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A TENSOR VIRIAL-EQUATION FOR STELLAR DYNAMICS

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Summary

A tensor virial-equation is derived for a system consisting of equal mass-points, appropriately for stellar dynamics, by starting with the Liouville equation governing an ensemble of such systems in the six N-dimensional phase space.

1. *Introduction.* The tensor form of the second-order virial equation has, in recent years, found numerous applications to a wide variety of problems (for a general account of these topics, see Chandrasekhar 1964). In the gravitational context (to which we shall restrict ourselves) there are two limiting classes of systems for which the virial equation has been explicitly written down: (1) systems consisting of a number of discrete mass points (or particles) $m^{(\alpha)}$ ($\alpha = 1, 2, \ldots,$) under their mutual attractions and (2) systems described in terms of a density ρ and an isotropic pressure p and governed by the usual hydrodynamic equations. For a system belonging to the first class, the equation takes the form

$$\frac{1}{2}\frac{d^2}{dt^2}\sum_\alpha m^{(\alpha)}x_i^{(\alpha)}x_j^{(\alpha)} = \sum_\alpha m^{(\alpha)}u_i^{(\alpha)}u_j^{(\alpha)} - \frac{1}{2}G\sum_\alpha\sum_{\alpha\neq\beta} m^{(\alpha)}m^{(\beta)}\frac{[x_i^{(\alpha)}-x_i^{(\beta)}][x_j^{(\alpha)}-x_j^{(\beta)}]}{|\mathbf{x}^{(\alpha)}-\mathbf{x}^{(\beta)}|^3}, \tag{1}$$

where the Greek superscripts distinguish the different particles and the Latin subscripts distinguish the different Cartesian components of the position $\mathbf{x}$ and the velocity $\mathbf{u}$ of the particle α. And for a system belonging to the second class, the equation takes the form

$$\frac{1}{2}\frac{d^2}{dt^2}\int_V \rho x_i x_j\, d\mathbf{x} = 2\mathfrak{T}_{ij}+\Pi\,\delta_{ij}+\mathfrak{W}_{ij}, \tag{2}$$

where

$$\mathfrak{T}_{ij} = \frac{1}{2}\int_V \rho u_i u_j\, d\mathbf{x} \tag{3}$$

and

$$\mathfrak{W}_{ij} = -\frac{1}{2}\int_V \rho\mathfrak{B}_{ij}\, d\mathbf{x} = -\frac{1}{2}G\int_V\int_V \rho(\mathbf{x})\rho(\mathbf{x}')\frac{(x_i-x_i')(x_j-x_j')}{|\mathbf{x}-\mathbf{x}'|^3}\,d\mathbf{x}\,d\mathbf{x}' \tag{4}$$

are the kinetic-energy and the potential-energy tensors; also

$$\Pi = \int_V p\, d\mathbf{x}. \tag{5}$$

(In equations (2)–(5) the integrations are effected over the entire volume V occupied by the fluid.)

While the forms of the two equations (1) and (2) are very similar, the essential difference between them is reflected in the separation, in equation (2), of the contributions to the kinetic-energy tensor by the macroscopic (or mean) and the microscopic (or molecular) motions: the former is included in $\mathfrak{T}_{ij}$ and the latter is manifested by the pressure term $\Pi\,\delta_{ij}$. But neither of these two equations is suitable under the normal circumstances of stellar dynamics. For in the context of stellar dynamics, the idealization in terms of the strict N-body problem (which does not permit a separation between the 'mean' and the 'peculiar' motions) or in terms of a fully relaxed hydrodynamic system (which does permit a unique separation of the two) are both unrealistic.

A recent discussion by Camm (1967) has suggested that it might be worthwhile to draw attention to a form of the tensor virial-equation that is 'exact' in the context of stellar dynamics in that it follows, rigorously, without any assumptions or approximations, from the six N-dimensional Liouville equation governing an ensemble of systems.

2. *The six N-dimensional Liouville equation and its integrated form.* We consider an ensemble of systems consisting of a (large) number N of discrete mass points (or particles) under their mutual gravitational attractions. For the sake of simplicity, we shall suppose that the masses of all the particles are the same so that any statistical property of the ensemble in which we may be interested, may be assumed to be symmetric in all the particles. And this assumption of symmetry will be made in this paper.

In the six N-dimensional phase space of the system, we describe an ensemble by the density function

$$f^{(N)} \equiv f^{(N)}(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots, \mathbf{x}^{(N)}; \mathbf{u}^{(1)}, \mathbf{u}^{(2)}, \ldots, \mathbf{u}^{(N)}; t), \tag{6}$$

where the superscripts (for which we shall use Greek letters) distinguish the different particles; and in accordance with our earlier remark, $f^{(N)}$ will be assumed to be symmetric in the variables of the different particles.

The Liouville equation governing $f^{(N)}$ is

$$\frac{\partial f^{(N)}}{\partial t} + \sum_{\alpha} u_k^{(\alpha)} \frac{\partial f^{(N)}}{\partial x_k^{(\alpha)}} + Gm \sum_{\alpha} \sum_{\beta \neq \alpha} \frac{x_k^{(\beta)} - x_k^{(\alpha)}}{|\mathbf{x}^{(\beta)} - \mathbf{x}^{(\alpha)}|^3} \frac{\partial f^{(N)}}{\partial u_k^{(\alpha)}} = 0; \tag{7}$$

and on integrating this equation over the coordinates and the velocities of all the particles except one, say the first, we obtain

$$\frac{\partial f^{(1)}}{\partial t} + u_k^{(1)} \frac{\partial f^{(1)}}{\partial x_k^{(1)}} + (N-1)Gm \int\int \frac{x_k^{(2)} - x_k^{(1)}}{|\mathbf{x}^{(2)} - \mathbf{x}^{(1)}|^3} \frac{\partial f^{(2)}}{\partial u_k^{(1)}} d\mathbf{x}^{(2)}\, d\mathbf{u}^{(2)} = 0, \tag{8}$$

where

$$f^{(2)} \equiv f^{(2)}(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}; \mathbf{u}^{(1)}, \mathbf{u}^{(2)}; t) = \int \ldots \int f^{(N)}\, d\mathbf{x}^{(3)}\, d\mathbf{x}^{(4)} \ldots d\mathbf{x}^{(N)}\, d\mathbf{u}^{(3)}\, d\mathbf{u}^{(4)} \ldots d\mathbf{u}^{(N)}, \tag{9}$$

and

$$f^{(1)} \equiv f^{(1)}(\mathbf{x}^{(1)}; \mathbf{u}^{(1)}; t) = \int\int f^{(2)}\, d\mathbf{x}^{(2)}\, d\mathbf{u}^{(2)} = \int \ldots \int f^{(N)}\, d\mathbf{x}^{(2)}\, d\mathbf{x}^{(3)} \ldots d\mathbf{x}^{(N)}\, d\mathbf{u}^{(2)}\, d\mathbf{u}^{(3)} \ldots d\mathbf{u}^{(N)}. \tag{10}$$

The factor $(N-1)$ in the last term in equation (8) originates in the assumed symmetry of $f^{(N)}$ in the variables of the different particles: the result of integrating the term in the summand involving the pair of particles (α, β), with α fixed and β variable, is independent of β.

3. *The hydrodynamic equations.* By integrating equation (8) over the velocities of the particle 1, as well, we obtain (cf. the corresponding derivation based on the six-dimensional 'collisionless' Boltzmann equation in Chandrasekhar (1942, see pp. 185, 186))

$$\frac{\partial n^{(1)}}{\partial t}+\frac{\partial}{\partial x_k^{(1)}}[n^{(1)}\langle u_k^{(1)}\rangle] = 0, \tag{11}$$

where

$$n^{(1)} \equiv n^{(1)}(\mathbf{x}^{(1)}, t) = \int f^{(1)}\, d\mathbf{u}^{(1)}, \tag{12}$$

and $\langle u_k^{(1)}\rangle$ is the average 1-particle velocity defined by

$$n^{(1)}\langle u_k^{(1)}\rangle = \int f^{(1)} u_k^{(1)}\, d\mathbf{u}^{(1)}. \tag{13}$$

It should be noted that here (and in the sequel) 'averages' (indicated by angular brackets) are averages over an ensemble.

Next, multiplying equation (8) by $u_i^{(1)}$ and integrating over the velocities of the particle 1, we obtain (cf. once again, the corresponding derivation based on the six-dimensional Liouville equation in Chandrasekhar (1942))

$$\frac{\partial}{\partial t}[n^{(1)}\langle u_i^{(1)}\rangle]+\frac{\partial}{\partial x_k^{(1)}}[n^{(1)}\langle u_i^{(1)}u_k^{(1)}\rangle] \\ -(N-1)Gm\int\int\int f^{(2)}\frac{x_i^{(2)}-x_i^{(1)}}{|\mathbf{x}^{(2)}-\mathbf{x}^{(1)}|^3}\, d\mathbf{x}^{(2)}\, d\mathbf{u}^{(2)}\, d\mathbf{u}^{(1)} = 0, \tag{14}$$

where in reducing the last term on the right-hand side of equation (8), an integration by parts has been effected and an assumption, appropriate to ignoring the integrated part, has been made.

In terms of the two-particle function,

$$n^{(1,2)} \equiv n^{(1,2)}(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, t) = \int\int f^{(2)}\, d\mathbf{u}^{(1)}\, d\mathbf{u}^{(2)}, \tag{15}$$

we can rewrite equation (14) in the form

$$\frac{\partial}{\partial t}[n^{(1)}\langle u_i^{(1)}\rangle]+\frac{\partial}{\partial x_k^{(1)}}[n^{(1)}\langle u_i^{(1)}u_k^{(1)}\rangle] \\ -(N-1)Gm\int n^{(1,2)}(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, t)\frac{x_i^{(2)}-x_i^{(1)}}{|\mathbf{x}^{(2)}-\mathbf{x}^{(1)}|^3}\, d\mathbf{x}^{(2)} = 0, \tag{16}$$

where it should be noted that $n^{(1,2)}$ (on our original assumption concerning $f^{(N)}$) is symmetric in $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$.

We can now separate the 'mean' $(\mathbf{U}^{(1)})$ from the 'peculiar' $(\mathbf{v}^{(1)})$ velocities and define a *pressure-tensor* $P_{ik}^{(1)}$ by means of the relations

$$u_i^{(1)} = v_i^{(1)}+\langle u_i^{(1)}\rangle = v_i^{(1)}+U_i^{(1)}, \tag{17}$$

and

$$n^{(1)}\langle u_i^{(1)}u_k^{(1)}\rangle = n^{(1)}\langle v_i^{(1)}v_k^{(1)}\rangle + n^{(1)}U_i^{(1)}U_k^{(1)}$$
$$= P_{ik}^{(1)} + n^{(1)}U_i^{(1)}U_k^{(1)}. \tag{18}$$

With these definitions, equations (11) and (16) take the forms

$$\frac{\partial}{\partial t}[n^{(1)}U_i^{(1)}] + \frac{\partial}{\partial x_k^{(1)}}[n^{(1)}U_i^{(1)}U_k^{(1)}] = -\frac{\partial P_{ik}^{(1)}}{\partial x_k^{(1)}}$$
$$+(N-1)Gm \int n^{(1,\,2)} \frac{x_i^{(2)} - x_i^{(1)}}{|\mathbf{x}^{(2)} - \mathbf{x}^{(1)}|^3} d\mathbf{x}^{(2)} \tag{19}$$

and

$$\frac{\partial n^{(1)}}{\partial t} + \frac{\partial}{\partial x_k^{(1)}}[n^{(1)}U_k^{(1)}] = 0. \tag{20}$$

4. *The tensor virial-equation.* From equation (19) we may derive a 'virial equation' by multiplying it by $x_j^{(1)}$ and integrating over the coordinates of the particle 1. The left-hand side of the equation gives, after an integration by parts,

$$\frac{d}{dt}\int n^{(1)}U_i^{(1)}x_j^{(1)}\, d\mathbf{x}^{(1)} - 2\mathfrak{T}_{ij}, \tag{21}$$

where

$$\mathfrak{T}_{ij} = \tfrac{1}{2}\int n^{(1)}U_i^{(1)}U_j^{(1)}\, d\mathbf{x}^{(1)}. \tag{22}$$

And the terms on the right-hand side give

$$-\int x_j^{(1)} \frac{\partial P_{ik}^{(1)}}{\partial x_k^{(1)}}\, d\mathbf{x}^{(1)} = \int P_{ij}^{(1)}\, d\mathbf{x}^{(1)} = \Pi_{ij} \text{ (say)}, \tag{23}$$

and

$$(N-1)Gm \int\int n^{(1,\,2)}(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, t)\, \frac{[x_i^{(2)} - x_i^{(1)}]x_j^{(1)}}{|\mathbf{x}^{(2)} - \mathbf{x}^{(1)}|^3}\, d\mathbf{x}^{(1)}\, d\mathbf{x}^{(2)}$$
$$= -\tfrac{1}{2}(N-1)Gm \int\int n^{(1,\,2)}(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, t)\, \frac{[x_i^{(2)} - x_i^{(1)}][x_j^{(2)} - x_j^{(1)}]}{|\mathbf{x}^{(2)} - \mathbf{x}^{(1)}|^3}\, d\mathbf{x}^{(1)}\, d\mathbf{x}^{(2)}$$
$$= \mathfrak{W}_{ij} \text{ (say)}, \tag{24}$$

where in integrating by parts in equation (23), it has been assumed that $P_{ik}^{(1)}$ vanishes on the boundary of the system; and, further, in passing from the first to the second line in equation (24), the symmetry of $n^{(1,\,2)}$ in $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ has been used and the expression for $\mathfrak{W}_{ij}$ is the average of the original form and the one obtained from it by interchanging the variables of integration $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$. Combining the results of these reductions, we obtain

$$\frac{d}{dt}\int n^{(1)}U_i^{(1)}x_j^{(1)}\, d\mathbf{x}^{(1)} = 2\mathfrak{T}_{ij} + \Pi_{ij} + \mathfrak{W}_{ij}. \tag{25}$$

The tensors on the right-hand side of equation (25) are manifestly symmetric in i and j. Therefore

$$\frac{d}{dt}\int n^{(1)}[U_i^{(1)}x_j^{(1)} - U_j^{(1)}x_i^{(1)}]\, d\mathbf{x}^{(1)} = 0; \tag{26}$$

and this equation represents the conservation of the angular momentum. And taking the symmetric part of the term on the left-hand side of equation (25), we have

$$\frac{1}{2}\frac{d}{dt}\int n^{(1)}[U_i^{(1)}x_j^{(1)}+U_j^{(1)}x_i^{(1)}]\,d\mathbf{x}^{(1)}$$

$$= \frac{1}{2}\frac{d^2}{dt^2}\int n^{(1)}x_i^{(1)}x_j^{(1)}\,d\mathbf{x}^{(1)} = \frac{1}{2}\frac{d^2I_{ij}}{dt^2}\ \text{(say)}. \quad (27)$$

Thus, we finally obtain

$$\frac{1}{2}\frac{d^2I_{ij}}{dt^2} = 2\mathfrak{T}_{ij}+\Pi_{ij}+\mathfrak{W}_{ij}. \quad (28)$$

While equation (28) is similar in form to the tensor virial-equation (2) for fluids, there are two important differences: *first*, the averages that have been taken before arriving at equation (28) are averages over an ensemble and *second*, $\mathfrak{W}_{ij}$ is *not* the potential-energy tensor (4) as defined in hydrodynamics: it is now defined in terms of the symmetric two-particle function $n^{(1,\,2)}\,(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, t)$ and not in terms of the product $n^{(1)}(\mathbf{x}^{(1)}, t)\,n^{(2)}(\mathbf{x}^{(2)}, t)$ of two one-particle functions.

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