ON STELLAR STATISTICS

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ABSTRACT

In this paper a method is proposed for treating the fundamental equation of stellar statistics which will take into account the known cloud structure of the interstellar absorbing matter. The method is based on the determination and the interpretation of the fluctuations in the numbers of stars, $N(m)$, per unit solid angle, and brighter than a given apparent magnitude, m, in the various parts of the sky. The marked dependence of these fluctuations on the galactic latitude confirms their relation to the interstellar clouds. Theoretical formulae are derived for the dispersions to be expected in $N(m)$ as a function of the galactic latitude on certain idealized distributions of stars and clouds. The results of star counts tabulated by van Rhijn and by Baker and Kiefer are analyzed in terms of these formulae, and a theoretical prediction based on them is verified. The analysis discloses the importance of taking into account the dispersion in the transparencies of the interstellar clouds as a factor in these considerations.

1. *Introduction.*—The principal quantity with which stellar statistics deals is the number of stars, $A(m)$, of apparent magnitude m, per unit solid angle and per unit magnitude interval in the various parts of the sky; to a less extent it also concerns itself with the mean parallax, $\pi(m)$, of these stars. In tabulating the quantities $A(m)$ and $\pi(m)$, one either includes all stars without regard to their spectral types or includes only such stars as belong to a given spectral type (or a range of spectral types). The latter refinement of subdividing the stars into various spectral classes is customary in the more recent investigations, though the division was not made in much of the earlier work.

As is well known, the quantities $A(m)$ and $\pi(m)$ are related to the luminosity function $\phi(M)$ governing the frequency of occurrence of stars with a given absolute magnitude M, the function $D(s)$ giving the density of the stars at a distance s from the observer and in the direction considered, and the function $a(s)$ giving the absorption in magnitudes by the intervening interstellar matter, by the fundamental equations of stellar statistics, namely,

$$A(m) = \int_0^\infty D(s)\,\phi(M)\,s^2 ds \tag{1}$$

and

$$\pi(m) = \frac{1}{A(m)} \int_0^\infty D(s)\,\phi(M)\,s\,ds\,. \tag{2}$$

In equations (1) and (2) the argument M in the luminosity function, $\phi(M)$, is related to the apparent magnitude, m, by the relation

$$M = m + 5 - 5 \log s - a(s)\,, \tag{3}$$

where it is assumed that the distances are measured in parsecs.

If, in making the counts $A(m)$, all stars without regard to their spectral types are included, we must take $\phi(M)$ to mean the general luminosity function; otherwise, $\phi(M)$ should be taken to mean the luminosity function appropriate for the restricted class of stars.

In practice one often considers, instead of $A(m)$, the number of stars,

$$N(m) = \int_{-\infty}^{m} A(m')\,dm'\,, \tag{4}$$

brighter than a given apparent magnitude m. A corresponding integration of equation (3) leads to the integral equation

$$N(m) = \int_0^\infty D(s)\,\Phi(M)\,s^2 ds\,, \tag{5}$$

where

$$\Phi(M) = \int_{-\infty}^{M} \phi(M')\,dM'\,, \tag{6}$$

and the argument, M, in $\Phi(M)$ is still related to m as in equation (3).

Tables of $N(m)$ for various galactic longitudes and latitudes have been given, for example, by van Rhijn.[1]

From the observed counts $N(m)$ (and also from the mean parallaxes, if the latter data are available) stellar statistics seeks to derive information concerning $\Phi(M)$, $D(s)$, and $a(s)$. In practice, $\Phi(M)$ is assumed to be known from a study of the near-by stars, and the principal use to which equations (2) and (5) are put is to determine the density function $D(s)$ and the variation of the absorption $a(s)$ with distance. With regard to $a(s)$, in most of the early investigations the assumption is made that the interstellar matter is uniformly distributed and that there exists a coefficient of interstellar absorption, A (mg/kpc), such that

$$a(s) = A s \times 10^{-3}\,. \tag{7}$$

In the more recent investigations this assumption of the uniform distribution of the interstellar matter is not made; instead, an attempt is made to determine $a(s)$ from the color excesses of the stars included in the counts. However, any assumption which regards $a(s)$ as a "smooth" function of s in the same sense that the density of stars, $D(s)$, is a smooth function of s is contrary to the well-established fact that interstellar matter occurs in the form of clouds and that, in consequence, $a(s)$ must be considered as a random function subject to fluctuations. In other words, consistent with our present knowledge, we should rather write

$$a(s) = \sum_{i=1}^{n(s)} \epsilon_i\,, \tag{8}$$

where ϵ_i is the absorption in magnitudes by the ith cloud in the line of sight and the summation in equation (5) is extended over all the clouds which occur in the line of sight.

It is currently estimated that the average number of interstellar clouds which a line of sight will intersect is of the order of seven per kiloparsec and that, on the average, each cloud will cut down the brightness of the stars immediately behind it by about 0.2 mag. It is evident from these figures that the fluctuations in $N(m)$ arising from the cloud structure of interstellar matter must be a dominant factor in the problem. This latter fact may, in part, explain why so few results of any definiteness have been obtained by the attempts which have been made so far in inverting the integral equation (5) by numerical methods. However, to avoid any misunderstanding, we should like to state that in our view the essential failure—for so it must be admitted—of the attempts to derive the density function by a numerical inversion of the integral equation (5) cannot be traced entirely to this source; rather, we believe that the failure must be attributed to the fact that the observations which are available are far from having the accuracy necessary for the method of solution adopted to give meaningful results.[2] Thus the observations are known only in the form of histograms, and the derivatives of the observed distribution of $N(m)$, on which the solution sensitively depends, are too imperfectly known to lead to

[1] *Groningen Pub.*, No. 43, 1924.

[2] We are greatly indebted to Dr. Nancy G. Roman for valuable discussions on these and other related problems in stellar statistics.

trustworthy results. Moreover, in the method of solution which is generally adopted, the physical quantities are averaged over such large intervals of distance that we cannot, in principle, expect the method to succeed.

Since the conventional methods of treating the equations of stellar statistics have not yielded concrete results commensurate with the efforts which have been expended on them, we may ask if a reorientation and reformulation of the basic objectives of stellar statistics is not possible which may redirect the investigations in this field along more fruitful lines. It would appear from the investigations of Ambarzumian,[3] Markarian,[4] and the more recent ones of the present authors[5] that results of significance can be obtained by directing our attention to the properties of interstellar clouds—their frequency of occurrence in a line of sight, their mean absorptive power, etc.—and away from the stars.

Since the average number of clouds in the line of sight is of the order of seven per kiloparsec, the actual numbers which will occur in particular instances will be subject to the fluctuations of a Poisson distribution; the pattern of the resulting fluctuations in $N(m)$ should therefore be predictable in principle. Moreover, by considering the fluctuations in $N(m)$ as a function of the galactic latitude, β, we should have a valuable check on the assumptions that may have been made (for reasons of analytical simplicity, for example), since the average number of the clouds in the direction β may be expected to vary as cosec β.

A further consequence of the dominant effect of the fluctuations in the number of clouds in the line of sight is that assumptions concerning $\Phi(M)$ and $D(s)$ will not materially affect the conclusions we may draw about the clouds. In view of this last circumstance, we shall assume in our subsequent analysis that

$$\Phi(M) = \Phi_0 e^{aM} \tag{9}$$

and

$$D(s) = D_0 e^{-s/h}, \tag{10}$$

where Φ_0, D_0, a, and h are suitably chosen constants. In our context neither of these two assumptions is as *ad hoc* as it may seem; for it is true that $\Phi(M)$ in the range of M which contributes to $N(m)$ can be represented with sufficient accuracy by a formula of the form (9). The same remark also applies to assumption (10) if we limit ourselves to distances not exceeding 1000 parsecs in the galactic plane; and for galactic latitudes greater than 30° there are good reasons for believing that a formula of the form (10) provides a suitable approximation.

In this paper we shall derive formulae for the dispersion in $N(m)$ which may be expected on account of the cloud structure of interstellar matter and for certain model distributions of stars and clouds. The analysis in this paper is a generalization and an extension of that contained in the paper by Markarian[4] to which we have already referred. The formulae derived in this paper are applied to van Rhijn's counts,[1] and certain conclusions regarding the mean absorptive power of the interstellar clouds are drawn; these differ from the conclusions Markarian drew from his more limited analysis.

2. *The dispersion to be expected in $N(m)$ for a model distribution of stars and clouds.*—In this paper we shall restrict ourselves to the following model distribution of stars and clouds: Both stars and clouds occur to a distance L in the line of sight. The density of the stars may vary with distance along the line of sight; but the clouds are uniformly distributed and occur in the line of sight on an average of ν per unit distance. On this picture of the distribution of stars and clouds, equation (5) becomes

$$N(m) = \int_0^L D(s)\,\Phi(M)\,s^2 ds\,, \tag{11}$$

[3] *Abastumani Obs.*, No. 2, p. 37, 1938; No. 4, p. 17, 1940.

[4] *Contr. Burakan Obs. Acad. Sci. Armenian S.S.R.*, No. 1, 1946.

[5] S. Chandrasekhar and G. Münch, *Ap.J.*, **112**, 380, 393, 1950; these papers will be referred to hereafter as "Paper I" and "Paper II," respectively.

where

$$M = 5 + m - 5 \log s - \sum_{i=1}^{n(s)} \epsilon_i , \tag{12}$$

and the occurrence of a particular number $n(s)$ of clouds in the distance s is governed by a Poisson distribution with a mean νs.

In our further analysis we shall suppose that the function $\Phi(M)$ can be approximated by a formula of the form (eq. [9])

$$\Phi (M) = \Phi_0 e^{aM} . \tag{13}$$

For $\Phi(M)$ given by equation (13), equation (11) becomes

$$N (m) = \Phi_0 e^{a(5+m)} \int_0^L D (s) \, s^{2-5a \log e} \exp \left\{ - a \sum_{i=1}^{n(s)} \epsilon_i \right\} d s . \tag{14}$$

Letting

$$\alpha = 3 - 5 a \log e \qquad \text{and} \qquad Q_i = e^{-a\epsilon_i} , \tag{15}$$

we can write equation (14) in the form

$$N (m) = \Phi_0 e^{a(5+m)} \int_0^L D (s) \, s^{\alpha-1} \prod_{i=1}^{n(s)} Q_i d s . \tag{16}$$[6]

In Paper I we introduced the fraction q by which an interstellar cloud cuts down the light of the stars immediately behind it. It is readily verified that q defined in this manner is related to Q by (cf. eq. [15])

$$Q = \exp (2.5 a \log_{10} q) = \exp (1.0859 a \log_e q) = q^{1.0859a} . \tag{17}$$

By letting $r = \nu s$, we can make the Poisson distribution governing the frequency of occurrence of clouds in the interval $(0, r)$ be

$$\frac{e^{-r} r^n}{n!} . \tag{18}$$

In terms of the variable r, equation (16) can be written in the form

$$\frac{\nu^\alpha}{\Phi_0} e^{-a(5+m)} N (m) = \int_0^\xi D \left(\frac{r}{\nu} \right) r^{\alpha-1} \prod_{i=1}^{n(r)} Q_i d r , \tag{19}$$

where

$$\xi = L\nu \tag{20}$$

is the average number of clouds to be expected in the distance L.

The problem now is to evaluate the moments of the quantity defined on the right-hand side of equation (19). We shall show how these moments can be evaluated by using the result established in the following lemma.

Lemma.—Consider the random function

$$g (\xi; x) = \int_0^\xi F (r + x) \prod_{i=1}^{n(r)} Q_i d r , \tag{21}$$

[6] If we treat the mean parallaxes, then the equation for $A(m)\ \pi(m)$ will differ from eq. (16) only by the fact that the exponent of s in the integrand will be $\alpha - 2$ instead of $\alpha - 1$ (cf. eqs. [1] and [2]). Consequently, the analysis of the rest of this section will apply to $A(m)\ \pi(m)$ if α is replaced by $\alpha - 1$ wherever it occurs.

where ξ and x are two constants and the Q's occur with a probability given by a function $\Psi(Q)$ $(0 \leqslant Q \leqslant 1)$ and the $n(r)$'s are subject to fluctuations governed by the Poisson distribution $e^{-r}r^n/n!$. Under these circumstances the mth moment, $\mu_m(\xi; 0)$ of

$$g(\xi; 0) = \int_0^\xi F(r) \prod_{i=1}^{n(r)} Q_i dr \tag{22}$$

can be derived from the $(m-1)$th moment $\mu_{m-1}(\xi; x)$ of $g(\xi; x)$ by the formula

$$\mu_m(\xi; 0) = m\int_0^\xi e^{-(1-Q_m)r} F(r)\, \mu_{m-1}(\xi - r; r)\, dr, \tag{23}$$

where

$$Q_m = \int_0^1 Q^m \Psi(Q)\, dQ. \tag{24}$$

Proof.—By definition

$$\mu_m(\xi; 0) = \int_0^\xi dr_m F(r_m) \int_0^\xi dr_{m-1} F(r_{m-1}) \int_0^\xi \cdots \int_0^\xi dr_1 F(r_1) \times \left\{ \prod_{j=1}^{m} \prod_{i=1}^{n(r_j)} Q_i \right\}_{\text{average}}. \tag{25}$$

The integrand in this m-fold integral is a symmetrical function of the variables. We can accordingly use the lemma established in Paper I (eq. [47]) and rewrite the foregoing expression for $\mu_m(\xi; 0)$ in the form

$$\mu_m(\xi; 0) = m\int_0^\xi dr_m F(r_m) \int_{r_m}^\xi dr_{m-1} F(r_{m-1}) \int_{r_m}^\xi dr_{m-2} F(r_{m-2}) \int_{r_m}^\xi \cdots \times \int_{r_m}^\xi dr_1 F(r_1) \left\{ \prod_{j=1}^{m} \prod_{i=1}^{n(r_j)} Q_i \right\}_{\text{average}}. \tag{26}$$

With the integration over the variables arranged in this fashion,

$$r_j \geqslant r_m \quad \text{for} \quad j \leqslant m-1. \tag{27}$$

Under these circumstances

$$n(r_j) - n(r_m) = n(r_j - r_m), \tag{28}$$

and (cf. Paper I, eqs. [52]–[58])

$$\left\{ \prod_{j=1}^{m} \prod_{i=1}^{n(r_j)} Q_i \right\}_{\text{average}} = \left\{ \prod_{i=1}^{n(r_m)} Q_i^m \right\}_{\text{average}} \times \left\{ \prod_{j=1}^{m-1} \prod_{i=1}^{n(r_j - r_m)} Q_i \right\}_{\text{average}} = e^{-(1-Q_m)r_m} \left\{ \prod_{j=1}^{m-1} \prod_{i=1}^{n(r_j - r_m)} Q_i \right\}_{\text{average}}. \tag{29}$$

Using this last result in equation (26) and writing r_j in place of $r_j - r_m$ $(j = m-1, \ldots, 1)$ and r in place of r_m, we have

$$\mu_m(\xi; 0) = m\int_0^\xi dr F(r)\, e^{-(1-Q_m)r} \left\{ \prod_{j=1}^{m-1} \int_0^{\xi - r} dr_j F(r_j + r) \prod_{i=1}^{n(r_j)} Q_i \right\}_{\text{average}}. \tag{30}$$

But the quantity in braces on the right-hand side of equation (30) is $\mu_{m-1}(\xi - r; r)$. Hence,

$$\mu_m(\xi; 0) = m \int_0^{\xi} d\,r F(r)\, \mu_{m-1}(\xi - r; r)\, e^{-(1-Q_m) r} . \tag{31}$$

This is the required relation.

It is evident that by using the relation expressed by equation (31) we can successively evaluate all the moments of $N(m)$. However, in this paper we shall limit ourselves to evaluating only the dispersion of $N(m)$ for the case (cf. eq. [10])

$$D(s) = D_0 e^{-s/h} . \tag{32}$$

For this purpose we need the first and the second moments of

$$\mathfrak{N} = \frac{\nu^a}{\Phi_0 D_0} e^{-a(5+m)} N(m) = \int_0^{\xi} e^{-\gamma r} r^{a-1} \prod_{i=1}^{n(r)} Q_i d\,r , \tag{33}$$

where

$$\gamma = (\nu h)^{-1} . \tag{34}$$

These moments can be found by using the lemma for $m = 1$ and 2 for the case

$$F(r) = e^{-\gamma r} r^{a-1} . \tag{35}$$

i) *The first moment of* $\mathfrak{N}$.—For $m = 1$, we conclude from equation (31) that

$$\mu_1(\xi; x) = \int_0^{\xi} d\,r F(r + x)\, e^{-(1-Q_1) r} , \tag{36}$$

since $\mu_0 \equiv 1$. For $F(r)$ given by equation (35), we have

$$\mu_1(\xi; x) = \int_0^{\xi} e^{-\gamma(r+x)} (r + x)^{a-1} e^{-(1-Q_1) r} d\,r . \tag{37}$$

Writing $(r + x) = y$ in this equation, we have

$$\mu_1(\xi; x) = e^{x(1-Q_1)} \int_x^{\xi+x} e^{-(\gamma+1-Q_1) y} y^{a-1} dy , \tag{38}$$

or, alternatively,

$$\mu_1(\xi; x) = \frac{e^{x(1-Q_1)}}{(\gamma+1-Q_1)^a} \int_{x(\gamma+1-Q_1)}^{(\xi+x)(\gamma+1-Q_1)} dy\, y^{a-1} e^{-y} . \tag{39}$$

In terms of the incomplete Γ-function,

$$\Gamma(a; z) = \int_0^{z} e^{-y} y^{a-1} dy , \tag{40}$$

we can express $\mu_1(\xi; x)$ in the form

$$\mu_1(\xi; x) = \frac{e^{x(1-Q_1)}}{(\gamma+1-Q_1)^a} \{\Gamma(a; [\xi + x][\gamma + 1 - Q_1]) - \Gamma(a; x[\gamma + 1 - Q_1])\} . \tag{41}$$

In particular, for $x = 0$, we have

$$\bar{\mathfrak{N}} = \mu_1(\xi; 0) = \frac{\Gamma(a; \xi[\gamma + 1 - Q_1])}{(\gamma+1-Q_1)^a} . \tag{42}$$

When $\gamma = 0$, $\xi = \infty$, and Q occurs with only one value, equation (42) reduces to a formula given by Markarian (*op. cit.*, eq. [5]).

Letting

$$\xi_1 = \xi\,(\gamma + 1 - Q_1)\,, \tag{43}$$

we can express equation (42) more conveniently in the form

$$\mu_1\,(\xi;\,0) = \left(\frac{\xi}{\xi_1}\right)^a \Gamma\,(a;\,\xi_1)\,. \tag{44}$$

The incomplete Γ-function (40) has the expansion

$$\Gamma\,(a;\,z) = e^{-z} z^a \sum_{j=0}^{\infty} \frac{z^j}{a_j} \qquad (a > 0)\,, \tag{45}$$

where

$$a_0 = a \qquad \text{and} \qquad a_j = a\,(a+1)\ldots(a+j) \qquad (j \geqslant 1)\,. \tag{46}$$

The corresponding expansion for $\mu_1(\xi;\,0)$ is

$$\mu_1\,(\xi;\,0) = \xi^a e^{-\xi_1} \sum_{j=0}^{\infty} \frac{\xi_1^j}{a_j}\,. \tag{47}$$

ii) *The second moment of* $\mathfrak{N}$.—Substituting for $\mu_1(\xi - r;\,r)$ in accordance with equation (39) in equation (31) for $m = 2$ and $F(r)$ given by equation (35), we have

$$\tfrac{1}{2}\mu_2\,(\xi;\,0) = \int_0^{\xi} d\,r\,r^{a-1} e^{-(1-Q_2)r-\gamma r} \frac{e^{r(1-Q_1)}}{(\gamma+1-Q_1)^a} \int_{r(\gamma+1-Q_1)}^{\xi(\gamma+1-Q_1)} d y\, y^{a-1} e^{-y}\,, \tag{48}$$

or

$$\mu_2\,(\xi;\,0) = \frac{2}{(\gamma+1-Q_1)^a} \int_0^{\xi} d\,r\,r^{a-1} e^{-(\gamma+Q_1-Q_2)r} \int_{r(\gamma+1-Q_1)}^{\xi(\gamma+1-Q_1)} d y\, y^{a-1} e^{-y}\,. \tag{49}$$

An alternative form of this equation is

$$\mu_2\,(\xi;\,0) = \frac{2\,\xi^{2a}}{\xi_1^a \xi_2^a} \int_0^{\xi_2} d\,z\,z^{a-1} e^{-z} \int_{z\xi_1/\xi_2}^{\xi_1} d y\, y^{a-1} e^{-y}\,, \tag{50}$$

where, in addition to ξ_1 (eq. [43]), we have let

$$\xi_2 = \xi\,(\gamma + Q_1 - Q_2)\,. \tag{51}$$

Expressing the integral over y in equation (50) in terms of the incomplete Γ-function (40), we obtain

$$\mu_2\,(\xi;\,0) = \frac{2\,\xi^{2a}}{\xi_1^a \xi_2^a} \left\{\Gamma\,(a;\,\xi_1)\,\Gamma\,(a;\,\xi_2) - \int_0^{\xi_2} d\,z\,z^{a-1} e^{-z}\,\Gamma\left(a;\,\frac{z\xi_1}{\xi_2}\right)\right\}. \tag{52}$$

Expanding $\Gamma(a;\,z\xi_1/\xi_2)$ according to equation (45), we can reduce the second term on the right-hand side of equation (52) as follows:

$$\frac{2\,\xi^{2a}}{\xi_1^a \xi_2^a}\int_0^{\xi_2} d\,z\,z^{a-1}e^{-z(1+\xi_1/\xi_2)}\left(\frac{z\,\xi_1}{\xi_2}\right)^a \sum_{j=0}^{\infty}\left(\frac{z\,\xi_1}{\xi_2}\right)^j \frac{1}{a_j}$$

$$\begin{aligned} &= 2\,\xi^{2a}\sum_{j=0}^{\infty}\frac{\xi_1^j}{a_j}\int_0^1 e^{-z(\xi_1+\xi_2)}\,z^{2a+j-1}d\,z \\ &= 2\,\xi^{2a}\sum_{j=0}^{\infty}\frac{\xi_1^j}{a_j\,(\xi_1+\xi_2)^{2a+j}}\,\Gamma\,(2\,a+j;\;\xi_1+\xi_2) \\ &= 2\,\xi^{2a}e^{-(\xi_1+\xi_2)}\sum_{j=0}^{\infty}\frac{\xi_1^j}{a_j}\left\{\sum_{k=0}^{\infty}\frac{(\xi_1+\xi_2)^k}{(2\,a+j)_k}\right\}. \end{aligned} \tag{53}$$

Combining this with the corresponding expansions for $\Gamma(a;\,\xi_1)$ and $\Gamma(a;\,\xi_2)$, we obtain

$$\mu_2\,(\xi;\,0)\;=2\,\xi^{2a}e^{-(\xi_1+\xi_2)}\left\{\left(\sum_{j=0}^{\infty}\frac{\xi_1^j}{a_j}\right)\left(\sum_{j=0}^{\infty}\frac{\xi_2^j}{a_j}\right)-\sum_{j=0}^{\infty}\frac{\xi_1^j}{a_j}\left[\sum_{k=0}^{\infty}\frac{(\xi_1+\xi_2)^k}{(2\,a+j)_k}\right]\right\}. \tag{54}$$

For $\xi=\infty$, $\mu_2(\infty\,;\,0)$ can be reduced to an incomplete beta-function: When $\xi\to\infty$, both ξ_1 and $\xi_2\to\infty$, but ξ_1/ξ_2 tends to a finite limit, namely, $(\gamma+1-Q_1)/(\gamma+Q_1-Q_2)$. The corresponding expression for μ_2 is (cf. eq. [50])

$$\mu_2\,(\,\infty\,;\,0)\;=\frac{2}{(\gamma+1-Q_1)^a\,(\gamma+Q_1-Q_2)^a}\int_0^{\infty}d\,z\,z^{a-1}e^{-z}\int_{zt}^{\infty}d y\,y^{a-1}e^{-y}, \tag{55}$$

where we have written

$$t=\frac{\gamma+1-Q_1}{\gamma+Q_1-Q_2}. \tag{56}$$

Writing

$$J\,(t)\;=\int_0^{\infty}d\,z\,z^{a-1}e^{-z}\int_{zt}^{\infty}d y\,y^{a-1}e^{-y} \tag{57}$$

and differentiating with respect to t, we have

$$\frac{dJ}{dt}=-\,t^{a-1}\int_0^{\infty}d\,z\,z^{2a-1}e^{-z(1+t)}, \tag{58}$$

or

$$\frac{dJ}{dt}=-\frac{t^{a-1}}{(1+t)^{2a}}\,\Gamma\,(2\,a)\,. \tag{59}$$

Integrating this last equation over t and noting that $J\to 0$ as $t\to\infty$, we have

$$J\,(t)\;=\Gamma\,(2\,a)\int_t^{\infty}\frac{x^{a-1}}{(1+x)^{2a}}\,d\,x\,. \tag{60}$$

With the substitution $y=x/(1+x)$, $J(t)$ becomes

$$J\,(t)\;=\Gamma\,(2\,a)\int_{t/(1+t)}^{1}y^{a-1}\,(1-y)^{a-1}d y\,. \tag{61}$$

Alternatively, we can also write

$$J(t) = \Gamma(2a) \int_0^{1/(1+t)} x^{a-1} (1-x)^{a-1} dx . \tag{62}$$

Hence in this case the formula for the second moment takes the form

$$\mu_2(\infty;0) = \frac{2\Gamma(2a)}{(\gamma+1-Q_1)^a (\gamma+Q_1-Q_2)^a} \int_0^{(\gamma+Q_1-Q_2)/(1+2\gamma-Q_2)} x^{a-1} (1-x)^{a-1} dx . \tag{63}$$

When $\gamma = 0$ and Q occurs with only one value, equation (63) reduces to a formula given by Markarian (*op. cit.*, eq. [21]).

iii) *The dispersion in* $N(m)$.—Since $N(m)$ and $\mathfrak{N}$ differ only by a constant factor, the dispersion, δ^2, in $N(m)$ can be expressed in terms of the first two moments of $\mathfrak{N}$, which we have evaluated. Thus

$$\delta^2(\xi) = \frac{\mu_2(\xi;0)}{\mu_1^2(\xi;0)} - 1 . \tag{64}$$

Using expressions (47) and (54) for $\mu_1(\xi; 0)$ and $\mu_2(\xi; 0)$, we have

$$\delta^2(\xi) = 2e^{\xi_1-\xi_2} \left\{ \frac{\sum_{j=0}^{\infty} \xi_2^j / a_j}{\sum_{j=0}^{\infty} \xi_1^j / a_j} - \frac{\sum_{j=0}^{\infty} (\xi_1^j / a_j) \left[\sum_{k=0}^{\infty} (\xi_1+\xi_2)^k / (2a+j)_k \right]}{\left(\sum_{j=0}^{\infty} \xi_1^j / a_j \right)^2} \right\} - 1 . \tag{65}$$

It is seen that the dispersion depends on γ, Q_1, and Q_2 only through the combinations ξ_1 and ξ_2.

For $\xi = \infty$ the expression for the dispersion takes the form (cf. eqs. [42] and [63])

$$\delta^2(\infty) = \frac{2\Gamma(2a)}{\Gamma^2(a)} \left(\frac{\gamma+1-Q_1}{\gamma+Q_1-Q_2} \right)^a \int_0^{(\gamma+Q_1-Q_2)/(1+2\gamma-Q_2)} x^{a-1} (1-x)^{a-1} dx - 1 . \tag{66}$$

In this case δ^2 depends on γ, Q_1, and Q_2 only through the single combination

$$\chi = \frac{\gamma+Q_1-Q_2}{1+2\gamma-Q_2} = \frac{\xi_2}{\xi_1+\xi_2}, \tag{67}$$

for we can write

$$\delta^2(\infty) = \frac{2\Gamma(2a)}{\Gamma^2(a)} \left(\frac{1-\chi}{\chi} \right)^a \int_0^{\chi} x^{a-1} (1-x)^{a-1} dx - 1 . \tag{68}$$

It is found that, for ξ_1 and ξ_2 each less than 2, equation (65) with not too many terms suffices to determine δ^2. In Table 1 we have listed the dispersions computed with the help of this formula for certain ranges of a, ξ_1, and ξ_2 which are required in the applications of the theory to be discussed in § 3. An examination of Table 1 shows that for $\xi_1 < 1$, the dependence of $\delta^2(a, \xi_1; \xi_1 - \xi_2)$ on ξ_1 is so slight that it effectively depends only on a and $\xi_1 - \xi_2$. Also the dependence on a is not very pronounced (see Fig. 1). These facts have an important bearing when we come to discuss the observations in § 3.

For $\xi = \infty$, the dispersion is given by equation (68) and depends only on χ. The dispersions evaluated with the aid of this formula for a range of values of χ and a are given in Table 2.

3. *An analysis of star counts based on the formulae of* § *2*.—The paper by Markarian[4] to which we have already referred contains an analysis of the results of star counts tabu-

TABLE 1

VALUES OF $\delta^2(\alpha, \xi_1; \xi_1 - \xi_2)$

$\alpha=1.65$				$\alpha=1.50$			$\alpha=1.20$			
$\xi_1-\xi_2$	$\xi_1=0.3$	$\xi_1=0.4$	$\xi_1=0.5$	$\xi_1-\xi_2$	$\xi_1=0.4$	$\xi_1=0.8$	$\xi_1-\xi_2$	$\xi_1=0.5$	$\xi_1=1.0$	$\xi_1=2.0$
0.00	0	0	0	0.00	0	0	0.0	0	0	0
.02	0.0091	0.0089	0.0086	.05	0.0213	0.0197	.1	0.0354	0.0314	0.0246
.04	.0184	.0179	.0174	.10	.0431	.0399	.2	.0726	.0644	.0502
.06	.0278	.0271	.0264	.15	.0655	.0606	.3	.1117	.0990	.0769
.08	.0373	.0365	.0357	.20	.0885	.0819	.4	.1529	.1352	.1048
0.10	0.0470	0.0461	0.0453	0.25	0.1123	0.1039	0.5	0.1962	0.1730	0.1339

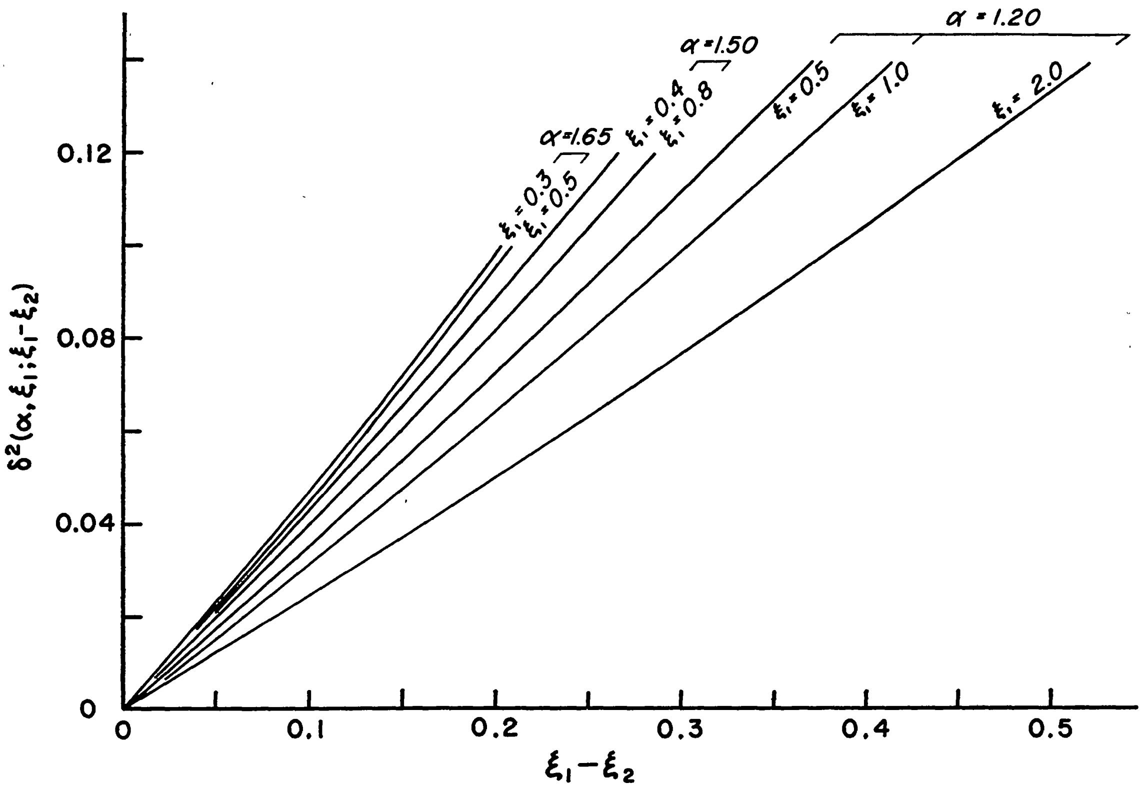

FIG. 1.—Illustrating the dependence of the dispersion $\delta^2(\alpha, \xi_1; \xi_1 - \xi_2)$ on its arguments

TABLE 2

THE DISPERSION $\delta^2(\alpha, \chi)$ FOR $\xi = \infty$

α	χ					
	0.50	0.48	0.46	0.44	0.42	0.40
1.0.	0	0.0400	0.0800	0.1200	0.1600	0.2000
1.1.	0	.0457	.0918	.1381	.1846	.2314
1.2.	0	.0516	.1039	.1568	.2103	.2644
1.3.	0	.0577	.1164	.1762	.2370	.2989
1.4.	0	.0639	.1293	.1963	.2648	.3349
1.5.	0	0.0702	0.1425	0.2170	0.2936	0.3724

lated by van Rhijn and by Baker and Kiefer.[7] Considering various regions with nearly the same galactic latitude but distributed over a range of galactic longitudes, Markarian determined the dispersions,

$$\delta^2(m) = \frac{\overline{N^2(m)}}{\overline{N(m)}^2} - 1, \tag{69}$$

of the tabulated counts. In view of the general inaccessibility of Markarian's paper, we have summarized the results of his analysis in Table 3.

TABLE 3

THE RESULTS OF MARKARIAN'S ANALYSIS

A. VAN RHIJN'S COUNTS

Group	β	λ	$\delta^2(m)$						$\delta^2_{\text{average}}$
			13	14	15	16	17	18	
I.	0	100°	0.053	0.069	0.096	0.090	0.087	0.115	0.085±0.020
II.	±10°	100°	.041	.046	.068	.092	.098	.078	.071± .021
III.	±30°	100°	.030	.023	.034	.041	.042	.023	.032± .009
IV.	±40°	100°	0.034	0.020	0.028	0.019	0.012	0.019	0.022±0.007

B. BAKER AND KIEFER'S COUNTS

Group	β	λ	$\delta^2(m)$						$\delta^2_{\text{average}}$
			10	11	12	13	14	15	
V.	−10°	160°	0.084	0.107	0.124	0.117	0.106	0.079	0.103±0.016
VI.	±10°	130°	.045	.070	.081	.100	.128	.133	.093± .031
VII.	+ 0°	130°	0.076	0.133	0.144	0.151	0.135	0.149	0.131±0.026

Now if assumption (13) concerning the luminosity function is strictly true, then $\delta^2(m)$ should be independent of m. Actually, the dispersions as determined by Markarian show an appreciable scatter with m. But the averages for the different groups (given in the last column of Table 3) show such a marked dependence on β that we may suppose that the formulae of § 2 can be applied to these averages. This supposition also underlies Markarian's discussion, though, since he had developed the theory only for the case $\xi = \infty$, he was unable to use the dispersions except for the regions with very low galactic latitudes.[8] However, with the more general formulae we have derived, we can discuss the dispersions at all galactic latitudes; we shall see that this enables a check on the basic assumptions underlying the present method of analysis. Also we shall find that Markarian's conclusions based on the dispersions for $\beta = 0$ require revision.

Before we proceed to apply the formulae of § 2 to the dispersions listed in Table 3, we

[7] *Ap. J.*, **94**, 482, 1941.

[8] Actually, Markarian applied his formulae also for the groups at $\beta = \pm 10°$. But, as we shall see, this application of the formulae valid for $\xi = \infty$ to latitudes even as low as 10° is an invalid procedure.

shall relate the means Q_1 and Q_2 (eq. [24]), which occur in these formulae, with the corresponding means q_1 and q_2 introduced in Paper I. Since (cf. eq. [17])

$$Q = q^A, \qquad A = 1.0859a,$$

we have

$$Q_k = \int_0^1 \psi(q)\, q^{kA} dq, \tag{70}$$

where $\psi(q)$ governs the frequency of occurrence of interstellar clouds with a transparency factor q.

In practice we shall be interested only in values of $q_1 (= \bar{q})$ close to unity. In this case it readily follows from equations (70) that

$$Q_k = 1 - kA(1 - q_1) + \tfrac{1}{2}kA(kA - 1)(1 - 2q_1 + q_2) + O[\overline{(1 - q)^3}], \tag{71}$$

where

$$q_j = \int_0^1 \psi(q)\, q^j dq. \tag{72}$$

Hence, to order $\overline{(1 - q)^3}$,

$$1 - Q_1 = A(1 - q_1) - \tfrac{1}{2}A(A - 1)(1 - 2q_1 + q_2),$$

and

$$1 - Q_2 = 2A(1 - q_1) - A(2A - 1)(1 - 2q_1 + q_2). \tag{73}$$

These equations provide entirely sufficient accuracy for our purposes; they also simplify the further discussion very considerably.

Now if ξ_0 is the average number of clouds to be expected in a direction perpendicular to the galactic plane, then the number of clouds, ξ, to be expected in a direction β is

$$\xi = \xi_0 \operatorname{cosec} \beta. \tag{74}$$

Further, if Δ is the (mean) absorption in magnitudes perpendicular to the galactic plane, then (cf. Paper I, eqs. [67]–[70])

$$\xi_0 = -\frac{\Delta}{2.5 \log_{10} q_1} = 0.9210 \frac{\Delta}{1 - q_1} + O(1). \tag{75}$$

It has been estimated by Hubble[9] from his counts of extragalactic nebulae that

$$\Delta = 0^{\mathrm{m}}.25; \tag{76}$$

however, for the present we shall leave Δ unspecified.

With ξ given by equation (74), the (exact) expressions for ξ_1 and ξ_2 are (eqs. [43] and [51])

$$\xi_1 = \xi_0(\gamma + 1 - Q_1) \operatorname{cosec} \beta$$

and

$$\xi_2 = \xi_0(\gamma + Q_1 - Q_2) \operatorname{cosec} \beta. \tag{77}$$

With $(1 - Q_1)$ given by equation (73), we have

$$\xi_1 = \xi_0 \{\gamma + A(1 - q_1) + O[\overline{(1 - q)^2}]\} \operatorname{cosec} \beta. \tag{78}$$

Writing

$$\gamma = A(1 - q_1)\, c \tag{79}$$

and substituting for ξ_0 from equation (73), we obtain

$$\xi_1 = 0.9210(1 + c)\Delta A \operatorname{cosec} \beta. \tag{80}$$

[9] *Ap. J.*, **79**, 8, 1934.

From this expression for ξ_1 it is evident that the principal uncertainty in estimating ξ_1 will arise from a lack of knowledge of c, i.e., of γ. Now we may safely exclude density gradients, which will lead us to expect appreciable differences in density over distances much less than a kiloparsec. According to equation (34), this excludes values of γ greater than 0.15 if we accept the currently estimated number of seven clouds per kiloparsec. We shall presently see that the values of A which we shall encounter are of order unity; and, since we expect values of $(1 - q_1)$ in the neighborhood of 0.15, we conclude that the uncertainty in our knowledge of the density gradients implies an uncertainty in ξ_1 of a factor of the order of 2. This must be kept in mind when we come to interpret the observed dispersions.

i) *The discussion of the groups centered at galactic latitudes $\beta = \pm 10°$, $\pm 30°$, and $\pm 40°$.*—In § 2 we saw that, for $\xi_1 < 1$, the principal argument on which the dispersion depends is

$$\xi_1 - \xi_2 = \xi_0 (1 - 2Q_1 + Q_2) \operatorname{cosec} \beta . \tag{81}$$

In the framework of the approximations represented by equations (73), this argument is

$$\xi_1 - \xi_2 = A^2 \xi_0 (1 - 2q_1 + q_2) \operatorname{cosec} \beta . \tag{82}$$

Consequently, if $(\xi_1 - \xi_2)/A^2$ *can be determined from the observed dispersions at various galactic latitudes, we shall find that it varies linearly with cosec* β. This is a definite prediction of the theory and is capable of verification. We shall analyze the dispersions for groups II–VI given in Table 3 with this in view.

The reduction of the data is exhibited in Table 4. The mean dispersions $(\delta^2)_{\text{av}}$ given in the third column are Markarian's values taken from Table 3. The constant a in the empirical representation of the luminosity function (eq. [13]) was determined for the regions included in each of the groups by the formula

$$a = 2.303 [\log \overline{N(m+1)} - \log \overline{N(m)}]_{\text{average}} . \tag{83}$$

The values of a determined in this fashion are given in the fourth column; estimates of the uncertainties in these constants were also made and are indicated. The constants A and α now follow from equations (70) and (15) and are given in the fifth and sixth columns. The values of ξ_1 given by equation (80) for $\Delta = 0^{\text{m}}.25$ and $c = 0$, and the values ξ_1 (in view of the uncertainty in γ which we have described earlier) for which $\xi_1 - \xi_2$ was determined according to equation (65) are given in the seventh and eighth columns. From the values of α and ξ_1 given in the sixth and eighth columns we observe that we are in the range of the parameters included in Table 1; the values of $\xi_1 - \xi_2$ determined with the aid of this table are given in the ninth column; the uncertainties in these entries are largely due to the scatter in $\delta^2(m)$. It will be noticed that the derived values of $\xi_1 - \xi_2$ are not sensitive to the assumed values of ξ_1; this is particularly true of the high galactic-latitude groups III and IV. Finally, the last column gives the values of $(\xi_1 - \xi_2)/A^2$.

In Figure 2 we have plotted the values of $(\xi_1 - \xi_2)/A^2$ given in Table 4 against cosec β. It is seen that the predicted linear relation between $(\xi_1 - \xi_2)/A^2$ and cosec β is confirmed very satisfactorily. If, for the sake of definiteness, we adopt for $|\beta| = 10°$ the mean of the $(\xi_1 - \xi_2)/A^2$ values determined for groups II, V, and VII for $\xi_1 = 1.0$, we find that the straight line,

$$\frac{\xi_1 - \xi_2}{A^2} = 0.06 \operatorname{cosec} \beta , \tag{84}$$

fits the observations as well as may be expected; this line is also shown in Figure 2.

Finally, combining equation (82) with the empirically determined relation (84), we have

$$\xi_0 (1 - 2q_1 + q_2) = 0.06 . \tag{85}$$

TABLE 4

THE REDUCTION OF THE DATA ON THE DISPERSIONS ($\beta = \pm 10°, \pm 30°, \pm 40°$)

GROUP	COSEC β	$(\delta^2)_{av}$	a	A	α	$0.2303\ A \times$ COSEC β	ξ_1	$\xi_1 - \xi_2$	$(\xi_1 - \xi_2)/A^2$
	A. Van Rhijn's Regions								
IV...	1.56	0.022	0.62±0.09	0.67±0.09	1.65	0.24	{0.2 0.4	0.046 .048}	0.102±0.050
III...	2.00	.032	.70± .04	.76± .04	1.48	0.35	{0.4 0.8	.074 .080}	.132± .045
II....	5.76	0.071	0.82±0.05	0.89±0.06	1.22	1.18	{1.0 2.0	.22 0.28	.29 ± .10 0.35 ±0.10
	B. Baker and Kiefer's Regions								
V....	5.76	0.103	0.83±0.04	0.90±0.04	1.2	1.19	{1.0 2.0	0.31 .39	0.39 ±0.09 .48 ± .09
VI...	5.76	0.093	0.83±0.12	0.90±0.13	1.2	1.19	{1.0 2.0	.28 0.36	.35 ± .13 0.44 ±0.13

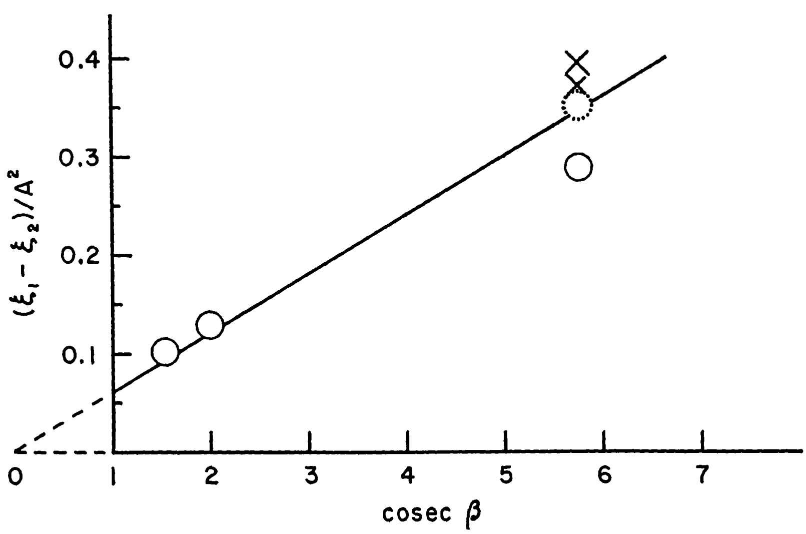

FIG. 2.—Illustrating the linear relation which exists between the observationally deduced values of $(\xi_1 - \xi_2)/A^2$ and cosec β. The circles and the crosses represent, respectively, the points deduced from van Rhijn's counts (groups II, III, and IV) and Baker and Kiefer's counts (groups V and VI). The straight line (eq. [85]) has been drawn to represent these points. The dotted circle refers to group II ($\beta = \pm 10°$) if a density gradient corresponding to $c = 1$ (eq. [80]) is assumed to exist.

ii) *The discussion of the groups on the galactic plane.*—Turning, now, to the dispersions for groups I and VII, which include only regions within 5° of the galactic plane, we first observe that, according to equation (68), the dispersion is expected to depend only on the single variable,

$$\chi=\frac{\gamma+Q_1-Q_2}{2\gamma+1-Q_2}. \tag{86}$$

With Q_1 and Q_2 given by the approximate formulae (73), χ becomes

$$\chi=\frac{1}{2}\left\{1-\tfrac{1}{2}A^2\,\frac{1-2q_1+q_2}{\gamma+A\,(1-q_1)}\right\}. \tag{87}$$

Writing γ as in equation (79), we can rewrite equation (87) in the form

$$\tfrac{1}{2}-\chi=\tfrac{1}{4}A\,\frac{1-2q_1+q_2}{(1+c)\,(1-q_1)}. \tag{88}$$

Multiplying the numerator and the denominator of the quantity on the right-hand side of equation (88) by ξ_0 and substituting for $\xi_0(1-q_1)$ according to equation (75), we have

$$\tfrac{1}{2}-\chi\simeq 0.271\,\frac{A}{\Delta\,(1+c)}\,\xi_0\,(1-2q_1+q_2). \tag{89}$$

As we have already seen, $\xi_0(1-2q_1+q_2)$ can be determined from the observed dispersions at the higher galactic latitudes. For $\Delta=0.25$ and $\xi_0(1-2q_1+q_2)$ given by equation (85), relation (89) reduces to

$$\tfrac{1}{2}-\chi=0.065\,\frac{A}{1+c}. \tag{90}$$

Accordingly, by combining the results of the analysis of the regions very close to the galactic plane ($|\beta|<5°$) with the results for the higher galactic latitudes, we can determine whether there is any effect arising from a density gradient in the distribution of the stars. The observed dispersions for groups I and VII were analyzed with this in view.

The constants A and α appropriate for groups I and VII were determined as for the other groups and are listed in Table 5. With these constants and with the aid of Table 2,

TABLE 5

THE REDUCTION OF THE DATA ON THE DISPERSIONS

($\beta=0°$)

Group	$(\delta^2)_{av}$	a	A	α	χ	$\frac{1}{2}-\chi$	$0.065\,A$	c
I.	0.085	0.79±0.05	0.86±0.06	1.28	0.472	0.028	0.056	1.0
VII. . . .	0.131	0.86±0.12	0.93±0.14	1.14	0.448	0.052	0.060	0.2

the corresponding values of χ were determined. The values of $\frac{1}{2}-\chi$ are then compared with $0.065A$ (cf. eq. [90]). From this comparison it appears that the observed dispersion for the regions included in group I indicates a density gradient corresponding to $c=1$; but the observed dispersion for group VII can be accounted for without assuming any appreciable density gradient. It is of interest to notice in this connection that a value of $c=1$ would make the deduced value of $(\xi_1-\xi_2)/A^2$ for group II (cf. Table 4 and Fig. 2) agree better with the values for groups V and VI and with the straight line (84). However, it should be stated that the uncertainties in δ^2, A, and α are such that too much reliance cannot be placed on this conclusion; for the observations can be combined in a manner in which the difference in the values of c for groups I and VII can be made to appear illusory. In any event, we are here concerned only with illustrating the use of the formulae we have derived.

iii) *The mean transparency of the interstellar clouds.*—The foregoing discussion of the observed dispersions in $N(m)$ leads to the one definite result expressed by equation (85). Using equation (75) (with $\Delta = 0.25$), we can write it alternatively in the form

$$1 - 2q_1 + q_2 = \frac{0.06}{0.2303}(1 - q_1) = 0.260(1 - q_1), \tag{91}$$

or

$$q_2 = 1.740 q_1 - 0.740 . \tag{92}$$

If we adopt the value $q_1 = 0.85$, then from equation (92) we deduce that

$$q_2 = 0.739 . \tag{93}$$

This agrees with the value (0.733) determined in Paper I (eq. [74]) from the fluctuations in the total brightness of the Milky Way and confirms the importance of including the dispersion in the transparencies of the interstellar clouds in these discussions.