THE THEORY OF THE FLUCTUATIONS IN BRIGHTNESS OF THE MILKY WAY. IV

G. MÜNCH AND S. CHANDRASEKHAR
Yerkes Observatory
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ABSTRACT

In this paper the integral equation governing the fluctuations in brightness of the Milky Way is solved, quite generally, for the case when the system of the stars and clouds extends to infinity in the direction of the line of sight and the transparency factor, q, characterizing the clouds is governed by an arbitrary frequency function, $\psi(q)(0 \leqslant q \leqslant 1)$. The solution is obtained in the form of a series in Laguerre polynomials with coefficients depending only on the moments of $\psi(q)$. It is further shown that the solutions found in Paper II for two particular forms of $\psi(q)$ can be obtained as special cases of the general solution given here.

1. *Introduction.*—In this paper, which is a continuation of three earlier papers[1] devoted to the theory of the fluctuations in brightness of the Milky Way, we shall return to the case considered in Paper II, namely, when the system of the stars and interstellar clouds extends to infinity in the direction of the line of sight. In this case the integral equation governing the distribution of brightness is (I, eq. [17], or III, eq. [3])

$$f(u)+\frac{df}{du}=\int_0^1 f\left(\frac{u}{q}\right)\psi(q)\,dq\,, \tag{1}$$

where u is a measure of the observed brightness, $f(u)$ is the probability that the brightness (in the chosen units) exceeds the assigned value u, and q is the transparency factor which is assumed to occur with a frequency given by $\psi(q)$. Regarding the solution of equation (1), we know that the moments of the distribution,

$$\mu_n=-\int_0^\infty u^n d f(u)\,, \tag{2}$$

are given by

$$\mu_n=n!\prod_{j=1}^{n}(1-\varpi_j)^{-1}\,, \tag{3}$$

where

$$\varpi_j=\int_0^1 q^j\psi(q)\,dq\,. \tag{4}$$ [2]

In Paper II we showed how equation (1)—or, rather, the equation

$$g(u)+\frac{dg}{du}=\int_0^1 g\left(\frac{u}{q}\right)\psi(q)\frac{dq}{q}\,, \tag{5}$$

governing the corresponding frequency function of u—can be solved for the two cases in which all the clouds are equally transparent and when $\psi(q)=(n+1)q^n$. In this paper we shall show how equation (1) can be solved quite generally.

[1] *Ap. J.*, **112**, 380, 393, 1950; **114**, 110, 1951. These papers will be referred to as "Papers I, II, and III," respectively.

[2] In Papers I and II we denoted the moments of q by q_j. We are now denoting them by ϖ_j, since we wish to retain q_1, q_2, etc., for denoting the running variable when integrating over q (cf. eqs. [12] and [13] below).

2. *The solution of equation (1) by the method of successive iterations.*—Letting

$$f(u) = e^{-u}F(u), \tag{6}$$

we can write equation (1) more conveniently in the form

$$\frac{dF(u)}{du} = \int_0^1 e^{-(1-q)u/q} F\left(\frac{u}{q}\right) d\phi(q), \tag{7}$$

where, for the sake of brevity, we have further written

$$d\phi(q) = \psi(q)\, dq\,. \tag{8}$$

[In eq. (7) the integral on the right-hand side may now be regarded as a Stieltjes integral.]

Equation (7) can be formally integrated to give

$$F(u) = 1 + \int_0^1 d\phi(q) \int_0^u dt\, e^{-(1-q)t/q} F\left(\frac{t}{q}\right), \tag{9}$$

where the constant of integration has been chosen to satisfy the normalizing condition,

$$F(0) = f(0) = 1\,. \tag{10}$$

According to equation (9), we have

$$F\left(\frac{t}{q}\right) = 1 + \int_0^1 d\phi(q_1) \int_0^{t/q} dt_1\, e^{-(1-q_1)t_1/q_1} F\left(\frac{t_1}{q_1}\right). \tag{11}$$

Substituting this expression for $F(t/q)$ in equation (9), we obtain

$$\begin{aligned} F(u) = 1 &+ \int_0^1 d\phi(q_1) \int_0^u dt_1\, e^{-(1-q_1)t_1/q_1} \\ &+ \int_0^1 d\phi(q_2) \int_0^1 d\phi(q_1) \int_0^u dt_2\, e^{-(1-q_2)t_2/q_2} \int_0^{t_2/q_2} dt_1\, e^{-(1-q_1)t_1/q_1} F\left(\frac{t_1}{q_1}\right). \end{aligned} \tag{12}$$

In this last equation, we can again substitute for $F(t_1/q_1)$ according to equation (11), and in the resulting equation we can again make the same substitution. In this manner, after m such substitutions, we shall obtain

$$\begin{aligned} F(u) = 1 &+ \sum_{n=1}^{m} \int_0^1 d\phi(q_n) \int_0^1 d\phi(q_{n-1}) \int_0^1 \ldots \int_0^1 d\phi(q_1) \int_0^u dt_n\, e^{-(1-q_n)t_n/q_n} \\ &\times \int_0^{t_n/q_n} dt_{n-1}\, e^{-(1-q_{n-1})t_{n-1}/q_{n-1}} \int_0^{t_{n-1}/q_{n-1}} \ldots \int_0^{t_2/q_2} dt_1\, e^{-(1-q_1)t_1/q_1} \\ &+ \int_0^1 d\phi(q_{m+1}) \int_0^1 d\phi(q_m) \int_0^1 \ldots \int_0^1 d\phi(q_1) \int_0^u dt_{m+1} e^{-(1-q_{m+1})t_{m+1}/q_{m+1}} \\ &\times \int_0^{t_{m+1}/q_{m+1}} dt_m\, e^{-(1-q_m)t_m/q_m} \ldots \int_0^{t_2/q_2} dt_1\, e^{-(1-q_1)t_1/q_1} F\left(\frac{t_1}{q_1}\right). \end{aligned} \tag{13}$$

By defining the sequence of functions

$$K_1(u;\, q_1) = \int_0^u dt_1\, e^{-(1-q_1)t_1/q_1} = \frac{q_1}{1-q_1}\left[1 - e^{-(1-q_1)u/q_1}\right], \tag{14}$$

$$K_2(u;\, q_2,\, q_1) = \int_0^u dt_2\, e^{-(1-q_2)t_2/q_2} K_1\left(\frac{t_2}{q_2};\, q_1\right), \tag{15}$$

and

$$K_n(u;\, q_n,\, \ldots,\, q_1) = \int_0^u dt_n\, e^{-(1-q_n)t_n/q_n} K_{n-1}\left(\frac{t_n}{q_n};\, q_{n-1},\, \ldots,\, q_1\right) \qquad (n = 2,\, 3,\, \ldots), \tag{16}$$

we can rewrite equation (13) in the form

$$F(u) = 1 + \sum_{n=1}^{m} \int_0^1 d\phi(q_n) \int_0^1 \ldots \int_0^1 d\phi(q_1)\, K_n(u;\, q_n,\, \ldots,\, q_1) + R_m(u), \tag{17}$$

where $R_m(u)$ stands for the last term in equation (13) involving $F(t_1/q_1)$.

We shall now show that the infinite series,

$$F(u) = 1 + \sum_{n=1}^{\infty} \int_0^1 d\phi(q_n) \int_0^1 \ldots \int_0^1 d\phi(q_1)\, K_n(u;\, q_n,\, \ldots,\, q_1), \tag{18}$$

obtained by letting $m \to \infty$ in equation (17), actually converges uniformly for all $0 \leqslant u < \infty$ to the required solution of equation (7). In order to establish this, we need the following two lemmas.

Lemma 1.—The functions $K_n(u;\, q_n,\, \ldots,\, q_1)$ defined by equations (14) and (16) satisfy the recurrence relation,

$$K_n(u;\, q_n,\, \ldots,\, q_1) = \frac{q_1}{1-q_1}[K_{n-1}(u;\, q_n,\, \ldots,\, q_3,\, q_2) - K_{n-1}(u;\, q_n,\, q_{n-1},\, \ldots,\, q_3,\, q_2 q_1)] \qquad (n = 2,\, 3,\, \ldots). \tag{19}$$

Proof.—The proof is by induction. From equations (14) and (15) it follows that

$$\begin{aligned} K_2(u;\, q_2,\, q_1) &= \frac{q_1}{1-q_1} \int_0^u e^{-(1-q_2)t_2/q_2}[1 - e^{-(1-q_1)t_2/q_2 q_1}]\, dt_2 \\ &= \frac{q_1}{1-q_1}\left[\int_0^u e^{-(1-q_2)t_2/q_2} dt_2 - \int_0^u e^{-(1-q_2 q_1)t_2/q_2 q_1} dt_2\right]. \end{aligned} \tag{20}$$

Hence

$$K_2(u;\, q_2,\, q_1) = \frac{q_1}{1-q_1}[K_1(u;\, q_2) - K_1(u;\, q_2 q_1)]. \tag{21}$$

This verifies the lemma for $n = 2$. Now assume that the lemma is true for $n - 1$ and consider the integral expression (eq. [16]) for K_n in terms of K_{n-1} and substitute for K_{n-2} in accordance with the lemma. We obtain

$$\begin{aligned} K_n(u;\, q_n,\, \ldots,\, q_1) &= \frac{q_1}{1-q_1}\left[\int_0^u dt_n\, e^{-(1-q_n)t_n/q_n}\right. \\ &\left.\times \left\{K_{n-2}\left(\frac{t_n}{q_n};\, q_{n-1},\, \ldots,\, q_3,\, q_2\right) - K_{n-2}\left(\frac{t_n}{q_n};\, q_{n-1},\, \ldots,\, q_3,\, q_2 q_1\right)\right\}\right]; \end{aligned} \tag{22}$$

and, again using equation (16) (for $n - 1$), we can rewrite the foregoing as

$$K_n(u;\, q_n,\, q_{n-1},\, \ldots,\, q_1) = \frac{q_1}{1-q_1}[K_{n-1}(u;\, q_n,\, \ldots,\, q_2) - K_{n-1}(u;\, q_n,\, \ldots,\, q_3,\, q_2 q_1)]; \tag{23}$$

this establishes the lemma for n. The general truth of the lemma now follows by induction.

Lemma 2.—The function $K_n(u;q_n, \ldots, q_1)$ ($0 \leqslant u < \infty$ and $0 \leqslant q_j < 1$, $j = 1, \ldots, n$) has a uniformly convergent expansion in terms of the Laguerre polynomials, $L_k(u)$, given by

$$K_n(u;\, q_n, \ldots, q_1) = q_n q_{n-1} \ldots q_1 - \sum_{k=1}^{\infty} p_k(n;\, q_n, \ldots, q_1) L_k(u)\,, \tag{24}$$

where

$$p_k(n;\, q_n, \ldots, q_1) = \sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j} q_n^{j+2} \sum_{i_{n-1}=1}^{j+2} q_{n-1}^{i_{n-1}} \sum_{i_{n-2}=1}^{i_{n-1}} q_{n-2}^{i_{n-2}} \times \ldots \sum_{i_2=1}^{i_3} q_2^{i_2} \sum_{i_1=1}^{i_2} q_1^{i_1}\,, \tag{25}$$

are polynomials in the q_i's.

Proof.—One definition of the Laguerre polynomials, $L_k(u)$, is in terms of the generating function $\exp[-ux/(1-x)]/(1-x)$. Thus,[3]

$$e^{-ux/(1-x)} = (1-x) \sum_{k=0}^{\infty} x^k L_k(u)\,. \tag{26}$$

This expansion is uniformly convergent for $0 \leqslant u < \infty$ and $0 \leqslant x < 1$. Letting $x = 1 - q_1$ and remembering that $L_0(u) \equiv 1$, we can rewrite equation (26) in the form

$$e^{-(1-q_1)u/q_1} = q_1 + q_1(1-q_1) \sum_{k=1}^{\infty} (1-q_1)^{k-1} L_k(u)\,. \tag{27}$$

Using this expansion in equation (14), we obtain the following representation for $K_1(u;\, q_1)$:

$$K_1(u;\, q_1) = q_1 - \sum_{k=1}^{\infty} p_k(1;\, q_1) L_k(u)\,, \tag{28}$$

where

$$p_k(1;\, q_1) = q_1^2(1-q_1)^{k-1} = \sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j} q_1^{j+2}\,. \tag{29}$$

Equations (28) and (29) verify the lemma for $n = 1$.

Now, assuming that the lemma is true for n, we can establish its validity for $n + 1$. For, expressing K_{n+1} in terms of K_n in accordance with lemma 1 and using the polynomial representation given by equations (24) and (25) for K_n, we have

$$K_{n+1}(u;\, q_{n+1}, \ldots, q_1) = \frac{q_1}{1-q_1}[K_n(u;\, q_{n+1}, \ldots, q_2) - K_n(u;\, q_{n+1}, \ldots, q_3, q_2 q_1)] = \frac{q_1}{1-q_1}\Big[q_{n+1} q_n \ldots q_3 q_2 (1-q_1) - \sum_{k=1}^{\infty} L_k(u) \sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j} q_{n+1}^{j+2} \sum_{i_n=1}^{j+2} q_n^{i_n} \sum_{i_{n-1}=1}^{i_n} q_{n-1}^{i_{n-1}} \ldots \sum_{i_2=1}^{i_3} q_2^{i_2}(1-q_1^{i_2})\Big]$$

[3] See, e.g., G. Szego, *Orthogonal Polynomials* ("American Mathematical Society Colloquium Publications," Vol. XXIII [1939]), p. 97.

$$= q_{n+1} q_n \cdots q_2 q_1 - \sum_{k=1}^{\infty} L_k(u) \sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j} q_{n+1}^{j+2} \sum_{i_n=1}^{j+2} q_n^{i_n} \tag{30}$$

$$\cdots \sum_{i_2=1}^{i_3} q_2^{i_2} \frac{q_1(1-q_1^{i_2})}{1-q_1} = q_{n+1} \cdots q_1 - \sum_{k=1}^{\infty} L_k(u) \sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j}$$

$$\times\, q_{n+1}^{j+2} \sum_{i_n=1}^{j+2} q_n^{i_n} \sum_{i_{n-1}}^{i_n} q_n^{i_{n-1}} \cdots \sum_{i_2=1}^{i_3} q_2^{i_2} \sum_{i_1=1}^{i_2} q_1^{i_1}.$$

The truth of the lemma now follows by induction.

Returning to equation (18), we can now substitute for K_n its expansion as a power series in the q_i's. According to equations (24) and (25), the general term in the series on the right-hand side of equation (18) is

$$\int_0^1 d\phi(q_n) \int_0^1 d\phi(q_{n-1}) \int_0^1 \cdots \int_0^1 d\phi(q_1)\, K_n(u;\, q_n, \ldots, q_1)$$

$$= \varpi_1^n - \sum_{k=1}^{\infty} L_k(u) \sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j} \varpi_{j+2} \sum_{i_{n-1}=1}^{j+2} \varpi_{i_{n-1}} \sum_{i_{n-2}=1}^{i_{n-1}} \varpi_{i_{n-2}} \cdots \sum_{i_1=1}^{i_2} \varpi_{i_1}; \tag{31}$$

we thus have

$$F(u) = 1 + \sum_{n=1}^{\infty} \varpi_1^n$$

$$- \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} L_k(u) \sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j} \varpi_{j+2} \sum_{i_{n-1}=1}^{j+2} \varpi_{i_{n-1}} \sum_{i_{n-2}=1}^{i_{n-1}} \varpi_{i_{n-2}} \cdots \sum_{i_1=1}^{i_2} \varpi_{i_1}. \tag{32}$$

Letting

$$a_k = \sum_{n=1}^{\infty} \sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j} \varpi_{j+2} \sum_{i_{n-1}=1}^{j+2} \varpi_{i_{n-1}} \sum_{i_{n-2}=1}^{i_{n-1}} \varpi_{i_{n-2}} \cdots \sum_{i_1=1}^{i_2} \varpi_{i_1}, \tag{33}$$

we can rewrite equation (32) in the form

$$F(u) = \frac{1}{1-\varpi_1} - \sum_{k=1}^{\infty} a_k L_k(u). \tag{34}$$

The expression for the coefficient a_k in equation (34) can be simplified in the following manner: By inverting the order of the summations over n and j in equation (33), we have

$$a_k = \sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j} \varpi_{j+2} \Bigg\{ 1 + \sum_{i_1=1}^{j+2} \varpi_{i_1} + \sum_{i_2=1}^{j+2} \varpi_{i_2} \sum_{i_1=1}^{i_2} \varpi_{i_1}$$

$$+ \cdots + \sum_{i_n=1}^{j+2} \varpi_{i_n} \sum_{i_{n-1}=1}^{i_n} \varpi_{i_{n-1}} \cdots \sum_{i_2=1}^{i_3} \varpi_{i_2} \sum_{i_1=1}^{i_2} \varpi_{i_1} + \cdots \Bigg\}. \tag{35}$$

The quantity in braces on the right-hand side is clearly

$$\prod_{i=1}^{j+2} (1 + \varpi_i + \varpi_i^2 + \cdots + \varpi_i + \cdots) = \prod_{i=1}^{j+2} (1-\varpi_i)^{-1}. \tag{36}$$

Hence

$$a_k = \sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j} \varpi_{j+2} \prod_{i=1}^{j+2} (1-\varpi_i)^{-1} . \tag{37}$$

By some rearranging of the terms, we can simplify the foregoing expression for a_k still further to the form

$$a_k = -\sum_{j=0}^{k} (-1)^j \binom{k}{j} \prod_{i=1}^{j+1} (1-\varpi_i)^{-1} . \tag{38}$$

The fact that we have been able to carry out, explicitly, the summation over n in the expression for a_k establishes the convergence of the series on the right-hand side of equation (18); and from lemma 2 it follows that the convergence is uniform for $0 \leqslant u < \infty$. The series (18), therefore, represents the required solution of equation (7).

With a_k given by equation (37), equation (34) becomes

$$F(u) = \frac{1}{1-\varpi_1} + \sum_{k=1}^{\infty} L_k(u) \sum_{j=0}^{k} (-1)^j \binom{k}{j} \prod_{i=1}^{j+1} (1-\varpi_i)^{-1} ; \tag{39}$$

or, since $L_0(u) \equiv 1$, we can also write

$$F(u) = \sum_{k=0}^{\infty} L_k(u) \sum_{j=0}^{k} (-1)^j \binom{k}{j} \prod_{i=1}^{j+1} (1-\varpi_i)^{-1} . \tag{40}$$

The solution for $f(u)$ is therefore given by

$$f(u) = e^{-u} \sum_{k=0}^{\infty} L_k(u) \sum_{j=0}^{k} (-1)^j \binom{k}{j} \prod_{i=1}^{j+1} (1-\varpi_i)^{-1} ; \tag{41}$$

or, using expression (3) for the moments μ_j, we can write, alternatively,

$$f(u) = e^{-u} \sum_{k=0}^{\infty} L_k(u) \sum_{j=0}^{k} \frac{(-1)^j}{(j+1)!} \binom{k}{j} \mu_{j+1} . \tag{42}$$

3. *Relation with a formal solution of the problem of moments.*—It is of interest to verify that the solution for $f(u)$ obtained in § 2 is in agreement with a formal solution of the classical problem of moments in mathematics.[4]

If a function $f(u)$ $(0 \leqslant u < \infty)$ can be expanded in a series in Laguerre polynomials of the form

$$f(u) = e^{-u} \sum_{k=0}^{\infty} a_k L_k(u) , \tag{43}$$

then the coefficients a_k in the expansion will be given by

$$a_k = \int_0^{\infty} f(u) L_k(u)\, du . \tag{44}$$

[4] J. A. Shohat and J. D. Tamarkin, *The Problem of Moments* (New York: American Mathematical Society, 1943), esp. § 10, p. 96.

Since

$$L_k(u) = \sum_{j=0}^{k} \frac{(-1)^j}{j!} \binom{k}{j} u^j , \tag{45}$$

we have

$$\begin{aligned} a_k &= \sum_{j=0}^{k} \frac{(-1)^j}{j!} \binom{k}{j} \int_0^\infty f(u)\, u^j du \\ &= -\sum_{j=0}^{k} \frac{(-1)^j}{(j+1)!} \binom{k}{j} \int_0^\infty u^{j+1} d f(u) , \end{aligned} \tag{46}$$

or

$$a_k = \sum_{j=0}^{k} \frac{(-1)^j}{(j+1)!} \binom{k}{j} \mu_{j+1} . \tag{47}$$

The last expression for a_k is in agreement with the coefficient of L_k in equation (42). It should, however, be emphasized that the foregoing analysis, giving a solution of the Stieltjes problem of moments, is purely formal; it is known that, in general, such solutions converge only in some "mean" sense (cf. Shohat and Tamarkin, *op. cit.*).

4. *Special forms of the solution (41)*.—We shall now show how the solutions found in Paper II for two particular forms of $\psi(q)$ can be derived as special cases of the general solution obtained in § 2.

i) *The case when all the clouds are equally transparent*.—In this case

$$\varpi_i = q^i ; \tag{48}$$

and in the general solution given by equation (41) we must write

$$\prod_{i=1}^{j+1} (1-\varpi_i)^{-1} = \prod_{i=1}^{j+1} (1-q^i)^{-1} . \tag{49}$$

In virtue of the identity,

$$\prod_{i=1}^{\infty} (1-q^{i+j}) = \sum_{n=0}^{\infty} (-1)^n q^{nj} \prod_{r=1}^{n} \frac{q^r}{1-q^r} , \tag{50}$$

established in Paper II (eq. [18]), we can also write

$$\begin{aligned} \prod_{i=1}^{j+1} (1-\varpi_i)^{-1} &= \left[\prod_{i=1}^{\infty} (1-q^i) \right]^{-1} \prod_{i=1}^{\infty} (1-q^{i+j+1}) \\ &= \left[\prod_{i=1}^{\infty} (1-q^i) \right]^{-1} \sum_{n=0}^{\infty} (-1)^n q^{n(j+1)} \prod_{r=1}^{n} \frac{q^r}{1-q^r} . \end{aligned} \tag{51}$$

Now letting (cf. III, eqs. [49] and [51])

$$K = \left[\prod_{i=1}^{\infty} (1-q_i) \right]^{-1} \quad \text{and} \quad Q_n = (-1)^n \prod_{r=1}^{n} \frac{q^r}{1-q^r} , \tag{52}$$

we can express equation (51) in the form

$$\prod_{i=1}^{j+1}(1-\varpi_i)^{-1}=K\sum_{n=0}^{\infty}Q_n q^{n(j+1)}\,. \tag{53}$$

Hence, in this case, the general solution reduces to

$$\begin{aligned} f(u) &= K e^{-u}\sum_{k=0}^{\infty}L_k(u)\sum_{n=0}^{\infty}Q_n q^n\sum_{j=0}^{k}(-1)^j\binom{k}{j}q^{nj} \\ &= K e^{-u}\sum_{n=0}^{\infty}Q_n q^n\sum_{k=0}^{\infty}(1-q^n)^k L_k(u)\,. \end{aligned} \tag{54}$$

But (cf. eq. [27])

$$q^n\sum_{k=0}^{\infty}L_k(u)(1-q^n)^k=e^{-(1-q^n)u/q^n}\,. \tag{55}$$

Hence

$$f(u)=K e^{-u}\sum_{n=0}^{\infty}Q_n e^{-(1-q^n)u/q^n}\,. \tag{56}$$

This is in agreement with the solution found in Paper II (eq. [21]).

ii) *The case* $\psi(q)=(n+1)q^n$.—In this case

$$\varpi_i=\frac{n+1}{n+i+1}\,, \tag{57}$$

$$\begin{aligned} \prod_{i=1}^{j+1}(1-\varpi_i)^{-1} &= \prod_{i=1}^{j+1}\left(1-\frac{n+1}{n+i+1}\right)^{-1}=\binom{n+j+2}{n+1} \\ &= \sum_{r=0}^{n+1}\binom{r+j}{r}, \end{aligned} \tag{58}$$

and the coefficient of L_k in the solution (41) becomes

$$\sum_{r=0}^{n+1}\sum_{j=0}^{k}(-1)^j\binom{k}{j}\binom{r+j}{r}. \tag{59}$$

Now from combinatorial analysis it is known[5] that

$$\begin{aligned} \sum_{j=0}^{k}(-1)^j\binom{k}{j}\binom{r+j}{r} &= (-1)^k\binom{r}{k} \qquad \text{if} \quad k\leqslant r \\ &= 0 \qquad\qquad \text{otherwise}\,. \end{aligned} \tag{60}$$

Hence, by combining equations (41), (59), and (60) we have

$$f(u)=e^{-u}\sum_{r=0}^{n+1}\sum_{k=0}^{r}(-1)^k\binom{r}{k}L_k(u)\,. \tag{61}$$

[5] W. Feller, *An Introduction to Probability Theory and Its Applications* (New York: John Wiley & Sons, 1950), **1**, 48, problem 10.

But the expansion of u^r in Laguerre polynomials is

$$u^r = r! \sum_{k=0}^{r} (-1)^k \binom{r}{k} L_k(u) . \tag{62}$$

Hence

$$f(u) = e^{-u} \sum_{r=0}^{n+1} \frac{u^r}{r!} . \tag{63}$$

By differentiating this last expression, we obtain

$$g(u) = -\frac{df}{du} = \frac{u^{n+1}}{(n+1)!} e^{-u} . \tag{64}$$

And again this is in agreement with the solution found in Paper II (eq. [33]).