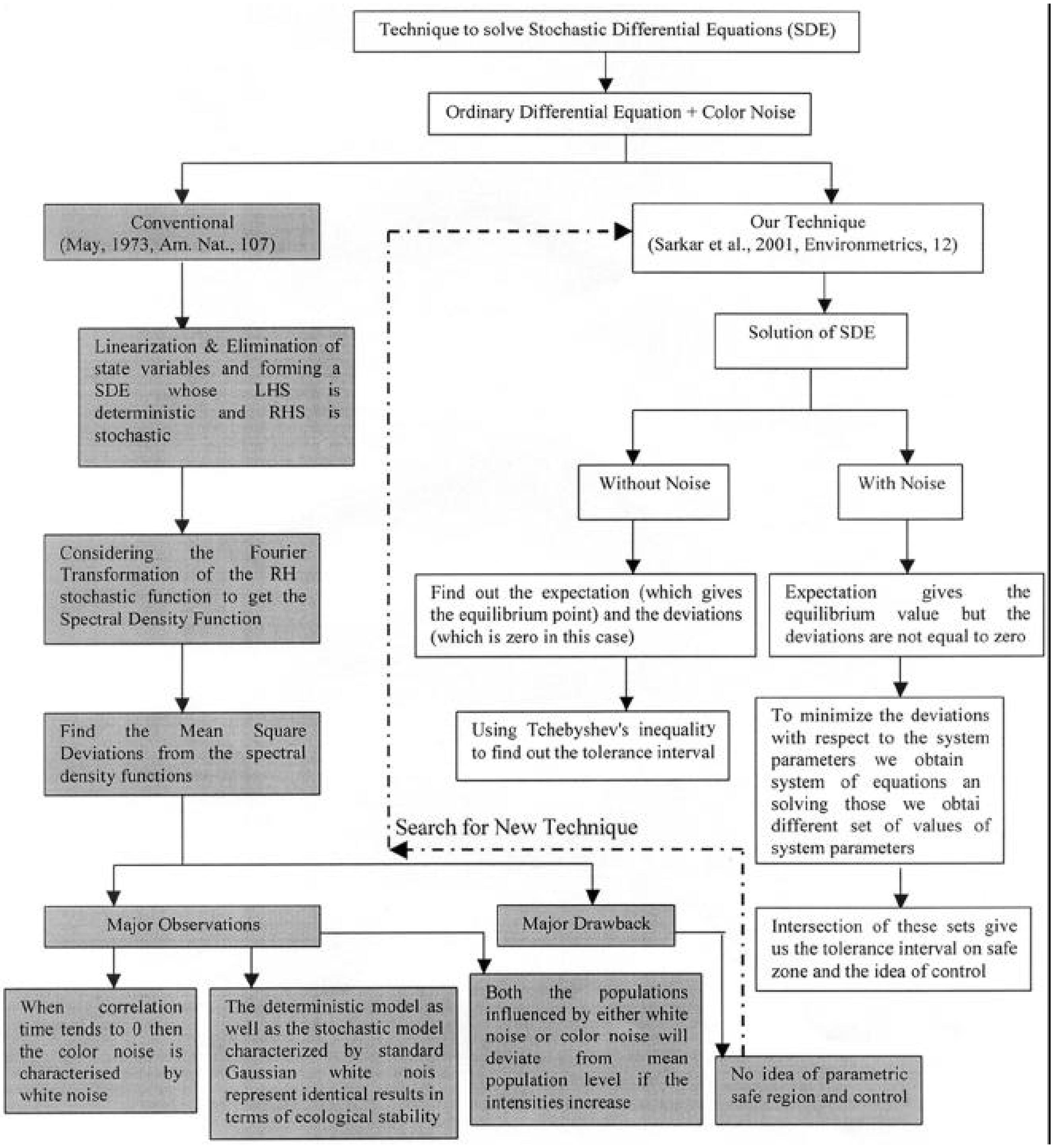

Figure 1. Schematic diagramme of the technique for finding safe parametric region (for detail mathematical analysis see, Sarkar *et al* 2001).

main of attraction of the equilibrium. Enhancement of intensities of random fluctuations increase the deviations of the population from mean population level (or equilibrium level) depicting instability of the system around the positive equilibrium. This sort of phenomena was also observed by May (1973) in a different system characterized by standard Gaussian white noise. But for the persistence of the species and for ecological stability of the system, the mean square deviations of the populations need to be minimum around the neighbourhood of the positive equilibrium and this gives rise to the idea of tolerance intervals around the positive equilibrium. The parameters of the system should be controlled in such a way that the entire population lie within these tolerance intervals.

The conventional technique (see, figure 1) only leads to the conclusion that the harvested system influenced by either white noise or colour noise is unstable. But for controlling the system we must have some idea on the maximum tolerance values of the system parameters. Maximum harvesting effort may be a key factor to regain the stability of the harvested system around the positive equilibrium. Unfortunately, in this context spectral density and mean square deviation analysis are not the appropriate tools. We have established a technique for finding the maximum (tolerance) values of the inaccessible parameters of the system which may give some idea for controlling the system to the system managers. The key technology for such a method lies in the solutions of stochastic differential equations, and an application of Tchebycheff's inequality.

Therefore, we have solved the linearized stochastic differential equation following the approach of Hoel *et al* (1993) and obtain

$$u(t) = u(0)\, f_1(t) + u'(0)\, f_2(t) + [(1 - \frac{ns}{d_0 L}) h_1(t) + \frac{ns}{L} x_1(t) + an\{\frac{1}{d_0'} h_2(t) - \sqrt{2e'}\, x_2(t)\}]$$

$$v(t) = v(0)\, f_1(t) + v'(0)\, f_2(t) + [(1 - \frac{mr}{d_0' K}) h_2(t) + \frac{mr}{K} x_2(t) + bm\{\frac{1}{d_0} h_1(t) - \sqrt{2e}\, x_1(t)\}]. \quad (9)$$

Here

$$a = \frac{rm}{K} + \frac{sn}{L},\; b = \frac{rsmn}{KL} - abmn, \text{ and}$$

$$f_1(t) = \frac{r_1' e^{r_1'' t} - r_1'' e^{r_1' t}}{r_1' - r_1''}, \qquad \text{when, } a^2 - 4b > 0.$$

$$= e^{a_1'} (\cos b_1' t - \frac{a_1'}{b_1'} \sin b_1' t), \quad \text{when, } a^2 - 4b < 0.$$

$$= e^{r_1 t}(1 - r_1 t), \qquad \text{when, } a^2 - 4b = 0.$$

$$f_2(t) = \frac{e^{r_1' t} - e^{r_1'' t}}{r_1' - r_1''}, \qquad \text{when, } a^2 - 4b > 0.$$

$$= \frac{e^{a_1'}}{b_1'} \sin b_1' t, \qquad \text{when, } a^2 - 4b < 0.$$

$$= te^{r_1 t}, \qquad \text{when, } a^2 - 4b = 0.$$

$$\text{with } r_1' = \frac{-a + \sqrt{a^2 - 4b}}{2},\quad r_1'' = \frac{-a - \sqrt{a^2 - 4b}}{2},$$

$$a_1' = \frac{-a}{2},\quad b_1' = \frac{\sqrt{4b - a^2}}{2},\quad r_1 = \frac{-a}{2} \text{ and}$$

$$u(0) = \log \frac{x(0)}{x*},\; u'(0) = [r - \frac{rx(0)}{K} - ay(0) - q_1 E],$$

$$v(0) = \log \frac{y(0)}{y*},\; v'(0) = [s - \frac{sy(0)}{L} - bx(0) - q_2 E].$$

The solution without noise takes the following form:

$$u(t) = u(0)\; f_1(t) + u'(0)\; f_2(t)$$

$$v(t) = v(0)\; f_1(t) + v'(0)\; f_2(t). \quad (10)$$

In this case, the ensemble average of the populations are given by

$$\langle u(t)\rangle = u(0)\,\langle f_1(t)\rangle + u'(0)\,\langle f_2(t)\rangle$$

and

$$\langle v(t)\rangle = v(0)\,\langle f_1(t)\rangle + v'(0)\,\langle f_2(t)\rangle.$$

For $t \to \infty$, $\langle f_1(t)\rangle \to 0$ and also $\langle f_2(t)\rangle \to 0$. Hence, we get $\langle u(t)\rangle = 0$ and $\langle v(t)\rangle = 0$. Using the inverse transformation, we obtain $\langle x(t)\rangle = x*$, $\langle y(t)\rangle = y*$ as well as the deviations of both the populations i.e. $s_x^2 = 0$ and $s_y^2 = 0$ (for $b > 0$).

We are now ready to present the technique for optimal harvesting by using Tchebycheff's inequality. This method was developed by Sarkar *et al* (2001) and has been successfully used by Chattopadhyay *et al* (2001). For ready reference, the out line of the technique is also discussed in figure 1.

We observe that when $s_x^2 = 0$ and $s_y^2 = 0$ are greater than a small neighbourhood around the positive equilibrium for different choice of system parameters, both the populations deviate from the tolerance level and the system becomes unstable around the positive equilibrium. In terms of system parameters, the deviations from the mean of two populations x and y are given by:

$$s_x^2 = ed_0 - \frac{2es[bK(r - q_1 E) - r(s - q_2 E)]}{(abKL - rs)} + (1 + \frac{e}{d_0}) \frac{s^2 [bK(r - q_1 E) - r(s - q_2 E)]^2}{(abKL - rs)^2}$$

$$+a^2e'(2+\frac{1}{d_0'})\frac{L^2[bK(r-q_1E)-r(s-q_2E)]^2}{(abKL-rs)^2}$$

$$s_y{}^2 = e'd_0' - \frac{2e'r[aL(s-q_2E)-s(r-q_1E)]}{(abKL-rs)} + (1+\frac{e'}{d_0'})$$

$$\frac{r^2[aL(s-q_2E)-s(r-q_1E)]^2}{(abKL-rs)^2}$$

$$+b^2e(2+\frac{1}{d_0})\frac{K^2[aL(s-q_2E)-s(r-q_1E)]^2}{(abKL-rs)^2}. \quad (11)$$

Ecological stability will persist if both the populations lie within the tolerance intervals. To obtain this we shall have to minimize the deviations from mean population level. Consequently we obtain two-sets of equations containing the system parameters. One can easily find out the solutions of these sets of equations by using Gauss elimination method (Danilina *et al* 1988) or any other method which will give two-sets of parameter values that are optimum for each population. The intersection of these two-sets will give the safe or stable zone in the parametric region of the system around the positive interior equilibrium (P_*).

3. Application of the method under different ecological scenario

In this section we will apply the above method in three different ecological settings to show its utility in both theory and practice.

(i) We aim to establish that the conclusion drawn through spectral density analysis and our method is identical. As optimal harvesting is the main factor for controlling the harvested system, we shall put our emphasis on the harvest function involving the parameters, catchability coefficients (q_1, q_2) and the effort (E).

(ii) We reinvestigate the deterministic two-species model of Bhattacharya and Begum (1996) under realistic environmental variability. We show that if the system managers chalk out suitable harvesting strategy based on their estimation, there is a high probability of extinction of the mother species, and consequently the entire population. So, there is a need for valid estimation of the system parameters under environmental variability.

(iii) We shall consider a study based on the experimental field observations carried out in Senegalese Artisanal Fisheries (Laloe *et al* 1998) and apply the technique for finding the inaccessible parameters in a systematic way without mathematical details.

Problem 1: Here, we consider the two-species harvested system (equation 2) with colour noise and will try to estimate the catchability coefficients (q_1 and q_2) by both the conventional spectral density analysis technique and our method (see, figure 1). The behaviour of the system in the plane of the catchability coefficients q_1 and q_2 is an interesting phenomenon in the study of harvesting strategy. For this problem, the result through spectral density analysis shows that in the q_1–q_2 plane if

$$q_1 \to \frac{r^+}{E} \text{ and } q_2 \to \frac{s^+}{E} \text{ along the straight lines}$$

$$q_2 = [\frac{s}{E} + \frac{es(abKL-rs)}{rE\{(1+\frac{e}{d_0})s^2 + a^2e'L^2(2+\frac{1}{d_0'})\}}] + \frac{bK}{rE^2}(q_1 - \frac{r}{E}) \text{ or}$$

$$q_2 = [\frac{s}{E} - \frac{e'r(abKL-rs)}{EaL\{1+\frac{e'}{d_0'})r^2 + b^2eK^2(2+\frac{1}{d_0})\}}] + \frac{s}{E^2aL}(q_1 - \frac{r}{E}),$$

then the system of interacting species in a random environment exhibits abnormally large fluctuations around the positive equilibrium with increasing time and a periodic background noise. Our objective is to show that the spectral density analysis and the method developed by us here give identical results in this particular situation, but our method give more precise values for the catchability coefficients.

Towards this we differente s_x^2 [given in equation (11)] with respect to q_1, q_2, E and obtain the following set of equations by equating it to zero:

$$q_1 = \frac{r}{E} - \frac{r(s-q_2E)}{EbK} - \frac{es(abKL-rs)}{EbK\{(1+\frac{e}{d_0})s^2 + a^2e'L^2(2+\frac{1}{d_0'})\}}$$

$$q_2 = \frac{s}{E} - \frac{bK(r-q_1E)}{rE} + \frac{es(abKL-rs)}{rE\{(1+\frac{e}{d_0})s^2 + a^2e'L^2(2+\frac{1}{d_0'})\}}$$

$$E = \frac{r(bK-s)}{bKq_1 - rq_2} - \frac{es(rs-abKL)}{(rq_2-bKq_1)\{(1+\frac{e}{d_0})s^2 + a^2e'L^2(2+\frac{1}{d_0'})\}}. \quad (12)$$

Similarly differentiating s_y^2 [given in equation (11)] with respect to q_1, q_2, E and equating to zero we obtain the following set of equations:

$$q_1 = \frac{r}{E} - \frac{aL(s-q_2E)}{sE} + \frac{e'r(abKL-rs)}{sE\{(1+\frac{e'}{d_0'})r^2 + b^2eK^2(2+\frac{1}{d_0})\}}$$

$$q_2 = \frac{s}{E} - \frac{s(r - q_1 E)}{EaL} - \frac{e'r(abKL - rs)}{EaL\{(1+\frac{e'}{d'_0})r^2 + b^2 eK^2 (2+\frac{1}{d_0})\}}$$

$$E = \frac{s(a'L - r)}{aLq_2 - sq_1} - \frac{e'r(rs - abKL)}{(sq_1 - aLq_2)\{(1+\frac{e'}{d'_0})r^2 + b^2 eK^2(2+\frac{1}{d_0})\}}. \quad (13)$$

Solving equations (12) and (13) we obtain

$$q_{1c} = \frac{r}{E}, \; q_{2c} = \frac{s}{E},$$

where, q_{1c} and q_{2c} are the maximum values marking the boundaries of the safe zone. It is obvious that if q_{1c} and q_{2c} do not lie in the safe zone i.e. when $q_{1c} \to r^+/E$ and $q_{2c} \to s^+/E$, then the system exhibits abnormally large fluctuations.

Ecological implication: This example clearly shows that conclusion drawn from the conventional technique and our method is the same. But for monitoring the harvested populations the exact critical values of the catchability coefficients and harvesting efforts should be known in advance. This new technique will be of considerable help to the resource managers in this aspect.

Problem 2: The method for finding the maximum value of effort in a harvested system is still an open question even in deterministic framework. For this, we consider a logistic growth model of two-species with competitive interaction between them as discussed in Bhattacharya and Begum (1996) where they have calculated the maximum efforts. We address the question if their estimated values for maximum harvesting of the species are valid under environmental variability or noise?

From equation (11), we see that

$$\frac{d^2 s_x^2}{dE^2} > 0 \text{ and } \frac{d^2 s_y^2}{dE^2} > 0.$$

Consequently, we obtain the maximum harvesting effort (E_{c1}) for species 1 as

$$E_{c1} = \frac{r(bK - s)}{bKq_1 - q_2} - \frac{es(rs - abKL)}{(rq_2 - bKq_1)[(1+\frac{e}{d_0})s^2 + a^2 e'L^2(2+\frac{1}{d'_0})]}$$

which is positive if

(i) $aL < r$, $bK < s$, $aLq_2 < sq_1 < rq_2$

(ii) $r < aL$, $s < bK$, $rq_2 < sq_1 < aLq_2$. $\quad (a)$

Also, the maximum harvesting effort (E_{c2}) for species 2 as:

$$E_{c2} = \frac{s(aL - r)}{aLq_2 - sq1} - \frac{e'r(rs - abKL)}{(sq_1 - aLq_2)[(1+\frac{e'}{d'_0})r^2 + b^2 eK^2(2+\frac{1}{d_0})]}$$

which is positive if

(i) $r < aL$, $s < bK$, $sq_1 < rq_2 < bKq_1$

(ii) $aL < r$, $bK < s$, $bKq_1 < rq_2 < sq_1$. $\quad (b)$

It is to be noted here that the maximum harvesting effort without noise, calculated by Bhattacharya and Begum (1996) are

$$\frac{r(bK - s)}{bKq_1 - rq_2} \text{ and } \frac{s(aL - r)}{aLq_2 - sq_1}$$

under the same conditions in (a) and (b) respectively. For stable equilibrium, if conditions (a) and (b) hold, the expressions

$$\frac{es(rs - abKL)}{(rq_2 - bKq_1)[(1+\frac{e}{d_0})s^2 + a^2 e'L^2(2+\frac{1}{d'_0})]} \text{ and}$$

$$\frac{e'r(rs - abKL)}{(sq_1 - aLq_2)[(1+\frac{e'}{d'_0})r^2 + b^2 eK^2(2+\frac{1}{d_0})]}$$

have a negative effect on E_{c1} and E_{c2} respectively. So in a real environment, the maximum harvesting effort is less than what was estimated by them. Hence in a realistic situation the condition for maximum harvesting effort in a two-species combined harvesting system will be,

$E_c = \min(E_{c1}, E_{c2})$.

Ecological implication: The above discussion reveals the fact that while monitoring a system it is necessary to consider the most realistic situation. If resource managers choose a harvesting strategy in a purely deterministic situation they may be in a wrong track and harvesting may lead to the extinction of the mother species and, ultimately, of the entire population. Hence, the estimation of maximum harvesting effort through our method provides a more realistic view for monitoring the system and may be helpful to save the species from exploitation.

Problem 3: In this problem we provide a systematic way of parameter estimation and gaining control on a practical problem. We consider the experimental field observations carried out on Senegalese artisanal fishery. In multifleet-multispecies fisheries exploitation, fishing units of some fleets may be an important factor. Laloe *et al* (1998) considered Lotka-Volterra model for multi-species harvested fishery system to represent the main features of the Senegalese artisanal fishery (Laloe and Samba 1991). They obtained some values of the parameters from experimental observations and estimated other inaccessible parameters of the model using least squares criterion. They also concluded that the equilibrium relations obtained from their model are similar to the models of Pella and Tomlinson (1969), Laloe (1988) and Schaefer (1957). The question addressed here is whether these estimations are still valid for the same model under environmental fluctuation?

We take the same two-species model considered by Laloe *et al* (1998) with environmental stochasticity in both the populations, and reconstruct the problem as an optimization problem. Our basic aim is to find out the range of inaccessible parameters in terms of known parameters which will give the safe parametric zone.

The model equations with noise are:

$$\frac{dx}{dt} = x(r + h_1(t) - \frac{rx}{K}) - axy - q_1 Ex$$

$$\frac{dy}{dt} = y(s + h_2(t) - \frac{sy}{L}) - bxy - q_2 Ey, \quad (14)$$

where the perturbed terms $h_i(t)$ ($i = 1, 2$) are coloured noise following the Ornstein-Uhlenbeck process described in equations (3) and (4). The constraint on the conditions for the existence and local stability of the deterministic system around the positive equilibrium are obtained in § 2.

The parametric values obtained from Laloe *et al* (1998) are $K = 10000$ tons, $L = 40000$ tons, $r = 2{\cdot}52$/day, $s = 0{\cdot}65$/day, $q_1 = 1{\cdot}81$/tons/day and $q_2 = 4{\cdot}08$/tons/day. The objective is to estimate the interaction parameters a, b and the effort E that are inaccessible to experimental validation.

To solve the above optimization problem we proceed in the following way:

Step 1: Minimize s_x^2 [equation (11)] considering the constraint inequality. This gives the following set of equations for a, b and E:

$$a = \frac{ersbK + (1+\frac{e}{d_0})\, sbKL\,[bK\,(r - q_1 E) - r\,(s - q_2 E)]}{eb^2 K^2 L - re' L^2 (2 + \frac{1}{d_0'})\,[bK\,(r - q_1 E) - r\,(s - q_2 E)]}$$

$$b = \frac{(1+\frac{e}{d_0})\, s^2 + a^2 e' \,(2 + \frac{1}{d_0'})\, L^2 + rs^2 e}{esaKL}$$

$$E = \frac{r(bK - s)}{bKq_1 - rq_2} - \frac{es(rs - abKL)}{(rq_2 - bKq_1)[(1+\frac{e}{d_0})s^2 + a^2 e' L^2 (2 + \frac{1}{d_0'})]}. \quad (15)$$

Similarly, minimizing s_y^2 [equation (11)], one can get:

$$a = \frac{(1+\frac{e'}{d_0'})r^2 + b^2 e(2 + \frac{1}{d_0})K^2 + r^2 se'}{e' rbKL}$$

$$b = \frac{e' rsaL + (1+\frac{e'}{d_0'})\, raKL[aL\,(s - q_2 E) - s(r - q_1 E)]}{e' a^2 KL^2 - seK^2 \,(2 + \frac{1}{d_0})\,[aL\,(s - q_2 E) - s(r - q_1 E)]}$$

$$E = \frac{s(aL - r)}{aLq_2 - sq_1} - \frac{e' r\,(rs - abKL)}{(sq_1 - aLq_2)[(1+\frac{e'}{d_0'})r^2 + b^2 e\, K^2\,(2 + \frac{1}{d_0})]}. \quad (16)$$

Step 2: Solving equation (15) and using the given parametric values one can obtain the tolerance values of a, b and E for which the deviation of x-population from the mean level is minimum. This gives $a_1 = 0{\cdot}003$, $b_1 = 246{\cdot}123$, $E_1 = 1{\cdot}184$. These values will give the safe region in the parametric space for which x-population will be stable around the positive equilibrium under environmental stochasticity.

Step 3: Similarly solution of equation (16) will give the tolerance values of a, b, E for which the deviation of y-population from the mean level is minimum. This yields $a_2 = 14{\cdot}842$, $b_2 = 0{\cdot}006$, $E_2 = 0{\cdot}136$. These values will give the safe region in the parametric space for which y-population will be stable around the positive equilibrium under environmental stochasticity.

Step 4: This step involves the tolerance values of a, b and E obtained from the above two-steps for which the entire system will remain stable around the positive equilibrium and both the populations will lie inside the tolerance level. These are:

$$a_c = \min\,\{a_1, a_2\} = 0{\cdot}003$$

$$b_c = \min\,\{b_1, b_2\} = 0{\cdot}006$$

$$E_c = \min\,\{E_1, E_2\} = 0{\cdot}136. \quad (17)$$

For the above set of parameter values, we observe that the y population lie, inside the tolerance level, depicting stable situation (see, figure 2). Keeping all the other accessible parameters fixed, if we further increase the catchability coefficient (q_1) about ten times than the previous one, then the values of the inaccessible parameters will change and they will be no longer in the safe parametric zone. In this situation, we observe that both the population will deviate from the tolerance level, depicting an unstable situation (see, figure 3). The safe parametric region (see, figure 4) for which both the populations will persist under environmental stochasticity can be obtained (using the values derived in step 2 and step 3) by calculating the intersection region $M \cap N$, where

$$M = \{a_1 = 0 - 0{\cdot}003,\ b_1 = 0 - 246{\cdot}123,\ E_1 = 0 - 1{\cdot}184\}$$

$$N = \{a_2 = 0 - 14{\cdot}842,\ b_2 = 0 - 0{\cdot}006,\ E_2 = 0 - 0{\cdot}136\}$$

and keeping all other accessible parameters fixed.

Ecological implication: It is quite interesting to note that the values of inaccessible parameters a and b estimated by Laloe *et al* (1998) have been increased by about 500% and decreased by 94% respectively by our method [the values of a and b estimated by Laloe *et al* (1998) were 0·0005 and 0·0965 respectively] and hence the differential impact of environmental fluctuation is clear. The above step wise discussion will be helpful to the system managers to chalk out a suitable harvesting strategy and to estimate the unknown system parameters (provided some of the other system parameters are known through experimental study) in fishery problems in practice.

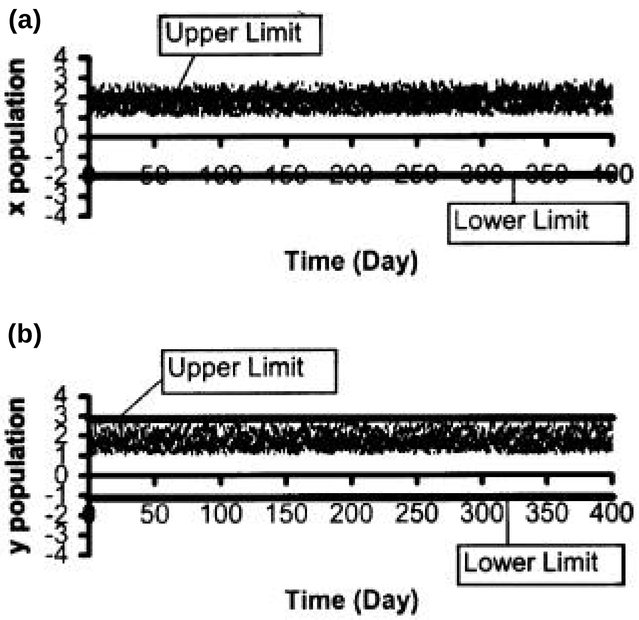

Figure 2. Numerical solutions of equation (9) show that the maximum number of populations lie within the tolerance interval for depicting a stable situation. Parameters as given in the text.

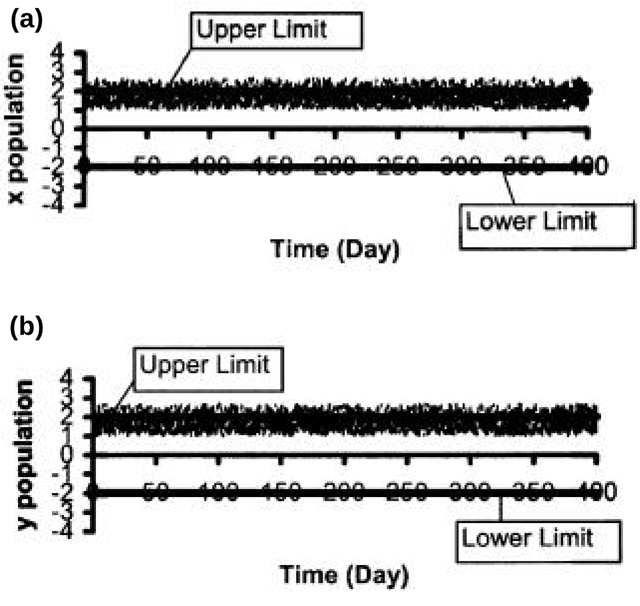

Figure 3. Numerical solutions of equation (9) show that the maximum number of populations lie outside the tolerance interval with $q_1 = 18{\cdot}1$/tons/day, depicting a unstable situation. Other parameters as given in the text.

4. Conclusion

Exploitation of biological resources and the harvest of population species are commonly practiced in fisheries, forestry and wild life management. In this paper, a two-species combined harvesting Lotka-Volterra competitive system has been considered. But as uncertainty is very

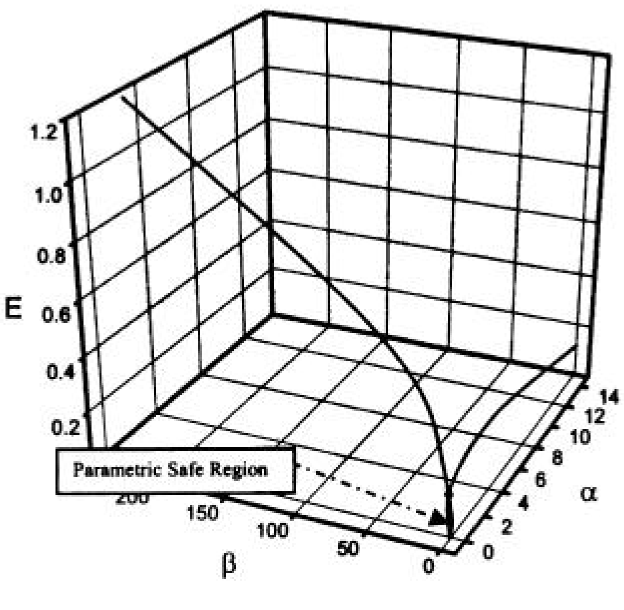

Figure 4. Safe parametric zone.

much prevalent in resource economics, random fluctuations in the form of colour noise has been considered in the system. The main purpose of this paper is to estimate the maximum harvesting effort and other inaccessible parameters of the system parameters. We have tried to solve this problem in two ways – the conventional approach by calculating mean square deviations of the populations with the help of spectral density analysis, and by using a new technique developed by us through solution of stochastic differential equations as well as using Tchebycheff's inequality. In any biological system the mechanism of regaining stability of the system is an important consideration but the spectral density analysis is not the appropriate tool in such a situation. For such situations, the parameters, specifically the inaccessible parameters of the system may play an important role in controlling the system. The method developed by us for finding inaccessible parameters involved in the system, enables us to obtain the maximum harvesting effort for which the system remains stable around the positive equilibrium under stochastic fluctuations. It has also been observed that the maximum harvesting effort estimated from deterministic situation is greater than in stochastic environment, which is an important finding from management view point. Moreover to substantiate the analytical results real life fishery data have been used to determine the inaccessible parameters in a systematic manner and a parametric safe region has been found for controlling the ecological stability of the system.

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