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THE VIRIAL EQUATIONS OF THE FOURTH ORDER

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ABSTRACT

The virial equations of the fourth order are derived; and the results of certain formal developments needed for their practical usefulness are also given. The equations are then used to locate the neutral points, along the Maclaurin and the Jacobian sequences, that belong to the fourth harmonics.

I. INTRODUCTION

In papers published in the *Astrophysical Journal* (Volumes **134–148**) during the past few years, the tensor virial equations of the second and the third orders have been made the basis of a systematic development of the theory of the equilibrium and the stability of the ellipsoidal figures which arise in various gravitational contexts. In particular, the first variations of the virial equations governing equilibrium were used to isolate the neutral points, belonging to the second and the third harmonics, along the different sequences. More recently, in connection with the construction of sequences, analogous to those of Maclaurin and Jacobi, in the post-Newtonian framework of general relativity, a further neutral point, along each of the two sequences and belonging to the fourth harmonics, was isolated (Chandrasekhar 1967*a*, *b*). However, along the Maclaurin sequence there must exist neutral points, besides the one already located, belonging to the fourth harmonics. To find them and, as well, for the sake of completing the general theory, we shall obtain in this paper the relevant virial equations of the fourth order and illustrate their usefulness.

II. THE FOURTH-ORDER VIRIAL EQUATIONS IN A ROTATING FRAME OF REFERENCE

For purposes of later applications (and also for greater generality) we shall obtain the basic equations in a frame of reference rotating uniformly with an angular velocity Ω about the x_3-axis (say). The appropriate form of the equation of motion is

$$\rho \frac{du_i}{dt} = -\frac{\partial p}{\partial x_i} + \rho \frac{\partial}{\partial x_i}[\mathfrak{V} + \tfrac{1}{2}\Omega^2(x_1^2 + x_2^2)] + 2\rho\Omega\epsilon_{im3}u_m , \tag{1}$$

where the various symbols have their usual meanings.

Multiplying equation (1) by $x_jx_kx_l$ and integrating over the volume V occupied by the fluid, we obtain by transformations familiar in this theory,

$$\begin{aligned}\frac{d}{dt}\int_V \rho u_i x_j x_k x_l d\boldsymbol{x} = {} & 2(\mathfrak{T}_{ij;kl} + \mathfrak{T}_{ik;lj} + \mathfrak{T}_{il;jk}) \\ & + \Omega^2(I_{ijkl} - \delta_{i3}I_{3jkl}) + \int_V \rho \frac{\partial \mathfrak{V}}{\partial x_i} x_j x_k x_l d\boldsymbol{x} \\ & + 2\Omega\epsilon_{im3}\int_V \rho u_m x_j x_k x_l d\boldsymbol{x} + \delta_{ij}\Pi_{kl} + \delta_{ik}\Pi_{lj} + \delta_{il}\Pi_{jk} ,\end{aligned} \tag{2}$$

where

$$I_{ijkl} = \int_V \rho x_i x_j x_k x_l d\boldsymbol{x} , \qquad \Pi_{jk} = \int_V p x_j x_k d\boldsymbol{x} ,$$

and

$$\mathfrak{T}_{ij;kl} = \tfrac{1}{2}\int_V \rho u_i u_j x_k x_l d\boldsymbol{x} . \tag{3}$$

The term involving $\partial\mathfrak{B}/\partial x_i$ can be transformed as follows. First writing

$$\int_V \rho \frac{\partial \mathfrak{B}}{\partial x_i} x_j x_k x_l d\boldsymbol{x} = -G\int_V\int_V \rho(\boldsymbol{x})\rho(\boldsymbol{x}') \frac{x_j x_k x_l (x_i - x_i')}{|\boldsymbol{x}-\boldsymbol{x}'|^3} d\boldsymbol{x} d\boldsymbol{x}'$$
$$= -\tfrac{1}{2}G\int_V\int_V \rho(\boldsymbol{x})\rho(\boldsymbol{x}') \frac{(x_j x_k x_l - x_j' x_k' x_l')(x_i - x_i')}{|\boldsymbol{x}-\boldsymbol{x}'|^3} d\boldsymbol{x} d\boldsymbol{x}', \tag{4}$$

we replace

$$(x_j x_k x_l - x_j' x_k' x_l')(x_i - x_i') \tag{5}$$

by

$$\tfrac{1}{3}[(x_j - x_j')x_k x_l + x_j'(x_k - x_k')x_l + x_j' x_k'(x_l - x_l')$$
$$+(x_k - x_k')x_l x_j + x_k'(x_l - x_l')x_j + x_k' x_l'(x_j - x_j')$$
$$+(x_l - x_l')x_j x_k + x_l'(x_j - x_j')x_k + x_l' x_j'(x_k - x_k')], \tag{6}$$

and obtain

$$\int_V \rho \frac{\partial \mathfrak{B}}{\partial x_i} x_j x_k x_l d\boldsymbol{x} = -\tfrac{1}{6}\int_V \rho(2x_k x_l \mathfrak{B}_{ij} + 2x_l x_j \mathfrak{B}_{ik} + 2x_j x_k \mathfrak{B}_{il}$$
$$+ x_l \mathfrak{D}_{ik;j} + x_j \mathfrak{D}_{il;k} + x_k \mathfrak{D}_{ij;l}) d\boldsymbol{x}, \tag{7}$$

where

$$\mathfrak{B}_{ij}(\boldsymbol{x}) = G\int_V \rho(\boldsymbol{x}') \frac{(x_i - x_i')(x_j - x_j')}{|\boldsymbol{x}-\boldsymbol{x}'|^3} d\boldsymbol{x}' \tag{8}$$

is the Newtonian tensor potential as usually defined and

$$\mathfrak{D}_{ij;k}(\boldsymbol{x}) = G\int_V \rho(\boldsymbol{x}') x_k' \frac{(x_i - x_i')(x_j - x_j')}{|\boldsymbol{x}-\boldsymbol{x}'|^3} d\boldsymbol{x}' \tag{9}$$

is the tensor potential due to a distribution of "density" ρx_k. In terms of the Newtonian potentials $\mathfrak{D}_i$ and $\mathfrak{D}_{ij}$, due to the distributions ρx_i and $\rho x_i x_j$, we can express $\mathfrak{D}_{ij;k}$ in the manner (cf. Chandrasekhar and Lebovitz 1962, eq. [71])

$$\mathfrak{D}_{ij;k} = -x_i \frac{\partial \mathfrak{D}_k}{\partial x_j} + \frac{\partial \mathfrak{D}_{ik}}{\partial x_j}. \tag{10}$$

Now define the tensors

$$\mathfrak{W}_{ij;kl} = -\tfrac{1}{2}\int_V \rho \mathfrak{B}_{ij} x_k x_l d\boldsymbol{x} \tag{11}$$

and

$$\mathfrak{W}_{ij;k;l} = -\tfrac{1}{2}\int_V \rho \mathfrak{D}_{ij;k} x_l d\boldsymbol{x} = -\tfrac{1}{2}\int_V \rho \mathfrak{D}_{ij;l} x_k d\boldsymbol{x}, \tag{12}$$

which (like the tensors $\mathfrak{T}_{ij;kl}$ and $\delta_{ij}\Pi_{kl}$) are symmetric in the indices (i,j) and (k,l), separately. In terms of these tensors, we can now write equation (2) in the form

$$\frac{d}{dt}\int_V \rho u_i x_j x_k x_l d\boldsymbol{x} = 2(\mathfrak{T}_{ij;kl} + \mathfrak{T}_{ik;lj} + \mathfrak{T}_{il;jk}) + \Omega^2(I_{ijkl} - \delta_{i3} I_{3jkl})$$
$$+\tfrac{1}{3}(2\mathfrak{W}_{ij;kl} + 2\mathfrak{W}_{ik;lj} + 2\mathfrak{W}_{il;jk} + \mathfrak{W}_{ij;k;l} + \mathfrak{W}_{ik;l;j} + \mathfrak{W}_{il;j;k}) \tag{13}$$
$$+\delta_{ij}\Pi_{kl} + \delta_{ik}\Pi_{lj} + \delta_{il}\Pi_{jk} + 2\Omega\epsilon_{im3}\int_V \rho u_m x_j x_k x_l d\boldsymbol{x}).$$

Equation (13) is the desired form of the virial equation of the fourth order. It provides a set of thirty moment equations in addition to the nine and the eighteen provided by the second- and the third-order equations, respectively.

Under conditions when no internal motions are present (in the rotating frame considered) and a stationary state prevails, equation (13) gives

$$\tfrac{1}{3}(2\mathfrak{W}_{ij;kl}+2\mathfrak{W}_{ik;lj}+2\mathfrak{W}_{il;jk}+\mathfrak{W}_{ij;k;l}+\mathfrak{W}_{ik;l;j}+\mathfrak{W}_{il;j;k})$$
$$+\Omega^2(I_{ijkl}-\delta_{i3}I_{3jkl})=-(\delta_{ij}\Pi_{kl}+\delta_{ik}\Pi_{lj}+\delta_{il}\Pi_{jk}). \quad (14)$$

a) The First Variations of $\mathfrak{W}_{ij;kl}$ and $\mathfrak{W}_{ij;k;l}$

For treating small departures from equilibrium by the virial equation (13), it is necessary that we know the first-order changes in $\mathfrak{W}_{ij;kl}$ and $\mathfrak{W}_{ij;k;l}$ that result from a deformation caused by the different elments having suffered different displacements. If $\xi(\boldsymbol{x},t)$ denotes the corresponding Lagrangian displacement, then the resulting first variations in $\mathfrak{W}_{ij;kl}$ and $\mathfrak{W}_{ij;k;l}$ can be readily evaluated by standard procedures. Thus,

$$-2\delta\mathfrak{W}_{ij;kl}=\delta\int_V\rho\mathfrak{B}_{ij}x_kx_l d\boldsymbol{x}$$
$$=\int_V\rho\mathfrak{B}_{ij}\xi_m\frac{\partial}{\partial x_m}(x_kx_l)d\boldsymbol{x}$$
$$+G\int_V\int_V\rho(\boldsymbol{x})\rho(\boldsymbol{x}')x_kx_l\Big[\xi_m(\boldsymbol{x})\frac{\partial}{\partial x_m}+\xi_m(\boldsymbol{x}')\frac{\partial}{\partial x_m'}\Big]\frac{(x_i-x_i')(x_j-x_j')}{|\boldsymbol{x}-\boldsymbol{x}'|^3}d\boldsymbol{x}d\boldsymbol{x}' \quad (15)$$
$$=\int_V\rho\Big[\mathfrak{B}_{ij}\xi_m\frac{\partial}{\partial x_m}(x_kx_l)+x_kx_l\xi_m\frac{\partial\mathfrak{B}_{ij}}{\partial x_m}\Big]d\boldsymbol{x}$$
$$+\int_V d\boldsymbol{x}'\rho(\boldsymbol{x}')\xi_m(\boldsymbol{x}')\frac{\partial}{\partial x_m'}\int_V d\boldsymbol{x}\,\rho(\boldsymbol{x})x_kx_l\frac{(x_i-x_i')(x_j-x_j')}{|\boldsymbol{x}'-\boldsymbol{x}|^3},$$

or, equivalently,

$$-2\delta\mathfrak{W}_{ij;kl}=\int\rho\xi_m\frac{\partial}{\partial x_m}(x_kx_l\mathfrak{B}_{ij}+\mathfrak{D}_{ij;kl})d\boldsymbol{x}, \quad (16)$$

where $\mathfrak{D}_{ij;kl}$ is the "$\mathfrak{B}_{ij}$" induced by the distribution ρx_kx_l and is determined by the Newtonian potentials $\mathfrak{D}_{kl}$ and $\mathfrak{D}_{ikl}$ due to the distributions ρx_kx_l and $\rho x_ix_kx_l$ by (cf. eq. [10])

$$\mathfrak{D}_{ij;kl}=-x_i\frac{\partial\mathfrak{D}_{kl}}{\partial x_j}+\frac{\partial\mathfrak{D}_{ikl}}{\partial x_j}. \quad (17)$$

Similarly,

$$-2\delta\mathfrak{W}_{ij;k;l}=\delta\int_V\rho x_k\mathfrak{D}_{ij;l}d\boldsymbol{x}$$
$$=\int_V\rho\xi_k\mathfrak{D}_{ij;l}d\boldsymbol{x}$$
$$+G\int_V\int_V\rho(\boldsymbol{x})\rho(\boldsymbol{x}')x_k\Big[\xi_m(\boldsymbol{x})\frac{\partial}{\partial x_m}+\xi_m(\boldsymbol{x}')\frac{\partial}{\partial x_m'}\Big]\frac{x_l'(x_i-x_i')(x_j-x_j')}{|\boldsymbol{x}-\boldsymbol{x}'|^3}d\boldsymbol{x}d\boldsymbol{x}' \quad (18)$$
$$=\int_V\rho\Big(\xi_k\mathfrak{D}_{ij;l}+\xi_l\mathfrak{D}_{ij;k}+x_k\xi_m\frac{\partial\mathfrak{D}_{ij;l}}{\partial x_m}+x_l\xi_m\frac{\partial\mathfrak{D}_{ij;k}}{\partial x_m}\Big)d\boldsymbol{x}$$

or, equivalently,

$$-2\,\delta\mathfrak{W}_{ij;k;l}=\int_V \rho\,\xi_m \frac{\partial}{\partial x_m}(x_k\mathfrak{D}_{ij;l}+x_l\mathfrak{D}_{ij;k})\,dx\,. \tag{19}$$

b) Explicit Expressions for $\delta\mathfrak{W}_{ij;kl}$ and $\delta\mathfrak{W}_{ij;k;l}$ for Homogeneous Ellipsoids

For homogeneous ellipsoids, the potentials $\mathfrak{D}_{ij}$ and $\mathfrak{D}_{ijk}$ are explicitly known (cf. Chandrasekhar and Lebovitz 1963*a*, eqs. [107]–[118] and Chandrasekhar 1967*a*, eq. [63]; in the former reference explicit expressions for $\mathfrak{D}_{ij;k}$ are also given). Making use of them, the formulae (16) and (19) can be directly evaluated and expressed in terms of the symmetrized second-order and fourth-order virials:

$$V_{ij}=\delta I_{ij}=\int_V \rho\,\xi_m \frac{\partial}{\partial x_m}(x_i x_j)\,dx$$

and (20)

$$V_{ijkl}=\delta I_{ijkl}=\int_V \rho\,\xi_m \frac{\partial}{\partial x_m}(x_i x_j x_k x_l)\,dx\,.$$

We find, distinguishing the various cases,[1]

$$\begin{aligned}
-2\delta\mathfrak{W}_{ii;ii} &= a_i^2(-\tfrac{15}{2}a_i^2B_{iii}+2A_i)V_{ii}-\tfrac{3}{2}a_i^4B_{iij}V_{jj}-\tfrac{3}{2}a_i^4B_{iik}V_{kk}\\
&\quad+(\tfrac{35}{4}a_i^4B_{iiii}+4B_{ii}-2A_i)V_{iiii}+\tfrac{3}{4}a_i^4B_{iijj}V_{jjjj}+\tfrac{3}{4}a_i^4B_{iikk}V_{kkkk}\\
&\quad+a_i^2(\tfrac{15}{2}a_i^2B_{iiij}-2A_{ij})V_{iijj}+a_i^2(\tfrac{15}{2}a_i^2B_{iiik}-2A_{ik})V_{iikk}+\tfrac{3}{2}a_i^4B_{iijk}V_{jjkk}\,,\\
-2\delta\mathfrak{W}_{ii;jj} &= a_j^2(-\tfrac{3}{2}a_i^2B_{iij}+B_{ij})V_{ii}+a_i^2(-\tfrac{1}{2}a_j^2B_{ijj}+a_j^4A_{ijj}+A_i)V_{jj}-\tfrac{1}{2}a_i^2a_j^2B_{ijk}V_{kk}\\
&\quad+a_j^2(\tfrac{5}{4}a_i^2B_{iiij}-B_{iij})V_{iiii}+a_i^2(\tfrac{1}{4}a_j^2B_{ijjj}-a_j^4A_{ijjj}-A_{ij})V_{jjjj}+\tfrac{1}{4}a_i^2a_j^2B_{ijkk}V_{kkkk}\\
&\quad+a_j^2(\tfrac{3}{2}a_i^2B_{iijk}-B_{ijk})V_{iikk}+a_i^2(\tfrac{1}{2}a_j^2B_{ijjk}-a_j^4A_{ijjk}-A_{ik})V_{jjkk}\\
&\quad+[3a_j^2(a_j^2+\tfrac{1}{2}a_i^2)B_{iijj}+3B_{ii}+B_{ij}-2A_i]V_{iijj}\,,\\
-2\delta\mathfrak{W}_{ij;ij} &= -\tfrac{3}{2}a_i^2a_j^2B_{iij}V_{ii}-\tfrac{3}{2}a_i^2a_j^2B_{ijj}V_{jj}-\tfrac{1}{2}a_i^2a_j^2B_{ijk}V_{kk}\\
&\quad+\tfrac{5}{4}a_i^2a_j^2B_{iiij}V_{iiii}+\tfrac{5}{4}a_i^2a_j^2B_{ijjj}V_{jjjj}+\tfrac{1}{4}a_i^2a_j^2B_{ijkk}V_{kkkk}\\
&\quad+(\tfrac{9}{2}a_i^2a_j^2B_{iijj}+2B_{ij})V_{iijj}+\tfrac{3}{2}a_i^2a_j^2B_{iijk}V_{iikk}+\tfrac{3}{2}a_i^2a_j^2B_{ijjk}V_{jjkk}\,,\\
-2\delta\mathfrak{W}_{ij;ik} &= -a_i^2a_k^2B_{ijk}V_{jk}+(3a_i^2a_k^2B_{iijk}+2B_{ij})V_{iijk}\\
&\quad+a_i^2a_k^2B_{ijjk}V_{jjjk}+a_i^2a_k^2B_{ijkk}V_{jkkk}\,,\\
-2\delta\mathfrak{W}_{ij;kk} &= a_k^2C_{ijk}V_{ij}+(3a_k^4B_{ijkk}-a_k^2B_{ijk}+2B_{ij})V_{ijkk}\\
&\quad-a_k^2C_{iijk}V_{iiij}-a_k^2C_{ijjk}V_{ijjj}\,,\\
-2\delta\mathfrak{W}_{ij;ii} &= a_i^2(-3a_i^2B_{iij}+B_{ij})V_{ij}+(5a_i^4B_{iiij}-a_i^2B_{iij}+2B_{ij})V_{iiij}\\
&\quad+a_i^2(3a_i^2B_{iijj}-B_{ijj})V_{ijjj}+a_i^2(3a_i^2B_{iijk}-B_{ijk})V_{ijkk}\,,\\
-2\delta\mathfrak{W}_{ii;jk} &= a_i^2(a_j^2a_k^2A_{ijk}+A_i)V_{jk}+(2B_{ii}-3a_i^2a_j^2a_k^2A_{iijk}+2a_j^2a_k^2A_{ijk}-a_i^2A_{ii})V_{iijk}\\
&\quad-a_i^2(a_j^2a_k^2A_{ijjk}+A_{ij})V_{jjjk}-a_i^2(a_j^2a_k^2A_{ijkk}+A_{ik})V_{jkkk}\,,\\
-2\delta\mathfrak{W}_{ii;ij} &= a_i^2(-a_j^2B_{iij}+B_{ij}+2a_i^2a_j^2A_{iij})V_{ij}+a_i^2(a_j^2B_{iijj}-B_{ijj}-2a_i^2a_j^2A_{iijj})V_{ijjj}\\
&\quad+a_i^2(a_j^2B_{iijk}-B_{ijk}-2a_i^2a_j^2A_{iijk})V_{ijkk}+(5a_i^2a_j^2B_{iiij}+2B_{ii}-a_i^2a_j^2A_{iij}-a_i^2A_{ii})V_{iiij}\,,
\end{aligned} \tag{21}$$

[1] The formulae (21) and (22) were also derived independently by Dr. C. E. Rosenkilde. I am grateful to him for providing a check on their correctness.

and

$$
\begin{aligned}
-2\delta\mathfrak{W}_{ii;i;i} &= 2a_i^2[(-2B_{ii}+a_i^2A_{ii})V_{ii}+(4B_{iii}-a_i^2A_{iii})V_{iiii} \\
&\quad +(2B_{iij}-a_i^2A_{iij})V_{iijj}+(2B_{iik}-a_i^2A_{iik})V_{iikk}]\,, \\
-2\delta\mathfrak{W}_{ii;j;j} &= 2a_j^2[a_i^2A_{ij}V_{jj}+(2B_{iij}-a_i^2A_{iij})V_{iijj}-a_i^2A_{ijj}V_{jjjj}-a_i^2A_{ijk}V_{jjkk}]\,, \\
-2\delta\mathfrak{W}_{ij;i;j} &= -a_j^2B_{ij}V_{ii}-a_i^2B_{ij}V_{jj}+a_j^2B_{iij}V_{iiii}+a_i^2B_{ijj}V_{jjjj} \\
&\quad +a_i^2B_{ijk}V_{jjkk}+a_j^2B_{ijk}V_{iikk}+3(a_i^2B_{iij}+a_j^2B_{ijj})V_{iijj}\,, \\
-2\delta\mathfrak{W}_{ij;i;k} &= -a_i^2B_{ij}V_{jk}+(3a_i^2B_{iij}+2a_k^2B_{ijk})V_{iijk}+a_i^2B_{ijj}V_{jjjk}+a_i^2B_{ijk}V_{jkkk}\,, \\
-2\delta\mathfrak{W}_{ij;k;k} &= 4a_k^2B_{ijk}V_{ijkk}, \\
-2\delta\mathfrak{W}_{ij;i;i} &= 2a_i^2(-B_{ij}V_{ij}+B_{ijj}V_{ijjj}+3B_{iij}V_{iiij}+B_{ijk}V_{ijkk})\,, \\
-2\delta\mathfrak{W}_{ii;j;k} &= a_i^2(a_j^2A_{ij}+a_k^2A_{ik})V_{jk}+(2a_j^2B_{iij}+2a_k^2B_{iik}-a_i^2a_j^2A_{iij}-a_i^2a_k^2A_{iik})V_{iijk} \\
&\quad -a_i^2(a_j^2A_{ijj}+a_k^2A_{ijk})V_{jjjk}-a_i^2(a_k^2A_{ikk}+a_j^2A_{ijk})V_{jkkk}\,, \\
-2\delta\mathfrak{W}_{ii;i;j} &= a_i^2(-2B_{ii}+a_i^2A_{ii}+a_j^2A_{ij})V_{ij}+(5a_i^2B_{iii}+2a_j^2B_{iij}-a_i^2a_j^2A_{iij}-a_i^2A_{ii})V_{iiij} \\
&\quad +a_i^2(3B_{iij}+B_{ijj}-2A_{ij})V_{ijjj}+a_i^2(3B_{iik}+B_{ijk}-2A_{ik})V_{ijkk}\,.
\end{aligned}
\tag{22}
$$

(In eqs. [21] and [22] the summation convention has been suspended; also $i \neq j \neq k$ represent distinct indices.)

In equations (21) and (22) the index symbols A_{ijk} .. and B_{ijk} .. are so normalized that $A_1 + A_2 + A_3 = 2$; and the symbol C_{ijk} . is related to the symbol B_{ijk} .. in the manner defined in Chandrasekhar (1967*b*), equation (67). Also, a common factor $\pi G\rho$ in the expressions for the potentials has been suppressed; and, consistently, Ω^2 will be measured in the unit $\pi G\rho$.

c) The Divergence Conditions on V_{ijkl} in Case of Incompressibility

When we are dealing with an incompressible ellipsoid, as we shall in §§ IV and V, the Lagrangian displacement considered in § II*b* should be divergence-free. This requirement leads to certain restrictions on the symmetrized virials V_{ijkl} even as there are restrictions on the second- and the third-order virials (cf. Lebovitz 1961, eq. [83]; Chandrasekhar and Lebovitz 1963*a*, § VI); and they can be similarly found. Thus, by an integration by parts we find

$$
\begin{aligned}
&\int_V \rho\, \xi \cdot \operatorname{grad}\left[x_i x_j \left(\sum_{k=1}^{3}\frac{x_k^2}{a_k^2}-1\right)\right] dx \\
&\quad = \int_S \rho x_i x_j \left(\sum_{k=1}^{3}\frac{x_k^2}{a_k^2}-1\right)\xi \cdot dS - \int_V \rho x_i x_j \left(\sum_{k=1}^{3}\frac{x_k^2}{a_k^2}-1\right) \operatorname{div}\xi\, dx = 0\,,
\end{aligned}
\tag{23}
$$

the surface integral (over S) vanishing on account of the equation satisfied by the ellipsoidal boundary and the volume integral (over V) vanishing on account of the solenoidal character of ξ; and we obtain, in view of the definitions (20),

$$
V_{ij} = \sum_{k=1}^{3}\frac{V_{ijkk}}{a_k^2}\,. \tag{24}
$$

(Note that we have suspended the summation convection over repeated indices. It will not be adopted in the rest of this paper. Summation will be indicated if required.)

Equation (24) provides six relations. On the other hand, since the divergence condition on the second-order virials requires

$$\sum_{k=1}^{3} \frac{V_{kk}}{a_k^2} = 0 , \tag{25}$$

we have the further relation

$$\sum_{k=1}^{3} \frac{V_{kkkk}}{a_k^4} + \sum_{i \neq j} \frac{V_{iijj}}{a_i^2 a_j^2} = 0 . \tag{26}$$

III. THE CONDITIONS TO BE SATISFIED AT A NEUTRAL POINT

At a point of bifurcation, or more generally at any neutral point, where the configuration allows a neutral mode of oscillation for some non-trivial displacement $\boldsymbol{\xi}$, the corresponding first variations of the equilibrium conditions given by the virial equations of the various orders must also be necessarily satisfied.

We shall now consider the conditions that follow from equation (14).

Equation (14) provides a total of thirty conditions to be satisfied. These conditions can be divided into four groups: a group of nine equations which are even in all three indices 1, 2, and 3; and three groups of seven equations each which are even in a particular index k and odd in the remaining two.

The nine even equations are, in turn, of two types: three equations of the form

$$2\delta\mathfrak{W}_{ii;ii} + \delta\mathfrak{W}_{ii;i;i} + \Omega^2(1 - \delta_{i3})V_{iiii} = -3\delta\Pi_{ii} \qquad (i = 1,2,3) \; ; \tag{27}$$

and six equations obtained by letting i and j ($i \neq j$) represent an *ordered pair* of indices in the equation

$$4\delta\mathfrak{W}_{ij;ij} + 2\delta\mathfrak{W}_{jj;ii} + 2\delta\mathfrak{W}_{ij;i;j} + \delta\mathfrak{W}_{jj;i;i} + 3\Omega^2(1 - \delta_{j3})V_{iijj} = -3\delta\Pi_{ii} . \tag{28}$$

(Note that eq. [28] is not symmetric in i and j and that the equations obtained by interchanging i and j are different.)

The seven equations, odd in a selected pair of indices i and j ($i \neq j$) and even in the index k ($\neq i \neq j$), are of four types. The equations

$$2\delta\mathfrak{W}_{ij;jj} + \delta\mathfrak{W}_{ij;j;j} + \Omega^2(1 - \delta_{i3})V_{ijjj} = 0 , \tag{29}$$

$$4\delta\mathfrak{W}_{ik;jk} + 2\delta\mathfrak{W}_{ij;kk} + 2\delta\mathfrak{W}_{ik;j;k} + \delta\mathfrak{W}_{ij;k;k} + 3\Omega^2(1 - \delta_{i3})V_{ijkk} = 0 , \tag{30}$$

and

$$4\delta\mathfrak{W}_{ii;ij} + 2\delta\mathfrak{W}_{ij;ii} + 2\delta\mathfrak{W}_{ii;i;j} + \delta\mathfrak{W}_{ij;i;i} + 3\Omega^2(1 - \delta_{i3})V_{iiij} = - 6\delta\Pi_{ij} , \tag{31}$$

together with the equations obtained from these by interchanging i and j, and the further equation

$$2\delta\mathfrak{W}_{kk;ij} + 2\delta\mathfrak{W}_{jk;ki} + 2\delta\mathfrak{W}_{ki;jk} + \delta\mathfrak{W}_{kk;i;j} + \delta\mathfrak{W}_{jk;k;i} + \delta\mathfrak{W}_{ki;j;k} + 3\Omega^2(1 - \delta_{k3})V_{ijkk} = -3\delta\Pi_{ij} \tag{32}$$

(symmetric in i and j) provide the seven equations.

When dealing with an incompressible configuration (and sometimes even quite generally) it is convenient to eliminate the six $\delta\Pi_{ij}$'s from equations (27)–(32). After the elimination, we shall be left with twenty-four equations. In the case of incompressible ellipsoids, the twenty-four equations must be supplemented by the divergence conditions (24)–(26); and in the general case, six equations including the different $\delta\Pi_{ij}$'s must be considered and the $\delta\Pi_{ij}$'s must be evaluated by some additional assumption, such as, that the perturbed motions take place adiabatically (cf. Chandrasekhar and Lebovitz 1963*c*, § VI).

In the case of the even equations, the six equations which remain after the elimination of the three $\delta\Pi_{ii}$'s are of two types:

$$2\delta\mathfrak{W}_{ii;ii}+\delta\mathfrak{W}_{ii;i;i}-(4\delta\mathfrak{W}_{ij;ij}+2\delta\mathfrak{W}_{jj;ii}+2\delta\mathfrak{W}_{ij;i;j}+\delta\mathfrak{W}_{jj;i;i}) +\Omega^2(1-\delta_{i3})V_{iiii}-3\Omega^2(1-\delta_{j3})V_{iijj}=0 \qquad (i\neq j)\,, \tag{33}$$

and

$$+(4\delta\mathfrak{W}_{ij;ij}+2\delta\mathfrak{W}_{jj;ii}+2\delta\mathfrak{W}_{ij;i;j}+\delta\mathfrak{W}_{jj;i;i})+3\Omega^2(1-\delta_{j3})V_{iijj} -(4\delta\mathfrak{W}_{ik;ik}+2\delta\mathfrak{W}_{kk;ii}+2\delta\mathfrak{W}_{ik;i;k}+\delta\mathfrak{W}_{kk;i;i})-3\Omega^2(1-\delta_{k3})V_{iikk}=0 \qquad (i\neq j\neq k)\,. \tag{34}$$

In an Appendix, we list, for the case of homogeneous ellipsoids, the explicit expressions for the various combinations of $\delta\mathfrak{W}_{ij;kl}$ and $\delta\mathfrak{W}_{ij;k;l}$ that occur in equations (29)–(34).

IV. THE NEUTRAL POINTS, BELONGING TO THE FOURTH HARMONICS, ALONG THE MACLAURIN SEQUENCE

One can readily convince oneself[2] that the neutral points, along the Maclaurin and the Jacobian sequences that belong to the fourth harmonics, are derived from displacements whose only non-vanishing fourth-order symmetrized virials are those that are even in the index 3. Accordingly, we need to consider only the nine equations which are even in all three indices and the seven equations which are odd in the indices 1 and 2.

In the case of the Maclaurin sequence, when $a_1 = a_2$ and the indices 1 and 2 can be identified in all the index symbols, the various equations simplify considerably.

Considering first the even equations (33) and (34) and using the formulae (A.1) and (A.2) given in the Appendix, we find that a non-trivial satisfaction of these equations can be accomplished in only one of two ways. *Either*

$$V_{1111}=V_{2222} \qquad \text{and} \qquad V_{1133}=V_{2233}\,, \tag{35}$$

and the fourth-order virials, after the elimination of the second-order virials with the aid of equation (24), satisfy the equations

$$(\Omega^2-2A_1+2a_1^6A_{1111})(V_{1111}-3V_{1122})=0\,, \tag{36}$$

$$\left(4.5a_1^2a_3^2B_{1113}-1.5a_3^4B_{1133}-6a_3^2B_{113}+3\frac{a_3^4}{a_1^2}B_{133}\right)(V_{1111}+V_{1122}) -\left(6a_1^2C_{1133}-24a_3^4B_{1133}-6\frac{a_3^2}{a_1^2}B_{33}\right)V_{1133} +(3.75a_1^2a_3^2B_{1333}-8.75a_3^4B_{3333}-1.5a_1^2B_{133}+2.5a_3^2B_{333}-2B_{33})V_{3333}=0\,, \tag{37}$$

$$(15a_1^4B_{1111}-9a_1^2a_3^2B_{1113}-12a_1^2B_{111}+6a_3^2B_{113})V_{1111} +(27a_1^4B_{1111}-9a_1^2a_3^2B_{1113}+6a_3^2B_{113}+12B_{11}-6\Omega^2)V_{1122} +6\left[3\frac{a_1^6}{a_3^2}B_{1113}-6a_1^2a_3^2B_{1133}+\frac{a_1^2}{a_3^2}(a_3^2-2a_1^2)B_{113}+B_{13}-3B_{33}\right]V_{1133} +\left(1.5a_1^4B_{1133}-7.5a_1^2a_3^2B_{1333}+9a_1^2B_{133}-3\frac{a_1^4}{a_3^2}B_{113}\right)V_{3333}=0\,, \tag{38}$$

and

$$\frac{2V_{1111}}{a_1^4}+\frac{2V_{1122}}{a_1^4}+\frac{4V_{1133}}{a_1^2a_3^2}+\frac{V_{3333}}{a_3^4}=0\,, \tag{39}$$

[2] I am grateful to Dr. N. Lebovitz for discussions relating to this point.

(where the last eq. [39] is the divergence condition [26] appropriate for this case), *or*

$$V_{1111} = -V_{2222},\ V_{1133} = -V_{2233}, \quad \text{and} \quad V_{1122} = V_{3333} = 0\,, \tag{40}$$

and the non-vanishing virials satisfy the equations

$$(\Omega^2 - 2B_{11} - a_1^2 C_{1113})V_{1111} + 6\,\frac{a_1^4}{a_3^2}\,C_{1113}V_{1133} = 0 \tag{41}$$

and

$$a_3^2 C_{1113} V_{1111} + (\Omega^2 - 2B_{11} - 6a_1^2 C_{1113})V_{1133} = 0\,. \tag{42}$$

It is manifest that equations (36)–(39) can, in turn, be satisfied non-trivially in one of two ways: *either* by letting

$$\Omega^2 = 2A_1 - 2a_1^6 A_{1111}\,, \tag{43}$$

and determining V_{1111}, V_{1122}, V_{1133}, and V_{3333}, apart from an arbitrary constant of proportionality, with the aid of equations (37)–(39), *or* by letting

$$V_{1122} = \tfrac{1}{3}V_{1111} \tag{44}$$

and requiring that the determinant of the system of equations

$$\begin{aligned}\Big(6a_1^2a_3^2B_{1113} - 2a_3^4B_{1133} - 8a_3^2B_{113} + 4\,\frac{a_3^4}{a_1^2}\,B_{133}\Big)V_{1111}\\ - 6\Big(a_1^2C_{1133} - 4a_3^4B_{1133} - \frac{a_3^2}{a_1^2}\,B_{33}\Big)V_{1133}\\ +(3.75a_1^2a_3^2B_{1333} - 8.75a_3^4B_{3333} - 1.5a_1^2B_{133} + 2.5a_3^2B_{333} - 2B_{33})V_{3333} = 0\,,\end{aligned} \tag{45}$$

$$\begin{aligned}(24a_1^4B_{1111} - 12a_1^2a_3^2B_{1113} + 8a_3^2B_{113} - 12a_1^2B_{111} + 4B_{11} - 2\Omega^2)V_{1111}\\ +6\Big[3\,\frac{a_1^6}{a_3^2}\,B_{1113} - 6a_1^2a_3^2B_{1133} + \frac{a_1^2}{a_3^2}(a_3^2 - 2a_1^2)B_{113} + B_{13} - 3B_{33}\Big]V_{1133}\\ +\Big(1.5a_1^4B_{1133} - 7.5a_1^2a_3^2B_{1333} + 9a_1^2B_{133} - 3\,\frac{a_1^4}{a_3^2}\,B_{113}\Big)V_{3333} = 0\,,\end{aligned} \tag{46}$$

and

$$\frac{8V_{1111}}{3a_1^4} + \frac{4V_{1133}}{a_1^2a_3^2} + \frac{V_{3333}}{a_3^4} = 0 \tag{47}$$

(which are the appropriate forms of eqs. [37]–[39] when the relation [44] obtains) vanish. The two neutral points, which follow in this fashion, occur for eccentricities of the Maclaurin spheroid given by

$$e_1^{(4)} = 0.93275 \qquad (\text{where } \Omega^2 = 2A_1 - a_1^6 A_{1111})\,, \tag{48}$$

and

$$e_2^{(4)} = 0.98531 \text{ (where the determinant of the system of eqs. [45]–[47] vanishes).} \tag{49}$$

The latter point at $e_2^{(4)}$ agrees with the point located earlier (Chandrasekhar 1967*a*) by a different method.[3]

An alternative form of the condition (43) is

$$\Omega^2 = 2(B_{11} + a_1^2 B_{111} + a_1^4 B_{1111})\,. \tag{50}$$

[3] The value 0.98526 determined earlier differs slightly from the value (49). But it is believed that the present determination is the more accurate of the two.

It is of interest to contrast this condition, for the occurrence of a neutral mode belonging to the fourth harmonics, with the conditions

$$\Omega^2 = 2B_{11} \quad \text{and} \quad \Omega^2 = 2(B_{11} + a_1^2 B_{111}) , \tag{51}$$ [4]

for the occurrence of neutral modes belonging to the second and the third harmonics. In view of these results, one may perhaps conjecture that a neutral mode belonging to the nth harmonics occurs where

$$\Omega^2 = 2(B_{11} + a_1^2 B_{11} + \ldots + a_1^{2(n-1)} B_{(n)}) . \tag{52}$$

Considering next equations (41) and (42), we infer that a neutral point occurs when the determinant of this system vanishes; and this condition gives

$$(\Omega^2 - 2B_{11})(\Omega^2 - 2B_{11} - 7a_1^2 C_{1113}) = 0 . \tag{53}$$

We must, therefore, have

$$\Omega^2 = 2B_{11} , \tag{54}$$

or

$$\Omega^2 = 2B_{11} + 7a_1^2 C_{1113} . \tag{55}$$

The occurrence of the "Jacobi point" (54) in this context is a consequence of the fact that neutral points belonging to the second harmonics are automatically included in the present analysis for the same reason that the *equilibrium* conditions (14) determine the same Maclaurin and Jacobi figures as the equations

$$\mathfrak{W}_{11} + \Omega^2 I_{11} = \mathfrak{W}_{22} + \Omega^2 I_{22} = \mathfrak{W}_{33} \tag{56}$$

provided by the second-order virial theorem.

The point determined by the condition (55) is, however, new and belongs, genuinely, to the fourth harmonics. It is found that the condition is satisfied for a Maclaurin spheroid with the eccentricity

$$e_3^{(4)} = 0.98097 \qquad (\text{where } \Omega^2 = 2B_{11} + 7a_1^2 C_{1113}) . \tag{57}$$

This completes the discussion of the even equations.

We turn now to a consideration of the seven equations odd in the indices 1 and 2. On inserting from equations (A.3)–(A.6) in equations (29)–(32), we find that the six equations remaining after the elimination of $\delta\Pi_{12}$ and V_{12} (the latter with the aid of eq. [24]) can be reduced to the following four equations:

$$(\Omega^2 - 2A_1 + 2a_1^6 A_{1111})(V_{1211} - V_{1222}) = 0 , \tag{58}$$

$$(\Omega^2 - 2B_{11} - a_1^2 C_{1113})(V_{1211} + V_{1222}) + 6\frac{a_1^4}{a_3^2} C_{1113} V_{1233} = 0 , \tag{59}$$

$$a_3^2 C_{1113}(V_{1211} + V_{1222}) + (\Omega^2 - 2B_{11} - 6a_1^2 C_{1113}) V_{1233} = 0 , \tag{60}$$

and

$$\begin{aligned} &\tfrac{1}{2}(5a_1^2 C_{1113} + 2B_{11} - \Omega^2)(V_{1211} + V_{1222}) \\ &-\tfrac{1}{2}(\Omega^2 - 2A_1 + 2a_1^6 A_{1111})(V_{1211} - V_{1222}) \\ &\qquad -\frac{a_1^2}{a_3^2}(4B_{11} + 15a_1^2 C_{1113} - 2\Omega^2) V_{1233} = 0 , \end{aligned} \tag{61}$$

[4] The first of these two points is, of course, the point of bifurcation where the Jacobian sequence branches off from the Maclaurin sequence; and for the location of the second point belonging to the third harmonics see Chandrasekhar and Lebovitz (1963*b*) and also Chandrasekhar (1963).

where it may be noted that equations (59) and (60) are each repeated twice among the original six equations.

It is clear that equations (58)–(60) can be satisfied non-trivially in one of two ways: *either* by requiring

$$V_{1211} = -V_{1222} \quad \text{and} \quad V_{1233} = 0 \tag{62}$$

or by requiring

$$V_{1211} = V_{1222} \quad \text{and} \quad V_{1233} \neq 0 . \tag{63}$$

In the former case, we are led to the same condition (43) that followed from equation (36); and in the latter case, we are led to the same characteristic equation (53) for Ω^2 that followed from equations (41) and (42). It remains to verify that equation (61) is satisfied under both these requirements. It is clearly satisfied under (62); that it is also satisfied under (63) can be established as follows. When the determinant of equations (59) and (60) vanishes (i.e., when eq. [53] holds)

$$\frac{V_{1211}}{V_{1233}} = -\frac{a_1^4}{a_3^2}\frac{3C_{1113}}{\Omega^2 - 2B_{11} - a_1^2 C_{1113}} = -\frac{\Omega^2 - 2B_{11} - 6a_1^2 C_{1113}}{2a_3^2 C_{1113}} . \tag{64}$$

In particular,

$$\begin{aligned} \frac{V_{1211}}{V_{1233}} &= \frac{3a_1^2}{a_3^2} \quad \text{when} \quad \Omega^2 = 2B_{11} \\ &= -\frac{a_1^2}{2a_3^2} \quad \text{when} \quad \Omega^2 = 2B_{11} + 7a_1^2 C_{1113} ; \end{aligned} \tag{65}$$

and we verify that under these circumstances equation (61) is indeed satisfied.

The neutral points $e_1^{(4)}$ and $e_3^{(4)}$ are, therefore, repeated as solutions of the odd equations. We conclude that the corresponding neutral states are degenerate. On the other hand, the point $e_2^{(4)}$ occurs only once as the solution of the even equations; and this indicates the non-degeneracy of the neutral state—a fact manifest from the uniqueness of the proper solution determined for this point in the earlier paper (Chandrasekhar 1967*a;* see particularly § IX).

V. THE NEUTRAL POINT ALONG THE JACOBIAN SEQUENCE BELONGING TO THE FOURTH HARMONICS

The six even equations provided by equations (33) and (34), after the elimination of the second-order virials with the aid of equation (24), represent a set of six homogeneous equations for the six fourth-order virials V_{1111}, V_{2222}, V_{3333}, V_{1122}, V_{2233}, and V_{3311}. However, only five of these six equations are linearly independent when they are supplemented by the further condition (26) that must also hold. Therefore, including equation (26), we have only six linear homogeneous equations between the six virials; and the vanishing of the determinant of these equations is the condition for the occurrence of a neutral point. It is found that the condition is met for a Jacobi ellipsoid for which

$$\cos^{-1}(a_3/a_1) = 75^\circ\!.068 ; \tag{66}$$

it is the same point that was determined earlier (Chandrasekhar 1967*b*) by a different method.[5] The uniqueness of this neutral point indicates the non-degeneracy of this neutral state—a fact that is in agreement with the unique proper solution that was explicitly determined for this point in the earlier paper.

Turning next to the equations that are odd in the indices 1 and 2, we find that they

[5] The value $\cos^{-1}(a_3/a_1) = 75^\circ\!.081$ found earlier differs slightly from the value (66); but the difference is not outside the limits of accuracy of the numerical evaluation.

provide no additional information: the six equations governing V_{1211}, V_{1222}, and V_{1233} (remaining after the elimination of $\delta\Pi_{12}$ and V_{12}) are all satisfied *identically* by the substitution

$$V_{1211} : V_{1222} : V_{1233} = 3a_1^2 : 3a_2^2 : a_3^2 \,, \tag{67}$$

by virtue of the properties of the Jacobi ellipsoid, namely, that

$$\Omega^2 = 2B_{12} \qquad \text{and} \qquad a_1^2 a_2^2 A_{12} = a_3^2 A_3 \,, \tag{68}$$

along the Jacobian sequence. The reason for this behavior is that the neutral mode of oscillation that characterizes the Maclaurin spheroid at its point of bifurcation persists through the entire Jacobian sequence.

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APPENDIX

The particular combinations of $\delta\mathfrak{W}_{ij;kl}$ and $\delta\mathfrak{W}_{ij;k;l}$ that occur in equations (29)–(34) can be written down by making use of the general formulae (21) and (22). In most cases, the resulting expressions can be simplified substantially if proper use is made of the various identities that relate the different index symbols. The expressions derived in this fashion are listed below.

$$\begin{aligned}
&2\delta\mathfrak{W}_{ii;ii} + \delta\mathfrak{W}_{ii;i;i} - (4\delta\mathfrak{W}_{ij;ij} + 2\delta\mathfrak{W}_{jj;ii} + 2\delta\mathfrak{W}_{ij;i;j} + \delta\mathfrak{W}_{jj;i;i}) \\
&\quad = 1.5a_i^2[5a_i^2 B_{iii} - (2a_i^2 + a_j^2)B_{iij}]V_{ii} \\
&\qquad +1.5a_i^2(a_i^2 B_{iij} - 3a_j^2 B_{ijj})V_{jj} + 1.5a_i^2(a_i^2 B_{iik} - a_j^2 B_{ijk})V_{kk} \\
&\qquad +[2B_{ij} - 4B_{ii} + 2a_j^2 B_{iij} - 4a_i^2 B_{iii} + a_i^2(a_i^2 + 2.75a_j^2)B_{iiij} - 8.75a_i^4 B_{iiii}]V_{iiii} \\
&\qquad +0.75a_i^2(5a_j^2 B_{ijjj} - a_i^2 B_{iijj})V_{jjjj} - 0.75a_i^2(a_i^2 B_{iikk} - a_j^2 B_{ijkk})V_{kkkk} \\
&\qquad -1.5a_i^2[5a_i^2 B_{iiik} - (2a_i^2 + a_j^2)B_{iijk}]V_{iikk} - 1.5a_i^2(a_i^2 B_{iijk} - 3a_j^2 B_{ijjk})V_{jjkk} \\
&\qquad +1.5\,[2B_{ij} + 2B_{jj} + 2(a_i^2 + a_j^2)B_{ijj} + a_i^2(2a_i^2 + 7a_j^2)B_{iijj} - 5a_i^4 B_{iiij}]V_{iijj} \,,
\end{aligned} \tag{A·1}$$

$$\begin{aligned}
&+ (4\delta\mathfrak{W}_{ij;ij} + 2\delta\mathfrak{W}_{jj;ii} + 2\delta\mathfrak{W}_{ij;i;j} + \delta\mathfrak{W}_{jj;i;i}) \\
&\qquad - (4\delta\mathfrak{W}_{ik;ik} + 2\delta\mathfrak{W}_{kk;ii} + 2\delta\mathfrak{W}_{ik;i;k} + \delta\mathfrak{W}_{kk;i;i}) \\
&\quad = 1.5a_i^2(a_j^2 - a_k^2)(B_{ijk} - 3a_i^2 B_{iijk})V_{ii} \\
&\qquad -1.5a_i^2(a_k^2 B_{ijk} - 3a_j^2 B_{ijj})V_{jj} + 1.5a_i^2(a_j^2 B_{ijk} - 3a_k^2 B_{ikk})V_{kk} \\
&\qquad +0.75a_i^2(a_j^2 - a_k^2)(5a_i^2 B_{iiijk} - B_{iijk})V_{iiii} \\
&\qquad -0.75a_i^2(5a_j^2 B_{ijjj} - a_k^2 B_{ijjk})V_{jjjj} + 0.75a_i^2(5a_k^2 B_{ikkk} - a_j^2 B_{ijkk})V_{kkkk} \\
&\qquad -4.5a_i^2(a_j^2 - a_k^2)C_{ijjkk}V_{jjkk} \\
&\qquad +[2B_{jk} - 5B_{ij} - 3B_{jj} + (a_i^2 + 2a_k^2)B_{ijk} - 3a_i^2 B_{iij} - 3(a_i^2 + a_j^2)B_{ijj} \\
&\qquad +4.5a_i^2 a_k^2 B_{iijk} - 3a_i^2(a_i^2 + 3.5a_j^2)B_{iijj}]V_{iijj} \\
&\qquad -[2B_{jk} - 5B_{ik} - 3B_{kk} + (a_i^2 + 2a_j^2)B_{ijk} - 3a_i^2 B_{iik} - 3(a_i^2 + a_k^2)B_{ikk} \\
&\qquad +4.5a_i^2 a_j^2 B_{iijk} - 3a_i^2(a_i^2 + 3.5a_k^2)B_{iikk}]V_{iikk} \,,
\end{aligned} \tag{A·2}$$

$$-(2\delta\mathfrak{W}_{ij;jj} + \delta\mathfrak{W}_{ij;j;j}) = 3a_j^4(-B_{ijj}V_{ij} + B_{iijj}V_{iiij} + B_{ijjk}V_{ijkk}) + (2B_{ij} + 2a_j^2B_{ijj} + 5a_j^4B_{ijjj})V_{ijjj}\,, \tag{A.3}$$

$$\begin{aligned}-(4\delta\mathfrak{W}_{ik;jk} &+ 2\delta\mathfrak{W}_{ij;kk} + 2\delta\mathfrak{W}_{ik;j;k} + \delta\mathfrak{W}_{ij;k;k})\\ &= 3a_j^2a_k^2(-B_{ijk}V_{ij} + B_{iijk}V_{iiij} + B_{ijjk}V_{ijjj})\\ &+ [2B_{ij} + 4B_{ik} + (2a_j^2 + a_k^2)B_{ijk} + 3a_k^2B_{ikk} + 3a_k^2(2a_j^2 + a_k^2)B_{ijkk}]V_{ijkk}\,,\end{aligned} \tag{A.4}$$

$$\begin{aligned}-(4\delta\mathfrak{W}_{ii;ij} &+ 2\delta\mathfrak{W}_{ij;ii} + 2\delta\mathfrak{W}_{ii;i;j} + \delta\mathfrak{W}_{ij;i;i})\\ &= 3a_i^2a_j^2[-(A_{ij} - 3a_i^2A_{iij})V_{ij} + (A_{ijj} - 3a_i^2A_{iijj})V_{ijjj} + (A_{ijk} - 3a_i^2A_{iijk})V_{ijkk}]\\ &+ 3(2A_i + a_i^2a_j^2A_{iij} - 5a_i^4a_j^2A_{iiij})V_{iiij}\,,\end{aligned} \tag{A.5}$$

$$\begin{aligned}-(2\delta\mathfrak{W}_{kk;ij} &+ 2\delta\mathfrak{W}_{ik;jk} + 2\delta\mathfrak{W}_{jk;ik} + \delta\mathfrak{W}_{kk;i;j} + \delta\mathfrak{W}_{ik;j;k} + \delta\mathfrak{W}_{jk;i;k})\\ &= 3a_i^2a_j^2a_k^2(A_{ijk}V_{ij} - A_{iijk}V_{ijii} - A_{ijjk}V_{ijjj})\\ &+ (6A_k - 9a_i^2a_j^2a_k^2A_{ijkk})V_{ijkk}\,.\end{aligned} \tag{A.6}$$

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