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CONSERVATION LAWS IN GENERAL RELATIVITY AND IN THE POST-NEWTONIAN APPROXIMATIONS

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ABSTRACT

It is shown how the exact conservation laws of general relativity, expressed in terms of the symmetric energy-momentum complex of Landau and Lifshitz, can be used to determine the various conserved quantities in the different post-Newtonian approximations. Particular attention is given to the conserved energy which emerges as the integral over the whole of space of the difference between the (0,0)-component of the Landau-Lifshitz complex and the energy of the conserved mass present. The method is illustrated in the framework of the first post-Newtonian equations of hydrodynamics.

I. INTRODUCTION

In an earlier paper (Chandrasekhar 1965; this paper will be referred to hereafter as Paper I) the equations of hydrodynamics governing a perfect fluid in the first post-Newtonian approximation to general relativity (i.e., in an approximation in which all terms of order c^{-2} are retained, consistently, with Einstein's field equations) were derived. These equations are no more than

$$T^{ij}{}_{;j} = 0 \qquad (1)^1$$

written out explicitly, correctly to order c^{-2}, with the aid of the metric coefficients and the Christoffel symbols derived earlier to the required orders. By a process essentially of inspection, it was shown that these equations allow the same number of integrals as the equations in the Newtonian limit; and, further, that they can be interpreted in the same way. However, when the equations of motion were derived (in Chandrasekhar and Nutku 1969) in the second post-Newtonian approximation (in which all terms inclusive of order c^{-4} are retained), the matter of finding by inspection whether equation (1), in this higher approximation, allowed similar integrals did not seem feasible. In particular, the analogue of the Newtonian energy integral (see eq. [14] below) was obtained in Paper I by contracting the α-component of equation (1) with v_α $(=dx_\alpha/dt)$. A similar manipulation of the equations of the second post-Newtonian approximation did not seem practical. For these reasons, it appeared that the conservation laws in the exact framework of general relativity and in particular their roles in the various post-Newtonian approximations required some clarification. This paper attempts such a clarification.

II. CONSERVATION LAWS IN THE NEWTONIAN HYDRODYNAMICS OF A PERFECT FLUID

As a preliminary to the discussion in the following sections, we shall first formulate the integrals, which the equations of hydrodynamics in the Newtonian limit admit, in a

[1] In this paper (as in Paper I) the convention will be adopted of letting Latin indices take the values 0, 1, 2, and 3 and the Greek indices take only the values 1, 2, and 3 referring to the spatial coordinates; and the summation convention over repeated indices will be restricted to their respective ranges. Also, the notations of semicolon $(;j)$ and comma $(,j)$ will be used to indicate covariant and (simple) partial differentiation (with respect to x_j), respectively. And finally, x_0 will be replaced by ct when the notation of ordinary Cartesian tensors is used; and when the notation of Cartesian tensors is used, the Greek indices will *always* be written as subscripts; and the summation convention over the repeated Greek indices will be assumed.

manner that their relations to the integrals in the various post-Newtonian approximations are manifest.

With the definition of the energy-momentum tensor,

$$T^{00} = \rho c^2 , \quad T^{0\alpha} = \rho c v_\alpha , \quad \text{and} \quad T^{\alpha\beta} = \rho v_\alpha v_\beta + p\delta_{\alpha\beta} , \tag{2}$$

where ρ denotes the density, p the pressure, and v_α the *Cartesian* components of the velocity, we can write the Eulerian equations of hydrodynamics in the form

$$T^{ij}{}_{,j} = 0 . \tag{3}$$

The existence of the conserved quantities

$$P^i = \int_V T^{0i} d\boldsymbol{x} = \text{constant} , \tag{4}$$

where the integration is extended over the volume V occupied by the fluid, follows directly from the *form* of equation (3) and the condition that the pressure vanishes on the boundary of V.

From the symmetry of the energy-momentum tensor, the constancy of the total angular momentum

$$L_\gamma = \epsilon_{\alpha\beta\gamma} \int_V \rho x_\alpha v_\beta d\boldsymbol{x} \tag{5}$$

follows from the space components of equation (3).

In addition to the seven integrals included in equations (4) and (5), the equations of motion allow an *eighth energy integral* if we make the additional assumption that every fluid element, during its motion, preserves its entropy. This assumption, by the first law of thermodynamics, requires that the change in the thermodynamic internal energy Π (per unit mass) which an element of the fluid experiences during its motion must be traceable directly to the work done by the pressure in expanding (or contracting) its volume; thus

$$\frac{d\Pi}{dt} = -p \frac{d}{dt}\left(\frac{1}{\rho}\right) . \tag{6}$$

On this assumption, we have the integral

$$\mathfrak{E} = \int_V \rho(\tfrac{1}{2}v^2 + \Pi) d\boldsymbol{x} = \text{constant} . \tag{7}$$

The foregoing equations apply when no external forces are acting on the fluid; indeed, no allowance has been made even for the forces resulting from its own gravitation. If we should now allow for these self-gravitational forces, then the space components of equation (3) must be modified to read

$$T^{\alpha j}{}_{,j} - \rho \frac{\partial U}{\partial x_\alpha} = 0 , \tag{8}$$

where U denotes the gravitational potential resulting from the prevailing distribution of ρ:

$$\nabla^2 U = -4\pi G \rho . \tag{9}$$

Equation (8) is *not* of the form (3); but it can be restored to that form by making use of the identity

$$-\rho \frac{\partial U}{\partial x_\alpha} = \frac{1}{16\pi G} \frac{\partial}{\partial x_\beta} \left[4 \frac{\partial U}{\partial x_\alpha} \frac{\partial U}{\partial x_\beta} - 2\delta_{\alpha\beta} \left(\frac{\partial U}{\partial x_\mu}\right)^2 \right] , \tag{10}$$

which obtains by virtue of the "field equation" (9). Thus, by defining the symmetric tensor t^{ij} by

$$t^{00}=0\,,\quad t^{0\alpha}=0\,,\quad \text{and}\quad t^{\alpha\beta}=\frac{1}{16\pi G}\left[4\frac{\partial U}{\partial x_\alpha}\frac{\partial U}{\partial x_\beta}-2\delta_{\alpha\beta}\left(\frac{\partial U}{\partial x_\mu}\right)^2\right], \tag{11}$$

and letting

$$\Theta^{ij}=T^{ij}+t^{ij}\,, \tag{12}$$

we can rewrite the equations governing the fluid, once again, in the form

$$\Theta^{ij}{}_{,j}=0\,. \tag{13}$$

Therefore, we may call t^{ij} the contribution of the gravitational field to the total *energy-momentum complex* Θ^{ij}.

With the equations of motion reduced to the form (13), it is manifest that the fluid subject to its own gravitation allows the *same* linear and angular momentum integrals. Again, if we should supplement equation (13) by the condition (6) that ensures that each fluid element preserves its entropy during its motion, then we should obtain the additional energy integral

$$\mathfrak{E}=\int_V \rho(\tfrac{1}{2}v^2+\Pi-\tfrac{1}{2}U)dx=\text{constant}\,. \tag{14}$$

It is important to draw attention to one fact which distinguishes the Newtonian theory: the integrals expressing the eight conserved quantities are all extended only over the volume V occupied by the fluid in which neither ρ nor p vanishes. We shall find that, except for the integral expressing the conserved baryon number, such a reduction is not, in general, possible in the exact framework of general relativity.

III. CONSERVATION LAWS IN GENERAL RELATIVITY

In general relativity, in contrast to the Newtonian theory, the physical character of a system is completely specified by the choice of the expression for the energy-momentum tensor. Thus, for a perfect fluid the expression for the energy-momentum tensor one generally assumes is

$$T^{ij}=\rho(c^2+\Pi+p/\rho)u^iu^j-pg^{ij}\,, \tag{15}$$

where u^i $(=dx_i/ds)$ is the four velocity, g^{ij} is the contravariant form of the metric tensor, and the remaining symbols have the same meanings as hitherto. While the expression (15) for T^{ij} is a natural "covariant" generalization of the one that one assumes in special relativity, the fact should not be overlooked that it involves the unspecified metric.

When the expression (15) for T^{ij} is inserted in the field equation

$$G^{ij}=R^{ij}-\tfrac{1}{2}Rg^{ij}=-\frac{8\pi G}{c^4}T^{ij}\,, \tag{16}$$

we no longer have the choices we had in Newtonian theory. Thus, the equation governing the fluid, namely,

$$T^{ij}{}_{;j}=0\,, \tag{17}$$

necessarily includes the effect of the gravitational field on the fluid motions: the terms in the covariant divergence of T^{ij}, besides those included in the ordinary divergence of T^{ij}, are to be attributed, even as in equation (8), to the effect of the gravitational field; indeed, in the Newtonian limit, the additional term is precisely that included in equation

(8), namely, $-\rho\partial U/\partial x_\alpha$. Similarly, the fact that the expression for T^{ij} includes no dissipative mechanism means that the motions must *necessarily* be consistent with requirements for isentropic flow: it should not be necessary to supplement equation (17) by any additional statement to ensure that each fluid element, during its motion, preserves its entropy. Thus, writing out equation (17) explicitly, we have

$$[\rho(c^2 + \Pi + p/\rho)u^j]_{;j}u^i + \rho(c^2 + \Pi + p/\rho)u^j u^i{}_{;j} - g^{ij}p_{,j} = 0\,; \tag{18}$$

and contracting this equation with u_i, we obtain

$$[\rho(c^2 + \Pi + p/\rho)u^j]_{;j} - u^j p_{,j} = 0\,. \tag{19}$$

Expanding this last equation and simplifying, we have

$$(\rho u^j)_{;j}(c^2 + \Pi + p/\rho) + \rho u^j\left[\Pi_{,j} + p\left(\frac{1}{\rho}\right)_{,j}\right] = 0\,. \tag{20}$$

From equation (20) it follows that *the conservation of mass required by the equation*

$$(\rho u^j)_{;j} = 0 \tag{21}$$

is compatible with the equations of motion if, and only if,

$$u^j\left[\Pi_{,j} + p\left(\frac{1}{\rho}\right)_{,j}\right] = 0\,; \tag{22}$$

and this last equation is no more than the requirement that the motion be isentropic (cf. eq. [6]). In other words, *the conservation of mass and the conservation of entropy are not independent requirements in the framework of general relativity*. And the reason for their *independence* in the Newtonian limit is that in this limit ("$c^2 \to \infty$") equation (20) reduces simply to the equation of continuity.

The question now arises as to the extent to which the conservation of mass required by equation (21) should be considered as a statement of a fundamental physical law that should supplement the field equations. From the standpoint of physics, a conservation law to which one might accede as "fundamental" is the conservation of the baryon number. In that case, one should rather write for the energy-momentum tensor the expression

$$T^{ij} = (\epsilon + p)\, u^i u^j - p g^{ij}\,, \tag{23}$$

where the energy density ϵ is some function of N (the number of baryons per unit volume), and p (the "equation of state"); thus

$$\epsilon \equiv \epsilon(N,p)\,. \tag{24}$$

And we must supplement equation (17) by the requirement

$$(Nu^j)_{;j} = 0\,. \tag{25}$$

Now treating equation (17), with T^{ij} given by equation (23), exactly as before, we find on using equation (25) that we must have

$$u^j \frac{\partial p}{\partial x_j} - \frac{\gamma p}{\sqrt{-g}} \frac{\partial}{\partial x_j}(u^j\sqrt{-g}) = 0\,, \tag{26}$$

where

$$\gamma = \frac{1}{p\partial N/\partial p}\left[N - (\epsilon + p)\frac{\partial N}{\partial \epsilon}\right] \tag{27}$$

is the "ratio of the specific heats" (cf. Chandrasekhar 1964, eq. [54]).

It is clear that the conservation of baryon number together with the statement of the first law of thermodynamics in the form of equation (26), and the conservation of mass and the statement of the first law in the form of equation (22) are entirely equivalent if we restrict ourselves to conditions in which the only baryons present are protons and neutrons and we agree to ignore their mass difference. In order not to complicate the analysis by matters that are inconsequential to our main purpose, we shall base our further considerations in this paper on equations (21) and (22).

An equivalent form of equation (21) is

$$(\rho u^j \sqrt{-g})_{,j} = 0\,; \tag{28}$$

and it follows from this equation that

$$M = \int_V \rho u^0 \sqrt{-g}\,dx = \text{constant} \tag{29}$$

is a conserved quantity.

Since in the framework of Einstein's equations the conservation of mass implies isentropic motion, and conversely, it follows that the energy integral (14) must be a derivable consequence of the field equations in an appropriate limit. And if this is the case, what is the generalization of this energy integral in the framework of the exact theory? To answer this question, we must turn to the exact conservation laws that obtain in the general theory.

The question of the conservation laws in general relativity is related to well-known questions concerning the "pseudo tensors" and the ambiguities associated with their definitions. While there is a vast literature on the subject (and the writer makes no claim that he understands it all), it appears that the essential *formal* content of the theory is very simple.

A mathematical identity that is directly and easily verifiable is (Synge 1960)

$$\tfrac{1}{2}(gU^{imkl})_{,l,m} = gG^{ik} - \frac{8\pi G}{c^4}\,g t^{ik}\,, \tag{30}$$

where

$$U^{imkl} = g^{il}g^{km} - g^{ik}g^{lm}\,, \tag{31}$$

$$t^{ik} = \frac{c^4}{16\pi G}\,(U^{imkl}D_{ml} + U^{ilmn}E^k{}_{.lmn} + U^{klmn}E^i{}_{.lmn} - U^{lmnp}\Gamma^i{}_{lp}\Gamma^k{}_{mn})\,, \tag{32}$$

$$D_{ml} = 2\Gamma^n{}_{lm}y_n - y_m y_l - \Gamma^n{}_{lp}\Gamma^p{}_{mn}\,,$$

$$E^i{}_{.lmn} = \Gamma^i{}_{lm}y_n + \Gamma^p{}_{lm}\Gamma^i{}_{pn}\,, \quad \text{and} \quad y_n = \frac{\partial \log\sqrt{-g}}{\partial x_n}\,; \tag{33}$$

(the factor $c^4/16\pi G$ has been introduced in the definition of t^{ik} for later convenience). If we now replace the Einstein tensor G^{ik} by $-8\pi G T^{ik}/c^4$, in accordance with the field equation, we obtain

$$\tfrac{1}{2}(gU^{imkl})_{,l,m} = \frac{8\pi G}{c^4}\,(-g)(T^{ik} + t^{ik})\,. \tag{34}$$

The essential features of equation (34) that are to be noted are (1) it is *not* a tensor equation, (2) U^{imkl} has the symmetries of the Riemann-Christoffel tensor: it is antisymmetric in (i,m) and (k,l) and symmetric for the simultaneous interchanges of i with k and l with m, (3) t^{ik} as defined in equation (32) is manifestly symmetric in i and k but it is *not* a tensor: it is in fact the "*pseudo tensor*" of Landau and Lifshitz (1962).

Defining the *energy-momentum complex*

$$\Theta^{ik} = (-g)(T^{ik} + t^{ik})\,, \tag{35}$$

we can rewrite equation (34) in the form

$$\tfrac{1}{2}(gU^{imkl})_{,l,m} = \frac{8\pi G}{c^4}\,\Theta^{ik}\,. \tag{36}$$

From the antisymmetry of U^{imkl} in (i,m) and (k,l) it now follows that

$$\Theta^{ik}{}_{,k} = \Theta^{ik}{}_{,i} = 0\,. \tag{37}$$

Thus the energy-momentum complex satisfies the same equation as in the Newtonian theory; and as in the Newtonian theory its derivation has required the explicit use of the field equation. And we may now, as then, call t^{ik} the contribution of the gravitational field to the total energy-momentum complex Θ^{ik}.

From the form of equations (37) we can infer the existence of the conserved quantities

$$P^i = \int \Theta^{0i} dx = \text{constant}$$

and

$$L_\gamma = \epsilon_{\alpha\beta\gamma}\int \Theta^{0\alpha} x_\beta dx = \text{constant}\,. \tag{38}$$

However, an important difference between these conserved quantities and those that occur in the Newtonian theory should be noted: the integrations in equations (38) and (39) have to be effected over the whole of the three-dimensional space and not only over the volume V occupied by the fluid: Θ^{0i}, unlike T^{0i}, need not necessarily vanish outside V.

As is well known, other complexes besides the one of Landau and Lifshitz can be defined. All these other complexes can be derived equally from the same mathematical identity (30) without any reference to variational principles or Lagrangians. Thus, multiplying equation (34) by $(-g)^{\nu/2} g_{jk}$, where ν is any arbitrary integer (positive or negative), we have

$$\tfrac{1}{2}(-g)^{\nu/2} g_{jk}(gU^{imkl})_{,l,m} = \frac{8\pi G}{c^4}(-g)^{(\nu+2)/2} g_{jk}(T^{ik} + t^{ik})\,. \tag{39}$$

We can rewrite this equation in the form

$$[\tfrac{1}{2}(-g)^{\nu/2} g_{jk}(gU^{imkl})_{,l}]_{,m} = \frac{8\pi G}{c^4}(-g)^{(\nu+2)/2}(T^i{}_{.j} + \tau^i{}_{.j})\,, \tag{40}$$

where

$$\tau^i{}_{.j} = g_{jk}t^{ik} + \frac{c^4}{16\pi G(-g)^{(\nu+2)/2}}[(-g)^{\nu/2} g_{jk}]_{,m}(gU^{imkl})_{,l}\,. \tag{41}$$

Since the antisymmetry in i and m of the quantity in square brackets on the left-hand side of the equation has been unaffected, it follows that

$$[(-g)^{(\nu+2)/2}(T^i{}_{.j} + \tau^i{}_{.j})]_{,i} = 0\,. \tag{42}$$

In this manner, all the "mixed" energy-momentum complexes can be derived; in particular, the choice $\nu = -1$ leads to the Einstein pseudo tensor and the Einstein complex. And the other possible "contravariant" complexes can be similarly deduced by simply suppressing g_{jk} in equation (39) and proceeding in the same way.

Thus, while equation (30) enables us to define all the possible complexes, it clearly distinguishes the complex of Landau and Lifshitz as the sole one which is symmetric in i and k; it is therefore the only one which will yield the angular momentum integral. Also Trautman (1958) and Cornish (1964) have shown that the conserved quantities can be given invariant meanings if space-time is asymptotically flat at infinity; and further that if the coordinates chosen are such that the metric also becomes Minkowskian at infinity, then the invariantly defined quantities are the same as those one obtains with the aid of the Landau-Lifshitz complex under the same circumstances. On these accounts, we shall restrict our further considerations to the Landau-Lifshitz complex.

We can now answer the question as to what the conserved quantity is that in the exact theory is the analogue of the energy integral (14) in the Newtonian theory. It is given by

$$\mathfrak{E} = \int (\Theta^{00} - c^2 \rho u^0 \sqrt{-g}) dx = \text{constant} . \tag{43}$$

(Note that in eq. [43] the integral must be extended over the whole of the three-dimensional space.)

An observation with respect to the energy integral (43) is here pertinent: it is, in the context of the perfect fluid, the content of the first law of thermodynamics implicit in the theory; and as such it gives to the Landau-Lifshitz complex a physical significance that cannot be evaded.[2]

From the form of equation (43) it is also clear why in the Newtonian framework the derivation of the energy integral explicitly requires one to supplement the equations of motion by equation (16) expressing the condition for isentropic flow: the Newtonian energy integral will emerge in general relativity only when Θ^{00} and $c^2\rho u^0\sqrt{-g}$ are known beyond their lowest (Newtonian!) orders. The Newtonian energy integral (14) is, from this point of view, a genuine post-Newtonian result. More generally, it follows from equation (43) that to obtain the energy integral in a given (say, nth) post-Newtonian approximation we must know Θ^{00} and $c^2\rho u^0\sqrt{-g}$ in one higher, $(n + 1)$th, approximation. It will appear that this is the most convenient way in which the conserved energy can be determined in post-Newtonian approximations beyond the first (cf. Chandrasekhar and Nutku 1969).

IV. THE CONSERVED QUANTITIES IN THE FIRST POST-NEWTONIAN APPROXIMATION

The different conserved quantities, in the first post-Newtonian approximation to the equations of general relativity governing a perfect fluid, were obtained in Paper I from an inspection of the equations of motion themselves. It will now be shown how these same conserved quantities can be obtained by a direct evaluation of the Landau-Lifshitz complex and the conserved density. However, it should be noted that the fact that it is possible to obtain the conserved quantities in this first post-Newtonian approximation without the aid of the Landau-Lifshitz complex is a peculiarity of this approximation: it does not obtain in the higher approximations (cf. Chandrasekhar and Nutku 1969).

a) The Conserved Mass and Energy

The conserved density in this approximation is (cf. Paper I, eqs. [15] and [53])

$$\begin{aligned} \rho u^0 \sqrt{-g} &= \rho\left[1 + \frac{1}{c^2}(\tfrac{1}{2}v^2 + U)\right]\left(1 + \frac{2U}{c^2}\right) + O(c^{-4}) \\ &= \rho\left[1 + \frac{1}{c^2}(\tfrac{1}{2}v^2 + 3U)\right] + O(c^{-4}) ; \end{aligned} \tag{44}$$

and this is precisely ρ^* as defined in Paper I, equation (118). The conserved mass is, accordingly,

$$M = \int_V \rho\left[1 + \frac{1}{c^2}(\tfrac{1}{2}v^2 + 3U)\right] dx . \tag{45}$$

[2] I am grateful to Dr. Kip S. Thorne for pointing out that this observation may be subject to misunderstanding and requires qualification. "*Practical usefulness*" is perhaps more appropriate than "*physical significance*" (!).

Considering next the conserved energy determined by the (0,0)-component of the Landau-Lifshitz complex, we find with the aid of equations (32) and (33) and of the Christoffel symbols listed in Paper I, equations (57), that

$$t^{00} = -\frac{7}{8\pi G}\left(\frac{\partial U}{\partial x_\mu}\right)^2 ; \tag{46}$$

while by equations (20) and (53) of Paper I,

$$\begin{aligned} -gT^{00} &= c^2\rho\left[1+\frac{1}{c^2}(v^2+2U+\Pi)\right]\left(1+\frac{4U}{c^2}\right)+O(c^{-2}) \\ &= c^2\rho\left[1+\frac{1}{c^2}(v^2+6U+\Pi)\right]+O(c^{-2}) . \end{aligned} \tag{47}$$

Therefore, in this approximation

$$\Theta^{00} = c^2\rho\left[1+\frac{1}{c^2}(v^2+6U+\Pi)\right]-\frac{7}{8\pi G}(\text{grad } U)^2 . \tag{48}$$

Since

$$(\text{grad } U)^2 = 4\pi G\rho U + \tfrac{1}{2}\,\text{div}\,(\text{grad } U^2) , \tag{49}$$

we can write

$$\Theta^{00} = c^2\rho\left[1+\frac{1}{c^2}(v^2+\tfrac{5}{2}U+\Pi)\right]-\frac{7}{16\pi G}\,\text{div}\,(\text{grad } U^2) . \tag{50}$$

It is now convenient to introduce the following terminology.

Definition: We shall say that two functions $f(x,t)$ and $g(x,t)$ are equal modulo divergence if they differ by the divergence of a vector which vanishes sufficiently rapidly at infinity that their integrals over the whole of space (assuming that they exist) are equal. We shall then write

$$f(x,t) \equiv g(x,t)\ (\text{mod div}) . \tag{51}$$

An immediate corollary of this definition is the following.

Corollary: If a function yields a conserved quantity when integrated over the whole of space, then any other function equal to it modulo divergence will yield the same conserved quantity.

According to the definition just stated,

$$\Theta^{00} \equiv c^2\rho\left[1+\frac{1}{c^2}(v^2+\tfrac{5}{2}U+\Pi)\right]\ (\text{mod div}) . \tag{52}$$

The conserved energy is therefore given by

$$P^0 = c^2\int_V \rho\left[1+\frac{1}{c^2}(v^2+\tfrac{5}{2}U+\Pi)\right]dx . \tag{53}$$

By equations (43), (45), and (53) we now have

$$\mathfrak{E} = P^0 - Mc^2 = \int_V \rho(\tfrac{1}{2}v^2 - \tfrac{1}{2}U+\Pi)dx ; \tag{54}$$

and this is indeed the energy integral in the Newtonian theory.

b) The Conserved Linear and Angular Momenta

Considering next the $(0,\alpha)$-component of the pseudo tensor t^{ik}, we find, again, with the aid of equations (32) and (33) and of the Christoffel symbols listed in Paper I, equations (57), that

$$t^{0\alpha} = \frac{1}{4\pi G c}\left[3\frac{\partial U}{\partial t}\frac{\partial U}{\partial x_\alpha} + 4\frac{\partial U}{\partial x_\beta}\left(\frac{\partial U_\beta}{\partial x_\alpha} - \frac{\partial U_\alpha}{\partial x_\beta}\right)\right]; \tag{55}$$

while by equations (20) and (53) of Paper I,

$$\begin{aligned} -gT^{0\alpha} &= c\rho\left[1 + \frac{1}{c^2}\left(v^2 + 2U + \Pi + \frac{p}{\rho}\right)\right]\left(1 + \frac{4U}{c^2}\right)v_\alpha + O(c^{-3}) \\ &= c\sigma v_\alpha + \frac{4}{c}\rho U v_\alpha + O(c^{-3}), \end{aligned} \tag{56}$$

where σ has the same meaning as in Paper I, equation (63). Therefore, in this approximation

$$\frac{1}{c}\Theta^{0\alpha} = \sigma v_\alpha + \frac{4}{c^2}\rho U v_\alpha + \frac{1}{4\pi G c^2}\left[3\frac{\partial U}{\partial t}\frac{\partial U}{\partial x_\alpha} + 4\frac{\partial U}{\partial x_\beta}\left(\frac{\partial U_\beta}{\partial x_\alpha} - \frac{\partial U_\alpha}{\partial x_\beta}\right)\right]. \tag{57}$$

By some straightforward reductions in which use is made of the equations governing U_α and χ (Paper I, eqs. [44] and [48]), the foregoing expression for $\Theta^{0\alpha}$ can be reduced to the form

$$\frac{1}{c}\Theta^{0\alpha} = \sigma v_\alpha + \frac{4}{c^2}\rho(U v_\alpha - U_\alpha) + \frac{1}{2c^2}\rho\frac{\partial^2\chi}{\partial t\partial x_\alpha} + \frac{1}{4\pi G c^2}\frac{\partial \theta_{\alpha\beta}}{\partial x_\beta}, \tag{58}$$

where

$$\begin{aligned} \theta_{\alpha\beta} &= 4U\left(\frac{\partial U_\alpha}{\partial x_\beta} + \frac{\partial U_\beta}{\partial x_\alpha}\right) + \tfrac{3}{2}\delta_{\alpha\beta}\frac{\partial U^2}{\partial t} - U\frac{\partial^3\chi}{\partial t\partial x_\alpha\partial x_\beta} \\ &\quad + \frac{\partial}{\partial x_\beta}\left[U\left(\tfrac{1}{2}\frac{\partial^2\chi}{\partial t\partial x_\alpha} - 4U_\alpha\right)\right]. \end{aligned} \tag{59}$$

Therefore,

$$\frac{1}{c}\Theta^{0\alpha} \equiv \pi_\alpha \text{ (mod div)}, \tag{60}$$

where π_α is exactly the quantity defined in Paper I, equation (128) whose integral over the volume V, occupied by the fluid, yields the conserved linear momentum. Therefore, $\Theta^{0\alpha}/c$ will yield the same conserved linear momentum. Moreover, since $\theta_{\alpha\beta}$ is equal, modulo divergence, to a symmetric tensor, $\Theta^{0\alpha}/c$ and π_α will also yield the same conserved angular momentum.

We have thus been able to determine the conserved linear and angular momenta by the simple and direct algorism of evaluating the Landau-Lifshitz complex. But this algorism has not enabled us to evaluate the conserved energy in this approximation with the knowledge of the solution of the field equations appropriate to this approximation. As we have already pointed out, to obtain the conserved energy in this approximation we need to know Θ^{00} and $c^2 \rho u^0 \sqrt{-g}$ in one higher approximation, i.e., in the second post-Newtonian approximation (see Chandrasekhar and Nutku 1969, where this evaluation is carried out).

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