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THE POST-NEWTONIAN EFFECTS OF GENERAL RELATIVITY ON THE EQUILIBRIUM OF UNIFORMLY ROTATING BODIES
III. THE DEFORMED FIGURES OF THE JACOBI ELLIPSOIDS

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ABSTRACT

The effects of general relativity, in the post-Newtonian approximation, on the Jacobian figures of equilibrium of uniformly rotating homogeneous masses are determined. It is shown, for example, that the post-Newtonian figure is obtained by a deformation of the Jacobi ellipsoid by a suitable Lagrangian displacement cubic in the coordinates.

The solution of the post-Newtonian equations exhibits an *indeterminacy* at the point of bifurcation M_2, where the Jacobian sequence branches off from the Maclaurin sequence, and a *singularity* at a point J_4, where the axes of the Jacobi ellipsoid are in the ratios 1:0 2972:0.2575. The indeterminacy in the solution at M_2 arises from the fact that at this point the Maclaurin spheroid is neutral to an infinitesimal deformation proportional to $(x_1, -x_2, 0)$; and the singularity at J_4 arises from the fact that at this point the Jacobi ellipsoid is unstable to the deformation induced by the effects of general relativity.

I. INTRODUCTION

In Part II of this series of papers[1] (Chandrasekhar 1965*b*, 1967; these papers will be referred to hereafter as "Papers I and II," respectively) the deformation of the Maclaurin spheroids by the effects of general relativity in the post-Newtonian approximation was determined. In this paper we shall be concerned with the corresponding effects on the figures of the Jacobi ellipsoids.

The consideration of uniformly rotating non-axisymmetric figures (such as the Jacobi ellipsoids are) in the framework of the post-Newtonian equations of hydrodynamics as they have been established (Chandrasekhar 1965*a;* this paper will be referred to hereafter as "PNE") requires some care: in the coordinate system in which the equations are written, the figures are not stationary; they are stationary only in a frame that is rotating with the system. On these accounts, we shall find it convenient to transform first the equations of PNE to a system of moving axes in the sense of Greenhill (1880; see also Lamb 1932). This transformation is accomplished in § II; and in § III we begin the consideration of the deformed Jacobi ellipsoids.

II. THE EQUATIONS OF POST-NEWTONIAN HYDRODYNAMICS IN A SYSTEM OF MOVING AXES

The equations of hydrodynamics governing an ideal fluid, in the post-Newtonian approximation, are

$$\begin{aligned}\frac{\partial}{\partial t}(\sigma v_\alpha) + \frac{\partial}{\partial x_\beta}(\sigma v_\alpha v_\beta) + \frac{\partial}{\partial x_\alpha}\left[\left(1+\frac{2U}{c^2}\right)p\right] - \rho\frac{\partial U}{\partial x_\alpha} + \frac{4}{c^2}\rho\frac{d}{dt}(v_\alpha U - U_\alpha)\\ + \frac{4}{c^2}\rho v_\beta\frac{\partial U_\beta}{\partial x_\alpha} + \frac{1}{2c^2}\rho\frac{\partial^3\chi}{\partial x_\alpha\partial t^2} - \frac{2}{c^2}\rho\left(\phi\frac{\partial U}{\partial x_\alpha} + \frac{\partial\Phi}{\partial x_\alpha}\right) = 0\end{aligned} \tag{1}$$

and

$$\frac{\partial\sigma}{\partial t} + \frac{\partial}{\partial x_\alpha}(\sigma v_\alpha) + \frac{1}{c^2}\left(\rho\frac{\partial U}{\partial t} - \frac{\partial p}{\partial t}\right) = 0\,, \tag{2}$$

[1] The following misprints in Paper II may be noted here: in equation (24) the first terms in the expression for α_0 *should be* $\frac{1}{2}a_3^4 A_3^2$ (*instead of* $\frac{1}{2}a_3^4 A_3$); in equation (65) *read* $(i = 2, 3)$ *instead of* $(i = 1, 3)$; and in equation (69) the second summation is over $n = 2$ and 3 (and *not* $n = 1$ and 2). Also the running head for the article should have been "Post-Newtonian Effects" (and not "Newtonian Effects").

where

$$\sigma = \rho\left[1 + \frac{1}{c^2}\left(v^2 + 2U + \Pi + \frac{p}{\rho}\right)\right] \tag{3}$$

$$\phi = v^2 + U + \tfrac{1}{2}\Pi + \tfrac{3}{2}\frac{p}{\rho}, \tag{4}$$

and χ, U_a, and Φ are defined as solutions of the equations

$$\nabla^2\chi = -2U, \qquad \nabla^2 U_a = -4\pi G\rho v_a, \qquad \text{and} \qquad \nabla^2\Phi = -4\pi G\rho\phi; \tag{5}$$

also ρ denotes the density, p the pressure, $\rho\Pi$ the internal energy, v_a the components of the fluid velocity in the chosen frame, and U is the Newtonian gravitational potential determined by the distribution of ρ.

By a series of transformations in which use is made of equation (2) and of the Newtonian forms of the equations of motion in the terms which are explicitly post-Newtonian, it can be shown that

$$\frac{\partial}{\partial t}(\sigma v_a) + \frac{\partial}{\partial x_\beta}(\sigma v_a v_\beta) = \sigma\frac{d v_a}{dt} + \frac{v_a}{c^2}\left[\frac{dp}{dt} + \rho\frac{d}{dt}\left(\tfrac{1}{2}v^2 - U\right)\right]. \tag{6}$$

Inserting this result in equation (1) and replacing σ and ϕ by their equivalents (3) and (4), we find after some simplifications and rearrangements that it can be brought to the form

$$\begin{aligned}\rho\left[1+\frac{1}{c^2}(v^2+4U)\right]\frac{d v_a}{dt}+\left[1-\frac{1}{c^2}\left(\Pi+\frac{p}{\rho}\right)\right]\frac{\partial p}{\partial x_a}-\rho\left(1+\frac{2v^2}{c^2}\right)\frac{\partial U}{\partial x_a}+\frac{4}{c^2}\rho v_\beta\frac{\partial U_\beta}{\partial x_a}\\ +\frac{1}{c^2}\rho\frac{\partial}{\partial x_a}\left(\frac{1}{2}\frac{\partial^2\chi}{\partial t^2}-2\Phi\right)-\frac{4}{c^2}\rho\frac{d U_a}{dt}+\frac{v_a}{c^2}\left[\frac{dp}{dt}+\rho\frac{d}{dt}\left(\tfrac{1}{2}v^2+3U\right)\right]=0.\end{aligned} \tag{7}$$

And in place of equation (2), we shall choose the alternative form (PNE, eq. [147])

$$\frac{d\rho}{dt} + \rho\frac{\partial v_a}{\partial x_a} + \frac{1}{c^2}\frac{d}{dt}\left(\tfrac{1}{2}v^2 + 3U\right) = 0. \tag{8}$$

We shall now refer the equations of motion to a system of axes which is itself in motion. For the sake of simplicity, we shall restrict our transformation to the case when the origin of the system of coordinates is at rest while the orientation of the axes is subject to a rotation $\boldsymbol{\Omega}$ (which can be a function of time).[2]

Let $\boldsymbol{x}$ and $\boldsymbol{u}$ denote the position and the velocity of a fluid element in the moving frame. If $\boldsymbol{v}$ denotes the velocity of the same fluid element in the stationary frame but resolved along the *instantaneous* directions of the moving axes, then

$$\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{\Omega} \times \boldsymbol{x}. \tag{9}$$

More generally, if $\boldsymbol{f}(\boldsymbol{x}, t)$ is any vector defined in the moving frame and $\boldsymbol{f}^{(0)}$ is the *same* vector resolved along the instantaneous directions of the moving frame, then

$$\frac{d\boldsymbol{f}^{(0)}}{dt} = \frac{d\boldsymbol{f}}{dt} + \boldsymbol{\Omega} \times \boldsymbol{f}. \tag{10}$$

[2] The generalization to the case when the origin is also subject to motion presents no difficulties (cf. Lamb, *loc. cit.*).

We recover eq. [9] when f is identified with the position vector; and identifying $f^{(0)}$ with the acceleration vector, we have the relation

$$\left(\frac{d\boldsymbol{v}}{dt}\right)_{\text{inertial frame}}=\left(\frac{d\boldsymbol{v}}{dt}+\boldsymbol{\Omega}\times\boldsymbol{v}\right)_{\text{moving frame}}. \tag{11}$$

By resolving equation (7) along the instantaneous directions of the moving axes and making use of the relation (11), we can write

$$\begin{gathered}\rho\left[1+\frac{1}{c^2}(v^2+4U)\right]\left(\frac{dv_\alpha}{dt}+\epsilon_{\alpha\beta\gamma}\Omega_\beta v_\gamma\right)+\left[1-\frac{1}{c^2}\left(\Pi+\frac{p}{\rho}\right)\right]\frac{\partial p}{\partial x_\alpha}-\rho\left(1+\frac{2v^2}{c^2}\right)\frac{\partial U}{\partial x_\alpha}\\+\frac{4}{c^2}\rho v_\beta\frac{\partial U_\beta}{\partial x_\alpha}-\frac{4}{c^2}\rho\left(\frac{dU_\alpha}{dt}+\epsilon_{\alpha\beta\gamma}\Omega_\beta U_\gamma\right)+\frac{1}{c^2}\rho\frac{\partial}{\partial x_\alpha}\left[\tfrac{1}{2}\left(\frac{d}{dt}-v_\beta\frac{\partial}{\partial x_\beta}\right)^2\chi-2\Phi\right]\\\frac{v_\alpha}{c^2}\left[\frac{dp}{dt}+\rho\frac{d}{dt}(\tfrac{1}{2}v^2+3U)\right]=0,\end{gathered} \tag{12}$$

where $\boldsymbol{v}$ is related to $\boldsymbol{u}$ (the velocity of the fluid element in the moving frame) by equation (10) and

$$\frac{d}{dt}=\frac{\partial}{\partial t}+u_\alpha\frac{\partial}{\partial x_\alpha}. \tag{13}$$

Also U_α is now governed by the equation

$$\nabla^2 U_\alpha=-4\pi G\rho(u_\alpha+\epsilon_{\alpha\beta\gamma}\Omega_\beta x_\gamma). \tag{14}$$

In view of equation (9), the equation of continuity (8) takes the form

$$\frac{d\rho}{dt}+\rho\frac{\partial u_\alpha}{\partial x_\alpha}+\frac{1}{c^2}\left[v_\alpha\left(\frac{dv_\alpha}{dt}+\epsilon_{\alpha\beta\gamma}\Omega_\beta v_\gamma\right)+3\frac{dU}{dt}\right]=0. \tag{15}$$

a) Equations Governing the Hydrostatic Equilibrium of a Uniformly Rotating System

If the system appears stationary and there are no fluid motions in the moving frame, then in equations (12) and (15) we may put

$$\frac{d}{dt}=0,\qquad \boldsymbol{u}=0,\qquad\text{and}\qquad \boldsymbol{v}=\boldsymbol{\Omega}\times\boldsymbol{x}. \tag{16}$$

Equation (12) then becomes

$$\begin{gathered}\rho\left\{1+\frac{1}{c^2}[(\boldsymbol{\Omega}\times\boldsymbol{x})^2+4U]\right\}\epsilon_{\alpha\beta\gamma}\epsilon_{\gamma\mu\nu}\Omega_\beta\Omega_\mu x_\nu+\left[1-\frac{1}{c^2}\left(\Pi+\frac{p}{\rho}\right)\right]\frac{\partial p}{\partial x_\alpha}\\-\rho\left[1+\frac{2}{c^2}(\boldsymbol{\Omega}\times\boldsymbol{x})^2\right]\frac{\partial U}{\partial x_\alpha}+\frac{4}{c^2}\rho\epsilon_{\beta\mu\nu}\Omega_\mu x_\nu\frac{\partial U_\beta}{\partial x_\alpha}-\frac{4}{c^2}\rho\epsilon_{\alpha\ \nu}\Omega_\mu U_\nu\\+\frac{1}{c^2}\rho\frac{\partial}{\partial x_\alpha}\left[\tfrac{1}{2}\left(\epsilon_{\alpha\beta\gamma}\Omega_\alpha x_\beta\frac{\partial}{\partial x_\gamma}\right)^2\chi-2\Phi\right]=0,\end{gathered} \tag{17}$$

while equation (15) is identically satisfied.

We shall now suppose that $\boldsymbol{\Omega}$ is directed along the x_3-axis Then (cf. Paper II, eqs. [6]–[8])

$$U_1=-\Omega\mathfrak{D}_2,\qquad U_2=+\Omega\mathfrak{D}_1,\qquad\text{and}\qquad U_3=0. \tag{18}$$

where $\mathfrak{D}_\alpha$ is the Newtonian potential due to the "fictitious" density ρx_α. With U_α given by equations (18), it can be readily verified that

$$\epsilon_{\beta\mu\nu}\Omega_\mu x_\nu \frac{\partial U_\beta}{\partial x_\alpha} - \epsilon_{\alpha\mu\nu}\Omega_\mu U_\nu = \Omega^2 \frac{\partial}{\partial x_\alpha}(x_1\mathfrak{D}_1 + x_2\mathfrak{D}_2) . \tag{19}$$

We also have

$$\epsilon_{\alpha\beta\gamma}\epsilon_{\gamma\mu\nu}\Omega_\beta\Omega_\mu x_\nu = -\tfrac{1}{2}\frac{\partial}{\partial x_\alpha}\varpi^2\Omega^2 \qquad (\varpi^2 = x_1^2 + x_2^2) . \tag{20}$$

Inserting the foregoing relations in equation (17), we find after some further reductions and rearrangements that it takes the form

$$\begin{aligned}\left[1 - \frac{1}{c^2}\left(\Pi + \frac{p}{\rho}\right)\right]\frac{\partial p}{\partial x_\alpha} = \rho \frac{\partial}{\partial x_\alpha}\Big\{ U + \tfrac{1}{2}\Omega^2\varpi^2 + \frac{1}{c^2}[\tfrac{1}{4}\Omega^4\varpi^4 + 2U\Omega^2\varpi^2 \\ + 2\Phi - 4\Omega^2(x_1\mathfrak{D}_1 + x_2\mathfrak{D}_2) - \tfrac{1}{2}(\boldsymbol{\Omega}\times \boldsymbol{x}\cdot \text{grad})^2\chi]\Big\} .\end{aligned} \tag{21}$$

The term in χ on the right-hand side of equation (21) can also be expressed in terms of the potentials $\mathfrak{D}_\alpha$. The term in question is

$$(\boldsymbol{\Omega}\times \boldsymbol{x}\cdot \text{grad})^2\chi = \Omega^2\left(x_2^2\frac{\partial^2\chi}{\partial x_1^2} + x_1^2\frac{\partial^2\chi}{\partial x_2^2} - 2x_1x_2\frac{\partial^2\chi}{\partial x_1\partial x_2} - x_2\frac{\partial\chi}{\partial x_2} - x_1\frac{\partial\chi}{\partial x_1}\right) . \tag{22}$$

With the aid of the relations given in Chandrasekhar and Lebovitz (1962*a*, eqs. [5]–[7]), equation (22) can be reduced to the form

$$(\boldsymbol{\Omega}\times \boldsymbol{x}\cdot \text{grad})^2\chi = \Omega^2\left[x_2^2\frac{\partial\mathfrak{D}_1}{\partial x_1} + x_1^2\frac{\partial\mathfrak{D}_2}{\partial x_2} - x_1x_2\left(\frac{\partial\mathfrak{D}_1}{\partial x_2} + \frac{\partial\mathfrak{D}_2}{\partial x_1}\right) - (x_1\mathfrak{D}_1 + x_2\mathfrak{D}_2)\right] . \tag{23}$$

Thus, the equation of hydrostatic equilibrium of a uniformly rotating mass in the post-Newtonian approximation takes the form

$$\begin{aligned}\left[1 - \frac{1}{c^2}\left(\Pi + \frac{p}{\rho}\right)\right]\frac{\partial p}{\partial x_\alpha} = \rho \frac{\partial}{\partial x_\alpha}\Big\{ U + \tfrac{1}{2}\Omega^2\varpi^2 + \frac{1}{c^2}\Big[\tfrac{1}{4}\Omega^4\varpi^4 + 2U\Omega^2\varpi^2 + 2\Phi \\ - \tfrac{7}{2}\Omega^2(x_1\mathfrak{D}_1 + x_2\mathfrak{D}_2) - \tfrac{1}{2}\Omega^2\left(x_2^2\frac{\partial\mathfrak{D}_1}{\partial x_1} + x_1^2\frac{\partial\mathfrak{D}_2}{\partial x_2} - x_1x_2\frac{\partial\mathfrak{D}_1}{\partial x_2} - x_1x_2\frac{\partial\mathfrak{D}_2}{\partial x_1}\right)\Big]\Big\} .\end{aligned} \tag{24}$$

Equation (24) is a generalization of the relation first derived by Krefetz (1966; cf. also Paper I, eq. [12]) under the assumption of axisymmetry.

b) The Integral of Equation (24) in the Case of Convective Equilibrium

If convective equilibrium prevails, then it follows from one of Maxwell's thermodynamic relations that

$$\frac{\partial p}{\partial x_\alpha} = \rho\frac{\partial}{\partial x_\alpha}\left(\Pi + \frac{p}{\rho}\right) , \tag{25}$$

and the right-hand side of equation (24) can be written in the form

$$\rho\frac{\partial}{\partial x_\alpha}\left[\left(\Pi + \frac{p}{\rho}\right) - \frac{1}{2c^2}\left(\Pi + \frac{p}{\rho}\right)^2\right] ; \tag{26}$$

and the equation directly integrates to give

$$\Pi+\frac{p}{\rho}=U+\tfrac{1}{2}\Omega^2\varpi^2+\frac{1}{c^2}\Big\{\tfrac{1}{2}\Big(\Pi+\frac{p}{\rho}\Big)^2+\tfrac{1}{4}\Omega^4\varpi^4+2U\Omega^2\varpi^2+2\Phi-\tfrac{7}{2}\Omega^2(x_1\mathfrak{D}_1+x_2\mathfrak{D}_2)$$
$$-\tfrac{1}{2}\Omega^2\Big[x_2^2\frac{\partial\mathfrak{D}_1}{\partial x_1}+x_1^2\frac{\partial\mathfrak{D}_2}{\partial x_2}-x_1x_2\Big(\frac{\partial\mathfrak{D}_1}{\partial x_2}+\frac{\partial\mathfrak{D}_2}{\partial x_1}\Big)\Big]\Big\}+\text{constant}. \tag{27}$$

III. THE TERM THAT IS EXPLICITLY POST-NEWTONIAN IN THE PRESSURE INTEGRAL FOR THE DEFORMED JACOBI ELLIPSOIDS

In the rest of this paper, we shall be concerned with configurations in which the energy density $\epsilon(=\rho c^2+\Pi)$ is a constant. This assumption, that ϵ is a constant, is formally equivalent to the assumption

$$\rho=\text{constant} \qquad \text{and} \qquad \Pi=0, \tag{28}$$

and the assignment to ρ the meaning of ϵ/c^2. On this understanding, the integral (27) of the equation of hydrostatic equilibrium takes the form

$$\frac{p}{\rho}=U+\tfrac{1}{2}\Omega^2\varpi^2+\frac{1}{c^2}\Big\{\tfrac{1}{2}\Big(\frac{p}{\rho}\Big)^2+\tfrac{1}{4}\Omega^4\varpi^4+2U\Omega^2\varpi^2+2\Phi-\tfrac{7}{2}\Omega^2(x_1\mathfrak{D}_1+x_2\mathfrak{D}_2)$$
$$-\tfrac{1}{2}\Omega^2\Big[x_2^2\frac{\partial\mathfrak{D}_1}{\partial x_1}+x_1^2\frac{\partial\mathfrak{D}_2}{\partial x_2}-x_1x_2\Big(\frac{\partial\mathfrak{D}_1}{\partial x_2}+\frac{\partial\mathfrak{D}_2}{\partial x_1}\Big)\Big]\Big\}+\text{constant}. \tag{29}$$

The problem now is to determine the equation governing the surface of the equilibrium figure such that the pressure given by equation (29) vanishes identically on it.

In evaluating the term that is explicitly post-Newtonian in equation (29), we may legitimately use relations and equations that are valid in the Newtonian limit when the equilibrium figure is a Jacobian ellipsoid. The relevant relations are (cf. Paper II, eqs. [15]–[20])

$$\frac{p}{\rho}=\pi G\rho a_3^2A_3\Big(1-\sum_{\mu=1}^{3}\frac{x_\mu^2}{a_\mu^2}\Big), \tag{30}$$

$$U=\pi G\rho\Big(I-\sum_{\mu=1}^{3}A_\mu x_\mu^2\Big) \qquad \Big(I=\sum_{\mu=1}^{3}A_\mu a_\mu^2\Big), \tag{31}$$

$$\phi=\pi G\rho\Big[(I+\tfrac{3}{2}a_3^2A_3)+\Big(\frac{7\Omega^2}{4\pi G\rho}-\tfrac{5}{2}A_1\Big)x_1^2+\Big(\frac{7\Omega^2}{4\pi G\rho}-\tfrac{5}{2}A_2\Big)x_2^2-\tfrac{5}{2}A_3x_3^2\Big], \tag{32}$$

and

$$\frac{\Phi}{\pi G\rho}=(I+\tfrac{3}{2}a_3^2A_3)U+\Big(\frac{7\Omega^2}{4\pi G\rho}-\tfrac{5}{2}A_1\Big)\mathfrak{D}_{11}+\Big(\frac{7\Omega^2}{4\pi G\rho}-\tfrac{5}{2}A_2\Big)\mathfrak{D}_{22}-\tfrac{5}{2}A_3\mathfrak{D}_{33}, \tag{33}$$

where the various symbols have the same meanings as in Paper II. In particular, it is to be noted that the index symbols are so normalized that $\Sigma A_a=2$. Inserting the foregoing relations in the terms that occur with a factor $1/c^2$ in equation (29) and making use of the known expressions for $\mathfrak{D}_a$ and $\mathfrak{D}_{a\beta}$ (Chandrasekhar and Lebovitz 1962*b*, eqs. [49], [68], [70]) we find that the equation takes the form

$$\frac{p}{\rho}=U+\tfrac{1}{2}\Omega^2\varpi^2+\frac{(\pi G\rho)^2}{c^2}\Big(a_0+\sum_{\mu=1}^{3}a_\mu x_\mu^2+\sum_{\mu=1}^{3}a_{\mu\mu}x_\mu^4+\sum_{\mu\nu}^{12,23,31}a_{\mu\nu}x_\mu^2x_\nu^2\Big), \tag{34}$$

where

$$
\begin{aligned}
\alpha_0 &= \tfrac{1}{2} a_3^4 A_3^2 - \tfrac{5}{4} a_3^2 A_3 B_3 + I(2I + 3a_3^2 A_3) + \tfrac{1}{4} a_1^2 (\tfrac{7}{2}\Omega^2 - 5A_1) B_1 \\
&\qquad + \tfrac{1}{4} a_2^2 (\tfrac{7}{2}\Omega^2 - 5A_2) B_2 , \\
\alpha_1 &= -\frac{a_3^4}{a_1^2} A_3^2 - \tfrac{1}{2}\Omega^2 (a_2^2 A_2 - a_1^2 A_1) + 2\Omega^2 (I - 2a_1^2 A_1) - (2I + 3a_3^2 A_3) A_1 \\
&\qquad + (\tfrac{7}{2}\Omega^2 - 5A_1)(a_1^4 A_{11} - \tfrac{1}{2} a_1^2 B_{11}) - \tfrac{1}{2} a_2^2 B_{12} (\tfrac{7}{2}\Omega^2 - 5A_2) + \tfrac{5}{2} a_3^2 A_3 B_{13} , \\
\alpha_2 &= -\frac{a_3^4}{a_2^2} A_3^2 + \tfrac{1}{2}\Omega^2 (a_2^2 A_2 - a_1^2 A_1) + 2\Omega^2 (I - 2a_2^2 A_2) - (2I + 3a_3^2 A_3) A_2 \\
&\qquad + (\tfrac{7}{2}\Omega^2 - 5A_2)(a_2^4 A_{22} - \tfrac{1}{2} a_2^2 B_{22}) - \tfrac{1}{2} a_1^2 B_{12} (\tfrac{7}{2}\Omega^2 - 5A_1) + \tfrac{5}{2} a_3^2 A_3 B_{23} , \\
\alpha_3 &= - a_3^2 A_3^2 - (2I + 3a_3^2 A_3) A_3 - \tfrac{1}{2} a_1^2 B_{13} (\tfrac{7}{2}\Omega^2 - 5A_1) \\
&\qquad - \tfrac{1}{2} a_2^2 B_{23} (\tfrac{7}{2}\Omega^2 - 5A_2) - 5A_3 (a_3^4 A_{33} - \tfrac{1}{2} a_3^2 B_{33}) , \\
\alpha_{12} &= \frac{a_3^4 A_3^2}{a_1^2 a_2^2} + \Omega^2 [\tfrac{1}{2}\Omega^2 - 2(A_1 + A_2) + 4(a_1^2 + a_2^2) A_{12}] - \tfrac{5}{2} a_3^2 A_3 B_{123} \\
&\qquad + (\tfrac{7}{2}\Omega^2 - 5A_1)(-a_1^4 A_{112} + \tfrac{1}{2} a_1^2 B_{112}) + (\tfrac{7}{2}\Omega^2 - 5A_2)(-a_2^4 A_{122} + \tfrac{1}{2} a_2^2 B_{122}) \\
&\qquad - \tfrac{3}{2}\Omega^2 [(a_1^2 + a_2^2) A_{12} - a_1^2 A_{11} - a_2^2 A_{22}] , \qquad (35) \\
\alpha_{23} &= \frac{a_3^2 A_3^2}{a_2^2} + \Omega^2 (4a_2^2 A_{23} - 2A_3) + (\tfrac{7}{2}\Omega^2 - 5A_2)(-a_2^4 A_{223} + \tfrac{1}{2} a_2^2 B_{223}) \\
&\qquad + (\tfrac{7}{2}\Omega^2 - 5A_1)(\tfrac{1}{2} a_1^2 B_{123}) - 5A_3(-a_3^4 A_{332} + \tfrac{1}{2} a_3^2 B_{332}) - \tfrac{1}{2}\Omega^2 (a_2^2 A_{23} - a_1^2 A_{13}) , \\
\alpha_{31} &= \frac{a_3^2 A_3^2}{a_1^2} + \Omega^2 (4a_1^2 A_{31} - 2A_3) + (\tfrac{7}{2}\Omega^2 - 5A_1)(-a_1^4 A_{113} + \tfrac{1}{2} a_1^2 B_{113}) \\
&\qquad + (\tfrac{7}{2}\Omega^2 - 5A_2)(\tfrac{1}{2} a_2^2 B_{123}) - 5A_3(-a_3^4 A_{331} + \tfrac{1}{2} a_3^2 B_{331}) + \tfrac{1}{2}\Omega^2 (a_2^2 A_{23} - a_1^2 A_{13}) , \\
\alpha_{11} &= \tfrac{1}{2} \frac{a_3^4 A_3^2}{a_1^4} + \Omega^2 (\tfrac{1}{4}\Omega^2 - 2A_1 + 4a_1^2 A_{11}) + (\tfrac{7}{2}\Omega^2 - 5A_1)(-a_1^4 A_{111} + \tfrac{1}{4} a_1^2 B_{111}) \\
&\qquad + (\tfrac{7}{2}\Omega^2 - 5A_2)(\tfrac{1}{4} a_2^2 B_{112}) - \tfrac{5}{4} a_3^2 A_3 B_{113} - \tfrac{1}{2}\Omega^2 (a_1^2 A_{11} - a_2^2 A_{12}) , \\
\alpha_{22} &= \tfrac{1}{2} \frac{a_3^4 A_3^2}{a_2^4} + \Omega^2 (\tfrac{1}{4}\Omega^2 - 2A_2 + 4a_2^2 A_{22}) + (\tfrac{7}{2}\Omega^2 - 5A_2)(-a_2^4 A_{222} + \tfrac{1}{4} a_2^2 B_{222}) \\
&\qquad + (\tfrac{7}{2}\Omega^2 - 5A_1)(\tfrac{1}{4} a_1^2 B_{221}) - \tfrac{5}{4} a_3^2 A_3 B_{223} - \tfrac{1}{2}\Omega^2 (a_2^2 A_{22} - a_1^2 A_{12}) , \\
\alpha_{33} &= \tfrac{1}{2} A_3^2 + (\tfrac{7}{2}\Omega^2 - 5A_1)(\tfrac{1}{4} a_1^2 B_{133}) + (\tfrac{7}{2}\Omega^2 - 5A_2)(\tfrac{1}{4} a_2^2 B_{233}) \\
&\qquad - 5A_3(-a_3^4 A_{333} + \tfrac{1}{4} a_3^2 B_{333}) .
\end{aligned}
$$

(Ω^2 is measured in the unit $\pi G\rho$ in the terms on the right-hand sides of the foregoing equations.)

It should be noted that in equation (34), U is the Newtonian gravitional potential of the deformed (as yet unknown) figure which the body assumes in the post-Newtonian approximation. Similarly, Ω (in the "centrifugal term" $\frac{1}{2}\Omega^2\varpi^2$) is not necessarily the same (to order $1/c^2$) as that of the Jacobi ellipsoid which was used in the evaluation of the terms of order $1/c^2$.

The Virial Relations

With the expression for p/ρ given by equation (34), the equations of hydrostatic equilibrium (for the case that is being considered) are

$$\frac{\partial p}{\partial x_1}=\rho\frac{\partial U}{\partial x_1}+\rho\Omega^2 x_1+\frac{2(\pi G\rho)^2\rho}{c^2}x_1(\alpha_1+\alpha_{12}x_2^2+\alpha_{31}x_3^2+2\alpha_{11}x_1^2), \tag{36}$$

$$\frac{\partial p}{\partial x_2}=\rho\frac{\partial U}{\partial x_2}+\rho\Omega^2 x_2+\frac{2(\pi G\rho)^2\rho}{c^2}x_2(\alpha_2+\alpha_{12}x_1^2+\alpha_{23}x_3^2+2\alpha_{22}x_2^2) \tag{37}$$

and

$$\frac{\partial p}{\partial x_3}=\rho\frac{\partial U}{\partial x_3}+\frac{2(\pi G\rho)^2\rho}{c^2}x_3(\alpha_3+\alpha_{31}x_1^2+\alpha_{23}x_2^2+2\alpha_{33}x_3^2). \tag{38}$$

Multiplying equations (36)–(38) by x_1, x_2, and x_3, respectively, and integrating over the volume of the configuration, we obtain

$$\begin{aligned}
&-\int_V p\,dx\\
&=\mathfrak{W}_{11}+\Omega^2 I_{11}+\frac{1}{c^2}(8\pi a_1a_2a_3\rho)(\pi G\rho)^2\left(\frac{a_1^2}{15}\alpha_1+\frac{a_1^2a_2^2}{105}\alpha_{12}+\frac{a_3^2a_1^2}{105}\alpha_{31}+\frac{2a_1^4}{35}\alpha_{11}\right)\\
&=\mathfrak{W}_{22}+\Omega^2 I_{22}+\frac{1}{c^2}(8\pi a_1a_2a_3\rho)(\pi G\rho)^2\left(\frac{a_2^2}{15}\alpha_2+\frac{a_1^2a_2^2}{105}\alpha_{12}+\frac{a_2^2a_3^2}{105}\alpha_{23}+\frac{2a_2^4}{35}\alpha_{22}\right)\\
&=\mathfrak{W}_{33}+\frac{1}{c^2}(8\pi a_1a_2a_3\rho)(\pi G\rho)^2\left(\frac{a_3^2}{15}\alpha_3+\frac{a_3^2a_1^2}{105}\alpha_{31}+\frac{a_2^2a_3^2}{105}\alpha_{23}+\frac{2a_3^4}{35}\alpha_{33}\right),
\end{aligned} \tag{39}$$

where $\mathfrak{W}_{ij}$ and I_{ij} denote the potential energy and the moment of inertia tensors.

From equations (39) we obtain the eliminant relations

$$\Omega^2=\frac{\mathfrak{W}_{33}-\mathfrak{W}_{11}}{I_{11}}+\frac{(\pi G\rho)^2}{c^2}E_{31}=\frac{\mathfrak{W}_{33}-\mathfrak{W}_{22}}{I_{22}}+\frac{(\pi G\rho)^2}{c^2}E_{23}=\frac{\mathfrak{W}_{22}-\mathfrak{W}_{11}}{I_{11}-I_{22}}+\frac{(\pi G\rho)^2}{c^2}E_{12}, \tag{40}$$

where

$$\begin{aligned}
E_{31}&=\frac{2}{a_1^2}(a_3^2\alpha_3-a_1^2\alpha_1+\tfrac{1}{7}a_2^2a_3^2\alpha_{23}-\tfrac{1}{7}a_1^2a_2^2\alpha_{12}+\tfrac{6}{7}a_3^4\alpha_{33}-\tfrac{6}{7}a_1^4\alpha_{11}),\\
E_{23}&=\frac{2}{a_2^2}(a_3^2\alpha_3-a_2^2\alpha_2+\tfrac{1}{7}a_3^2a_1^2\alpha_{31}-\tfrac{1}{7}a_1^2a_2^2\alpha_{12}+\tfrac{6}{7}a_3^4\alpha_{33}-\tfrac{6}{7}a_2^4\alpha_{22}),\\
E_{12}&=\frac{2}{a_1^2-a_2^2}(a_2^2\alpha_2-a_1^2\alpha_1+\tfrac{1}{7}a_2^2a_3^2\alpha_{23}-\tfrac{1}{7}a_3^2a_1^2\alpha_{31}+\tfrac{6}{7}a_2^4\alpha_{22}-\tfrac{6}{7}a_1^4\alpha_{11}),
\end{aligned} \tag{41}$$

and

$$(a_1^2-a_2^2)E_{12}=a_1^2E_{31}-a_2^2E_{23}. \tag{42}$$

IV. THE NATURE OF THE POST-NEWTONIAN DEFORMATION

We shall suppose that the post-Newtonian figure is obtained by a deformation of the classical Jacobi ellipsoid by the application of a suitable Lagrangian displacement at each point of its interior and boundary. Since the density has to remain constant, before and after the deformation, the displacement must, in all events, be divergence free. For the sake of convenience and also for emphasizing that we are considering an effect of order $1/c^2$, we shall denote the required Lagrangian displacement by

$$\frac{\pi G \rho a_1^2}{c^2}\xi(x). \tag{43}$$

In virtue of this displacement, the bounding surface,

$$S_J=\sum_{\mu=1}^{3}\frac{x_\mu^2}{a_\mu^2}-1=0, \tag{44}$$

of the Jacobi ellipsoid becomes

$$S=S_J-\frac{\pi G \rho a_1^2}{c^2}\xi_\mu\frac{\partial S_J}{\partial x_\mu}=0; \tag{45}$$

or explicitly

$$S=\sum_{\mu=1}^{3}\frac{x_\mu^2}{a_\mu^2}-1-\frac{2\pi G \rho a_1^2}{c^2}\sum_{\mu=1}^{3}\frac{\xi_\mu x_\mu}{a_\mu^2}=0. \tag{46}$$

The fact that the post-Newtonian term, already present, in the expression for p/ρ is quadratic in x_μ^2 restricts us to displacements which are linearly dependent on the following eight:

$$\xi^{(1)}=(x_1,0,-x_3), \qquad \xi^{(2)}=(0,x_2,-x_3), \tag{47}$$

$$\xi^{(3)}=\frac{1}{a_1^2}(\tfrac{1}{3}x_1^3,-x_1^2x_2,0), \xi^{(4)}=\frac{1}{a_1^2}(0,\tfrac{1}{3}x_2^3,-x_2^2x_3), \xi^{(5)}=\frac{1}{a_1^2}(-x_3^2x_1,0,\tfrac{1}{3}x_3^3), \tag{48}$$

$$\xi^{(6)}=\frac{1}{a_1^2}(0,x_1^2x_2,-x_1^2x_3), \xi^{(7)}=\frac{1}{a_1^2}(-x_2^2x_1,0,x_2^2x_3), \xi^{(8)}=\frac{1}{a_1^2}(x_3^2x_1,-x_3^2x_2,0). \tag{49}$$

While the foregoing displacements are linearly independent as vectors, they are *not* linearly independent modulo the ellipsoid[3] S_J. Thus

$$\frac{1}{a_1^2}\xi^{(6)}+\frac{1}{a_2^2}\xi^{(7)}+\frac{1}{a_3^2}\xi^{(8)}\equiv 0\,(\text{mod } S_J); \tag{50}$$

for a linear superposition of the displacements $\xi^{(6)}$, $\xi^{(7)}$, and $\xi^{(8)}$ in the proportion $a_1^{-2}:a_2^{-2}:a_3^{-2}$ does not deform the boundary S_J and there is in consequence no change in the gravitational potential. In addition to equation (50), we have the relations

$$\frac{1}{a_1^2}\xi^{(1)}\equiv\left[+\frac{3}{a_1^2}\xi^{(3)}-\frac{3}{a_3^2}\xi^{(5)}+\frac{3}{a_1^2}\xi^{(6)}-\frac{1}{a_2^2}\xi^{(7)}\right](\text{mod } S_J) \tag{51}$$

and

$$\frac{1}{a_1^2}\xi^{(2)}\equiv\left[-\frac{3}{a_3^2}\xi^{(5)}+\frac{3}{a_2^2}\xi^{(4)}-\frac{3}{a_3^2}\xi^{(8)}+\frac{1}{a_1^2}\xi^{(6)}\right](\text{mod } S_J). \tag{52}$$

The relation (51) follows, for example, from the fact that *to the first order in* ϵ, *the equations*

$$\sum_{\mu=1}^{3}\frac{x_\mu^2}{a_\mu^2}=1+\epsilon\left(\frac{x_1^2}{a_1^2}-\frac{x_3^2}{a_3^2}\right)=1+\epsilon\xi_\mu^{(1)}\frac{\partial S_J}{\partial x_\mu} \tag{53}$$

[3] Precisely, a set of vectors is said to be linearly independent modulo the ellipsoid if there is no nontrivial combination of them which has a vanishing component normal to S_J

and

$$\sum_{\mu=1}^{3} \frac{x_\mu^2}{a_\mu^2} = 1 + \epsilon \left(\frac{x_1^2}{a_1^2} - \frac{x_3^2}{a_3^2} \right) \sum_{\mu=1}^{3} \frac{x_\mu^2}{a_\mu^2} \tag{54}$$

define the same surface. Thus, rearranging the second term on the right-hand side of equation (54), we have

$$\begin{aligned} \left(\frac{x_1^2}{a_1^2} - \frac{x_3^2}{a_3^2} \right) \sum_{\mu=1}^{3} \frac{x_\mu^2}{a_\mu^2} &= \frac{3}{a_1^2} \left(\frac{x_1^4}{3a_1^2} - \frac{x_1^2 x_2^2}{a_2^2} \right) - \frac{3}{a_3^2} \left(\frac{x_3^4}{3a_3^2} - \frac{x_3^2 x_1^2}{a_1^2} \right) \\ &+ \frac{3}{a_1^2} \left(\frac{x_1^2 x_2^2}{a_2^2} - \frac{x_1^2 x_3^2}{a_3^2} \right) - \frac{1}{a_2^2} \left(\frac{x_2^2 x_3^2}{a_3^2} - \frac{x_2^2 x_1^2}{a_1^2} \right) \\ &= a_1^2 \left[\frac{3}{a_1^2} \xi_\mu^{(3)} - \frac{3}{a_3^2} \xi_\mu^{(5)} + \frac{3}{a_1^2} \xi_\mu^{(6)} - \frac{1}{a_2^2} \xi_\mu^{(7)} \right] \frac{\partial S_J}{\partial x_\mu}; \end{aligned} \tag{55}$$

and the relation in question follows. The relation (52) can be similarly established. Thus, *only five of the eight displacements* $\xi^{(1)}, \ldots, \xi^{(8)}$ *are linearly independent modulo the ellipsoid* S_J. And we shall select $\xi^{(1)}, \ldots, \xi^{(5)}$ as the displacements in terms of which to express the Lagrangian displacement ξ that is to serve the purpose of deforming S_J to its post-Newtonian figure.

We shall write then

$$\xi = \sum_{i=1}^{5} S_i \xi^{(i)}, \tag{56}$$

where $S_i (i = 1, \ldots, 5)$ are constants to be determined. For a displacement of this chosen form, the equation of the bounding surface is

$$\begin{aligned} S(x) = \sum_{\mu=1}^{3} \frac{x_\mu^2}{a_\mu^2} - 1 - \frac{2\pi G\rho}{c^2} \Big[S_1 a_1^2 \left(\frac{x_1^2}{a_1^2} - \frac{x_3^2}{a_3^2} \right) + S_2 a_1^2 \left(\frac{x_2^2}{a_2^2} - \frac{x_3^2}{a_3^2} \right) \\ + S_3 \left(\frac{x_1^4}{3a_1^2} - \frac{x_1^2 x_2^2}{a_2^2} \right) + S_4 \left(\frac{x_2^4}{3a_2^2} - \frac{x_2^2 x_3^2}{a_3^2} \right) + S_5 \left(\frac{x_3^4}{3a_3^2} - \frac{x_3^2 x_1^2}{a_1^2} \right) \Big] = 0. \end{aligned} \tag{57}$$

V. THE RELATION BETWEEN THE ANGULAR VELOCITIES OF THE NEWTONIAN AND THE POST-NEWTONIAN CONFIGURATIONS

Since the post-Newtonian configuration is to be obtained by the deformation of a Jacobi ellipsoid by the application of a Lagrangian displacement, we may write

$$\mathfrak{W}_{\alpha\beta} = \mathfrak{W}_{\alpha\beta}(J) + \delta\mathfrak{W}_{\alpha\beta} \quad \text{and} \quad I_{\alpha\beta} = I_{\alpha\beta}(J) + \delta I_{\alpha\beta}, \tag{58}$$

where $\mathfrak{W}_{\alpha\beta}$ and $I_{\alpha\beta}$ refer to the post-Newtonian configuration and $\delta\mathfrak{W}_{\alpha\beta}$ and $\delta I_{\alpha\beta}$ are the first variations in $\mathfrak{W}_{\alpha\beta}(J)$ and $I_{\alpha\beta}(J)$ (of the Jacobian ellipsoid) caused by the deformation. The insertion of the foregoing relations in the virial equations (40) gives

$$\begin{aligned} \delta\Omega^2 = \Omega^2 - \Omega_J^2 &= \frac{1}{I_{11}} [(\delta\mathfrak{W}_{33} - \delta\mathfrak{W}_{11}) - \Omega_J^2 \delta I_{11}] + \frac{(\pi G\rho)^2}{c^2} E_{31} \\ &= \frac{1}{I_{22}} [(\delta\mathfrak{W}_{33} - \delta\mathfrak{W}_{22}) - \Omega_J^2 \delta I_{22}] + \frac{(\pi G\rho)^2}{c^2} E_{23} \\ &= \frac{1}{I_{11} - I_{22}} [(\delta\mathfrak{W}_{22} - \delta\mathfrak{W}_{11}) - \Omega_J^2 (\delta I_{11} - \delta I_{22})] + \frac{(\pi G\rho)^2}{c^2} E_{12}, \end{aligned} \tag{59}$$

where it may be recalled that

$$\Omega_J^2 = 2B_{12}\pi G\rho \,. \tag{60}$$

From a relation quoted in Paper II (eqs. [49] and [50]) we find that

$$\frac{\delta \mathfrak{W}_{\alpha\alpha} - \delta \mathfrak{W}_{\beta\beta}}{\pi G\rho} = -(3B_{\alpha\alpha} - B_{\alpha\beta})V_{\alpha\alpha} + (3B_{\beta\beta} - B_{\alpha\beta})V_{\beta\beta} + (B_{\beta\gamma} - B_{\alpha\gamma})V_{\gamma\gamma} \,, \tag{61}$$

where

$$V_{\alpha\alpha} = 2\int_V \rho\xi_\alpha x_\alpha d\boldsymbol{x} = \delta I_{\alpha\alpha}$$

($\alpha \neq \beta \neq \gamma$, and no summation over repeated indices in eqs. [61]).

Making use of the relations (61), we find that the virial equations become

$$\begin{aligned}
\delta\Omega^2 &= \frac{\pi G\rho}{I_{11}} \left\{ \left[(3B_{11} - B_{13} - 2B_{12}) + \frac{a_3^2}{a_1^2}(3B_{33} - B_{13}) \right] V_{11} \right. \\
&\quad \left. + \left[(B_{12} - B_{32}) + \frac{a_3^2}{a_2^2}(3B_{33} - B_{13}) \right] V_{22} \right\} + \frac{(\pi G\rho)^2}{c^2} E_{31} \\
&= \frac{\pi G\rho}{I_{22}} \left\{ \left[(B_{12} - B_{31}) + \frac{a_3^2}{a_1^2}(3B_{33} - B_{23}) \right] V_{11} \right. \\
&\quad \left. + \left[(3B_{22} - B_{23} - 2B_{12}) + \frac{a_3^2}{a_2^2}(3B_{33} - B_{23}) \right] V_{22} \right\} + \frac{(\pi G\rho)^2}{c^2} E_{23} \\
&= \frac{\pi G\rho}{I_{11} - I_{22}} \left\{ \left[3(B_{11} - B_{12}) - \frac{a_3^2}{a_1^2}(B_{13} - B_{23}) \right] V_{11} \right. \\
&\quad \left. - \left[3(B_{22} - B_{12}) + \frac{a_3^2}{a_2^2}(B_{13} - B_{23}) \right] V_{22} \right\} + \frac{(\pi G\rho)^2}{c^2} E_{12} \,,
\end{aligned} \tag{62}$$

where V_{33} has been expressed in terms of V_{11} and V_{22} by means of the relation

$$\frac{V_{11}}{a_1^2} + \frac{V_{22}}{a_2^2} + \frac{V_{33}}{a_3^2} = 0 \tag{63}$$

(satisfied by virtue of the solenoidal character of $\boldsymbol{\xi}$).

For the chosen form of $\boldsymbol{\xi}$, namely, that given by equations (47), (48), and (56), we readily find that

$$V_{11} = \frac{8\pi a_1 a_2 a_3 \rho}{105} \frac{\pi G\rho}{c^2} (7a_1^4 S_1 + a_1^4 S_3 - a_1^2 a_3^2 S_5)$$

and

$$V_{22} = \frac{8\pi a_1 a_2 a_3 \rho}{105} \frac{\pi G\rho}{c^2} (7a_1^2 a_2^2 S_2 + a_2^4 S_4 - a_1^2 a_2^2 S_3) \,. \tag{64}$$

Also,

$$I_{11} = \frac{4\pi a_1 a_2 a_3 \rho}{15} a_1^2 \quad \text{and} \quad I_{22} = \frac{4\pi a_1 a_2 a_3 \rho}{15} a_2^2 \,. \tag{65}$$

Inserting these relations in equations (62), we finally obtain

$$
\begin{aligned}
\delta\Omega^2 &= \frac{(\pi G\rho)^2}{c^2}\Big\{\frac{2}{7a_1^2}[a_1^2(3B_{11}-B_{13}-2B_{12})+a_3^2(3B_{33}-B_{13})](7a_1^2S_1+a_1^2S_3-a_3^2S_5) \\
&+\frac{2}{7a_1^2}[a_2^2(B_{12}-B_{32})+a_3^2(3B_{33}-B_{13})](7a_1^2S_2+a_2^2S_4-a_1^2S_3)+E_{31}\Big\} \\
&= \frac{(\pi G\rho)^2}{c^2}\Big\{\frac{2}{7a_2^2}[a_1^2(B_{12}-B_{31})+a_3^2(3B_{33}-B_{23})](7a_1^2S_1+a_1^2S_3-a_3^2S_5) \\
&+\frac{2}{7a_2^2}[a_2^2(3B_{22}-B_{23}-2B_{12})+a_3^2(3B_{33}-B_{23})](7a_1^2S_2+a_2^2S_4-a_1^2S_3)+E_{23}\Big\} \qquad (66) \\
&= \frac{(\pi G\rho)^2}{c^2}\Big\{\frac{2}{7(a_1^2-a_2^2)}[3a_1^2(B_{11}-B_{12})-a_3^2(B_{13}-B_{23})](7a_1^2S_1+a_1^2S_3-a_3^2S_5) \\
&-\frac{2}{7(a_1^2-a_2^2)}[3a_2^2(B_{22}-B_{12})+a_3^2(B_{13}-B_{23})](7a_1^2S_2+a_2^2S_4-a_1^2S_3)+E_{12}\Big\},
\end{aligned}
$$

where it may be verified that the last equality is implied by the first two (cf. eq. [42]).

VI. THE CHANGE IN THE GRAVITATIONAL POTENTIAL CAUSED BY THE DEFORMATION

The deformation of the Jacobi ellipsoid by the displacement (56) will change the gravitational potential U by the amount

$$\delta U = \frac{\pi G\rho a_1^2}{c^2}\sum_{i=1}^{5} S_i\,\delta U^{(i)}, \qquad (67)$$

where (cf. Paper II, eq. [58])

$$\delta U^{(i)} = -G\frac{\partial}{\partial x_\mu}\int_V \frac{\rho(x')\,\xi_\mu^{(i)}(x')}{|x-x'|}\,dx'. \qquad (68)$$

For the particular displacements $\xi^{(i)}$ listed in equations (47) and (48), the required variations $\delta U^{(i)}$ can be expressed in terms of the potentials $\mathfrak{D}_\alpha$ and $\mathfrak{D}_{\alpha\beta\gamma}$ (cf. Paper II, eqs. [8] and [59]); thus

$$
\begin{gathered}
\delta U^{(1)} = -\frac{\partial\mathfrak{D}_1}{\partial x_1}+\frac{\partial\mathfrak{D}_3}{\partial x_3}, \quad \delta U^{(2)} = -\frac{\partial\mathfrak{D}_2}{\partial x_2}+\frac{\partial\mathfrak{D}_3}{\partial x_3}, \quad a_1^2\,\delta U^{(3)} = \frac{\partial\mathfrak{D}_{211}}{\partial x_2}-\frac{1}{3}\frac{\partial\mathfrak{D}_{111}}{\partial x_1}, \\
a_1^2\,\delta U^{(4)} = \frac{\partial\mathfrak{D}_{322}}{\partial x_3}-\frac{1}{3}\frac{\partial\mathfrak{D}_{222}}{\partial x_2}, \quad \text{and} \quad a_1^2\,\delta U^{(5)} = \frac{\partial\mathfrak{D}_{133}}{\partial x_1}-\frac{1}{3}\frac{\partial\mathfrak{D}_{333}}{\partial x_3}.
\end{gathered} \qquad (69)
$$

Evaluating $\delta U^{(i)}$ with the aid of the known expressions for $\mathfrak{D}_\alpha$ (Chandrasekhar and Lebovitz 1962*b*, eq. [49]) and $\mathfrak{D}_{\alpha\beta\gamma}$ (Paper II, eq. [63]), we find that they are expressible in the forms

$$\frac{\delta U^{(i)}}{\pi G\rho} = u_0^{(i)}+\sum_{\mu=1}^{3}u_\mu^{(i)}x_\mu^2 \qquad (i=1,2) \qquad (70)$$

and

$$\frac{a_1^2\,\delta U^{(i)}}{\pi G\rho} = u_0^{(i)}+\sum_{\mu=1}^{3}u_\mu^{(i)}x_\mu^2+\sum_{\mu=1}^{3}u_{\mu\mu}^{(i)}x_\mu^4+\sum_{\mu\nu}^{12,23,31}u_{\mu\nu}^{(i)}x_\mu^2x_\nu^2 \qquad (i=3,4,5), \qquad (71)$$

where

$$
\begin{aligned}
u_0^{(1)} &= B_1 - B_3\,; & u_0^{(2)} &= B_2 - B_3\,;\\
u_1^{(1)} &= B_{31} - B_{11} + 2a_1^2 A_{11}\,; & u_1^{(2)} &= B_{31} - B_{12}\,;\\
u_2^{(1)} &= B_{32} - B_{12}\,; & u_2^{(2)} &= B_{32} - B_{22} + 2a_2^2 A_{22}\,;\\
u_3^{(1)} &= B_{33} - B_{13} - 2a_3^2 A_{33}\,; & u_3^{(2)} &= B_{33} - B_{23} - 2a_3^2 A_{33}\,;
\end{aligned}
\tag{72}
$$

$$
\begin{aligned}
u_0^{(3)} &= \tfrac{1}{4}a_1^2(a_2^2 B_{21} - a_1^2 B_{11})\,,\\
u_1^{(3)} &= a_1^4(B_{111} - B_{211}) - \tfrac{1}{2}a_1^2(a_2^2 B_{211} - a_1^2 B_{111}) + a_1^4 B_{111}\,,\\
u_2^{(3)} &= -\tfrac{1}{2}a_1^2(a_2^2 B_{122} - a_1^2 B_{112}) - a_1^2 a_2^2 B_{221}\,,\\
u_3^{(3)} &= -\tfrac{1}{2}a_1^2(a_2^2 B_{123} - a_1^2 B_{113})\,,\\
u_{11}^{(3)} &= -a_1^4(B_{1111} - B_{2111}) + \tfrac{2}{3}a_1^6 A_{1111} + \tfrac{1}{4}a_1^2(a_2^2 B_{2111} - a_1^2 B_{1111}) - a_1^4 B_{1111}\,,\\
u_{22}^{(3)} &= \tfrac{1}{4}a_1^2(a_2^2 B_{1222} - a_1^2 B_{1122}) + a_1^2 a_2^2 B_{1222}\,,\\
u_{33}^{(3)} &= \tfrac{1}{4}a_1^2(a_2^2 B_{1233} - a_1^2 B_{1133})\,,\\
u_{12}^{(3)} &= -a_1^4(B_{1112} - B_{1122}) - 2a_1^4 a_2^2 A_{1122} + \tfrac{3}{2}a_1^2(a_2^2 B_{1122} - a_1^2 B_{1112})\,,\\
u_{23}^{(3)} &= \tfrac{1}{2}a_1^2(a_2^2 B_{1223} - a_1^2 B_{1123}) + a_1^2 a_2^2 B_{1223}\,,\\
u_{31}^{(3)} &= -a_1^4(B_{1113} - B_{1123}) + \tfrac{1}{2}a_1^2(a_2^2 B_{1123} - a_1^2 B_{1113}) - a_1^4 B_{1113}\,;
\end{aligned}
\tag{73}
$$

and the expressions for the coefficients belonging to the displacements $\boldsymbol{\xi}^{(4)}$ and $\boldsymbol{\xi}^{(5)}$ can be obtained from those belonging to $\boldsymbol{\xi}^{(3)}$ by a cyclical permutation of the indices 1, 2, 3.

Combining the foregoing results, we can write

$$
\begin{aligned}
\delta U = \frac{(\pi G\rho)^2}{c^2}\Bigg\{ & a_1^2 \sum_{i=1}^{2} S_i\Big[u_0^{(i)} + \sum_{\mu=1}^{3} u_\mu^{(i)} x_\mu^2\Big]\\
&+ \sum_{i=3}^{5} S_i\Big[u_0^{(i)} + \sum_{\mu=1}^{3} u_\mu^{(i)} x_\mu^2 + \sum_{\mu=1}^{3} u_{\mu\mu}^{(i)} x_\mu^4 + \sum_{\mu\nu}^{12,23,31} u_{\mu\nu}^{(i)} x_\mu^2 x_\nu^2\Big]\Bigg\}\,.
\end{aligned}
\tag{74}
$$

VII. THE DETERMINATION OF THE POST-NEWTONIAN FIGURE

Returning to equation (34), we shall now rewrite it in the form

$$
\begin{aligned}
\frac{p}{\rho} = {} & U_J + \tfrac{1}{2}\Omega_J^2(x_1^2 + x_2^2) + \delta U + \tfrac{1}{2}\delta\Omega^2(x_1^2 + x_2^2)\\
&+ \frac{(\pi G\rho)^2}{c^2}\Big[a_0 + \sum_{\mu=1}^{3} a_\mu x_\mu^2 + \sum_{\mu=1}^{3} a_{\mu\mu} x_\mu^4 + \sum_{\mu\nu}^{12,23,31} a_{\mu\nu} x_\mu^2 x_\nu^2\Big] + \text{constant}\,,
\end{aligned}
\tag{75}
$$

where $\delta\Omega^2$ and δU are given by equations (66) and (74). The first two terms on the right-hand side of equation (75) together with a suitably chosen additive constant can be combined to give the expression (31) appropriate for the Jacobi ellipsoid. We may therefore write

$$\frac{p}{\rho} = (\pi G\rho)\, a_3^2 A_3 \left(1 - \sum_{\mu=1}^{3} \frac{x_\mu^2}{a_\mu^2}\right) + \frac{(\pi G\rho)^2}{c^2}\left[\tfrac{1}{2}\delta\omega^2(x_1^2 + x_2^2) + \sum_{\mu=1}^{3} P_\mu x_\mu^2 + \sum_{\mu=1}^{3} P_{\mu\mu} x_\mu^4 + \sum_{\mu\nu}^{12,23,31} P_{\mu\nu} x_\mu^2 x_\nu^2 + \text{constant}\right], \tag{76}$$

where

$$P_\mu = \alpha_\mu + \sum_{i=1}^{2} S_i a_1^2 u_\mu^{(i)} + \sum_{i=3}^{5} S_i u_\mu^{(i)} \qquad (\mu = 1,2,3),$$

and

$$P_{\mu\nu} = \alpha_{\mu\nu} + \sum_{i=3}^{5} S_i u_{\mu\nu}^{(i)} \qquad (\mu\nu = 11,22,33,12,23,31). \tag{77}$$

The additive constant in equation (76) is now of order $1/c^2$; also $\delta\omega^2$ stands for any of the three quantities in curly brackets on the right-hand side of equations (66), i.e.,

$$\delta\Omega^2 = \frac{(\pi G\rho)^2}{c^2}\,\delta\omega^2. \tag{78}$$

It remains to apply the boundary condition to the solution (76), namely, that it must vanish on the surface (57). And in view of equation (57), the values *of* p/ρ on S is given by

$$\left(\frac{p}{\rho}\right)_S = \frac{(\pi G\rho)^2}{c^2}\left\{-2a_3^2 A_3\left[S_1 x_1^2 + \frac{a_1^2}{a_2^2} S_2 x_2^2 - \frac{a_1^2}{a_3^2}(S_1 + S_2) x_3^2 + S_3\left(\frac{x_1^4}{3a_1^2} - \frac{x_1^2 x_2^2}{a_2^2}\right) + S_4\left(\frac{x_2^4}{3a_2^2} - \frac{x_2^2 x_3^2}{a_3^2}\right) + S_5\left(\frac{x_3^4}{3a_3^2} - \frac{x_3^2 x_1^2}{a_1^2}\right)\right] + \tfrac{1}{2}\delta\omega^2(x_1^2 + x_2^2) + \sum_{\mu=1}^{3} P_\mu x_\mu^2 + \sum_{\mu=1}^{3} P_{\mu\mu} x_\mu^4 + \sum_{\mu\nu}^{12,23,31} P_{\mu\nu} x_\mu^2 x_\nu^2 + \text{constant}\right\}, \tag{79}$$

or

$$\left(\frac{p}{\rho}\right)_S = \frac{(\pi G\rho)^2}{c^2}\left(\sum_{\mu=1}^{3} Q_\mu x_\mu^2 + \sum_{\mu=1}^{3} Q_{\mu\mu} x_\mu^4 + \sum_{\mu\nu}^{12,23,31} Q_{\mu\nu} x_\mu^2 x_\nu^2 + \text{constant}\right), \tag{80}$$

where

$$Q_1 = P_1 + \tfrac{1}{2}\delta\omega^2 - 2a_3^2 A_3 S_1,$$

$$Q_2 = P_2 + \tfrac{1}{2}\delta\omega^2 - 2a_3^2 \frac{a_1^2}{a_2^2} A_3 S_2,$$

$$Q_3 = P_3 + 2a_1^2 A_3 (S_1 + S_2),$$

$$Q_{11} = P_{11} - \frac{2a_3^2 A_3}{3a_1^2} S_3, \qquad Q_{22} = P_{22} - \frac{2a_3^2 A_3}{3a_2^2} S_4, \qquad Q_{33} = P_{33} - \tfrac{2}{3} A_3 S_5, \tag{81}$$

$$Q_{12} = P_{12} + \frac{2a_3^2 A_3}{a_2^2} S_3, \qquad Q_{23} = P_{23} + 2A_3 S_4, \qquad Q_{31} = P_{31} + \frac{2a_3^2 A_3}{a} S_5.$$

Since $(p/\rho)_S$ is of order $1/c^2$, it will clearly suffice if we can arrange for it to vanish on the original ellipsoid S_J. And we can arrange for it in two steps: *first* we require that the expression

$$\sum_{\mu=1}^{3} Q_\mu x_\mu^2 + \sum_{\mu=1}^{3} Q_{\mu\mu} x_\mu^4 + \sum_{\mu\nu}^{12,23,31} Q_{\mu\nu} x_\mu^2 x_\nu^2 \tag{82}$$

remains constant on S_J; *then* we determine the additive constant in the solution (80) so that $(p/\rho)_S$ does vanish.

We readily verify that in order that the expression (82) remains constant on S_J it is necessary and sufficient that

$$a_1^4 Q_{11} + a_2^4 Q_{22} - a_1^2 a_2^2 Q_{12} = 0\,, \tag{83}$$

$$a_2^4 Q_{22} + a_3^4 Q_{33} - a_2^2 a_3^2 Q_{23} = 0\,, \tag{84}$$

$$a_3^4 Q_{33} + a_1^4 Q_{11} - a_3^2 a_1^2 Q_{31} = 0\,, \tag{85}$$

$$a_1^4 Q_{11} - a_2^4 Q_{22} + a_1^2 Q_1 - a_2^2 Q_2 = 0\,, \tag{86}$$

and

$$a_3^4 Q_{33} - a_1^4 Q_{11} + a_3^2 Q_3 - a_1^2 Q_1 = 0\,. \tag{87}$$

And these equations must be considered together with equations (66).

Now it can be shown (the proof is outlined in Appendix I) that the five equations (83)–(87) are not linearly independent if we use for $\frac{1}{2}(a_1^2 - a_2^2)\delta\omega^2$ and $-\frac{1}{2}a_1^2\delta\omega^2$, which occur in equations (86) and (87) via the terms $a_1^2Q_1 - a_2^2Q_2$ and $-a_1^2Q_1$, respectively, the values (cf. eqs. [66])

$$\begin{aligned}\tfrac{1}{2}(a_1^2 - a_2^2)\,\delta\omega^2 &= \tfrac{1}{7}[3a_1^2(B_{11} - B_{12}) - a_3^2(B_{13} - B_{23})](7a_1^2S_1 + a_1^2S_3 - a_3^2S_5) \\ &- \tfrac{1}{7}[3a_2^2(B_{22} - B_{12}) + a_3^2(B_{13} - B_{23})](7a_1^2S_2 + a_2^2S_4 - a_1^2S_3) + \tfrac{1}{2}(a_1^2 - a_2^2)E_{12}\end{aligned} \tag{88}$$

and

$$\begin{aligned}\tfrac{1}{2}a_1^2\,\delta\omega^2 &= \tfrac{1}{7}[a_1^2(3B_{11} - B_{13} - 2B_{12}) + a_3^2(3B_{33} - B_{13})](7a_1^2S_1 + a_1^2S_3 - a_3^2S_5) \\ &+ \tfrac{1}{7}[a_2^2(B_{12} - B_{32}) + a_3^2(3B_{33} - B_{13})](7a_1^2S_2 + a_2^2S_4 - a_1^2S_3) + \tfrac{1}{2}a_1^2E_{31}\,;\end{aligned} \tag{89}$$

for, with these substitutions, the coefficients of S_1 and S_2 are identically zero in equations (86) and (87) and, moreover,

$$\begin{aligned}(a_3^4Q_{33} + a_1^4Q_{11} - a_3^2a_1^2Q_{31}) - (a_2^4Q_{22} + a_3^4Q_{33} - a_2^2a_3^2Q_{23}) \\ = 7(a_1^4Q_{11} - a_2^4Q_{22} + a_1^2Q_1 - a_2^2Q_2)\end{aligned} \tag{90}$$

and

$$\begin{aligned}(a_2^4Q_{22} + a_3^4Q_{33} - a_2^2a_3^2Q_{23}) - (a_1^4Q_{11} + a_2^4Q_{22} - a_1^2a_2^2Q_{12}) \\ = 7(a_3^4Q_{33} - a_1^4Q_{11} + a_3^2Q_3 - a_1^2Q_1)\,.\end{aligned} \tag{91}$$

Thus, none of the equations (83)–(87) involves S_1 or S_2 and only three of the five equations are linearly independent; they, therefore, suffice to determine S_3, S_4, and S_5 uniquely. The coefficients S_1 and S_2 are left undetermined by these equations. However, the requirement that equations (88) and (89) are consistent with the same value of $\delta\omega^2$

leads to a single relation between S_1 and S_2; and the solution can be made determinate only by specifying[4] $\delta\omega^2$. The situation here is the same as that which was encountered in Paper II in the treatment of the deformed Maclaurin spheroids where, of the two constants S_1 and S_2 there introduced, only S_2 was uniquely determined while S_1 was left undetermined so long as $\delta\omega^2$ was unspecified. In Paper II, the solution was made determinate by setting (arbitrarily!) $S_1 = 0$ and then determining $\delta\omega^2$ which followed from this assumption. In the same way, in this paper we shall make the solution determinate by setting (again, arbitrarily)

$$S_1 = S_2\,, \tag{92}$$

and then determining $\delta\omega^2$ which follows from this assumption.

In Tables 1, 2, and 3, we present the results of numerical calculations based on the formulae derived in this section. (For the explanation of the entries, appropriate to the point of bifurcation, along the first lines of Tables 1 and 2 see § VIII below.)

From Tables 1, 2, and 3 it is manifest that the solution is singular for a Jacobi ellipsoid for which $\cos^{-1} a_3/a_1 = 75\overset{\circ}{.}081$ and whose axes are in the ratios

$$1:0.29720:0.25746\,. \tag{93}$$

The origin of this singularity is explained in § IX below.

TABLE 1

THE VALUES OF THE COEFFICIENTS S_3, S_4, AND S_5

$\cos^{-1}a_3/a_2$	S_3	S_4	S_5	$\cos^{-1}a_3/a_2$	S_3	S_4	S_5
54°35762	−0 0174266	+0 069706	+0 29081	68°	+ 0 35776	− 0 22956	+ 1 7209
56°	− 004696	+ 051285	+0 34646	69°8166	+ 0 49483	− 0 39681	+ 2 6348
58°	+ 023772	+ 028057	+0 42549	70°	+ 0 51285	− 0 42119	+ 2 7679
60°	+ 064135	+ 002302	+0 52622	72°	+ 0 82586	− 0 89671	+ 5 3493
62°	+ 11534	− 028711	+0 66396	75°	+27 81	−49 84	+266 6
64°	+ 17798	− 069736	+0 86541	75°081	± ∞	± ∞	± ∞
66°	+0 25550	−0 12998	+1 18124	77°	− 1 00141	+ 2 6770	− 13 507

TABLE 2

THE RELATIONS GOVERNING S_1, S_2, AND $\delta\omega^2$; AND THE VALUES OF THE CONSTANTS IN THE CASE $S_1 = S_2$

$\cos^{-1}a_3/a_2$		$\delta\omega^2/a_1^2$	$S_1=S_2$	$\delta\omega^2(S_1=S_2)/a_1^2$
54°35762	S_1+1 00000 S_2= 0 20170	0 S_2+0 43532	+0 10085	+0 43532
56°	S_1+1 03394 S_2= 0 1935	0 048866 S_2+0 39638	+0 095152	+0 40103
58°	S_1+1 07680 S_2= 0 18219	11060 S_2+0 34922	+0 087725	+0 35892
60°	S_1+1 12219 S_2= 0 17006	17307 S_2+0 30281	+0 080134	+0 31668
62°	S_1+1 16996 S_2= 0 15834	23490 S_2+0 25744	+0 072970	+0 27458
64°	S_1+1 22017 S_2= 0 14829	29435 S_2+0 21332	+0 066792	+0 23298
66°	S_1+1 27289 S_2= 0 14152	34944 S_2+0 17040	+0 062265	+0 19216
68°	S_1+1 32813 S_2= 0 14088	39798 S_2+0 12812	+0 060513	+0 15220
69°8166	S_1+1 38055 S_2= 0 15088	43437 S_2+0 088647	+0 063380	+0 11618
70°	S_1+1 38596 S_2= 0 15284	43755 S_2+0 084467	+0 064060	+0 11250
72°	S_1+1 44631 S_2= 0 19955	46560 S_2+0 031296	+0 081571	+0 069275
75°	S_1+1 5414 S_2= 5 7129	48016 S_2−2 15262	+2 2479	−1 0732
75°081	. . .	. . . . .	± ∞	± ∞
77°	S_1+1 9358 S_2=−0 29077	0 54455 S_2+0 14083	−0 099041	+0 086902

[4] An exception occurs at the point of bifurcation where $\delta\omega^2$ is *uniquely* determined and an *essential indeterminacy* in the solution is left The origin of this indeterminacy is discussed in § VIII below.

VIII. THE POST-NEWTONIAN CONFIGURATIONS AT THE POINT OF BIFURCATION

At the point ($e = 0.812670$) along the Maclaurin sequence where the Jacobian sequence branches off and $a_1 = a_2$, equation (88), as it is written, becomes indeterminate. However, $\delta\omega^2$ given by this equation tends to a definite limit; it is given by

$$\delta\omega^2 = \tfrac{2}{7}(a_3^2 B_{113} - 3a_1^2 B_{111})[7a_1^2(S_1 + S_2) + a_1^2 S_4 - a_3^2 S_5] + E_{12}\,, \tag{94}$$

where (cf. eq. [41])

$$E_{12} = \lim_{a_1 \to a_2} 2\left(\frac{a_2^2\alpha_2 - a_1^2\alpha_1}{a_1^2 - a_2^2} + \tfrac{1}{7}a_3^2\,\frac{a_2^2\alpha_{23} - a_1^2\alpha_{31}}{a_1^2 - a_2^2} + \tfrac{6}{7}\,\frac{a_2^4\alpha_{22} - a_1^4\alpha_{11}}{a_1^2 - a_2^2}\right). \tag{95}$$

TABLE 3

THE COEFFICIENTS P_μ (IN THE CASE $S_1 = S_2$) AND $P_{\mu\nu}$ IN THE EXPRESSION FOR THE PRESSURE (EQ. [76])

$\cos^{-1} a_3/a_2$	P_1	P_2	P_3	P_{11}	P_{22}	P_{33}	P_{12}	P_{23}	P_{31}
54°.35762	−1 65334	−1 63424	−4 16967	+0 39826	+0 39826	+1 77778	+ 0 79652	+1 53280	+1 53280
56°	−1 43801	−1 59358	−3 84130	+0 36023	+0 43948	+1 77789	+ 0 79138	+1 63505	+1 43124
58°	−1 20155	−1 53524	−3 45550	+0 31778	+0 49479	+1 77823	+ 0 77088	+1 76198	+1 30700
60°	−0 99272	−1 46687	−3 08576	+0 27975	+0 55571	+1 77891	+ 0 73460	+1 89126	+1 18222
62°	−0 81020	−1 38830	−2 73258	+0 24632	+0 62242	+1 77996	+ 0 68249	+2 02280	+1 05661
64°	−0 65274	−1 29964	−2 39649	+0 21778	+0 69500	+1 78149	+ 0 61448	+2 15645	+0 92946
66°	−0 51911	−1 20136	−2 07780	+0 19471	+0 77360	+1 78371	+ 0 53000	+2 29222	+0 79913
68°. . . .	−0 40836	−1 09414	−1 77668	+0 17841	+0 85851	+1 78716	+ 0 42642	+2 43060	+0 66172
69°.8166	−0 32751	−0 98965	−1 51780	+0 17241	+0 94205	+1 79240	+ 0 30814	+2 56035	+0 52221
70°	−0 32045	−0 97875	−1 49238	+0 17251	+0 95088	+1 79313	+ 0 29425	+2 57385	+0 50666
72°	−0 25989	−0 85527	−1 22070	+0 19058	+1 05578	+1 80642	+ 0 09287	+2 73143	+0 29473
75°. . .	−1 781	−0 296	+0 363	+3 680	+2 593	+3 046	−11 05	+5 56	−9 92
75°.081	± ∞	± ∞	± ∞	± ∞	± ∞	± ∞	± ∞	± ∞	± ∞
77°	−0 15132	−0 74504	−0 44611	−0 00517	+1 58871	+2 38364	+ 1 21229	−0 0524	−0 31968

,

By making use of the explicit expressions for the α_μ's and $\alpha_{\mu\nu}$'s given in equations (35) we find after some considerable reductions that

$$\begin{aligned}
E_{12} = 2[&- a_1^2\Omega^2 B_{11} - 2\Omega^2 I + 4a_1^2\Omega^2(a_1^2 A_{11} + 2B_{11}) + (2I + 3a_3^2 A_3)B_{11} \\
&+ a_1^2(\tfrac{7}{2}\Omega^2 - 5A_1)(3C_{111} - 2B_{11} - a_1^2 A_{11}) - 5a_1^4(A_1 - \tfrac{3}{2}B_{11})A_{11} \\
&- \tfrac{5}{2}a_1^4 B_{11}A_{11} - \tfrac{5}{2}a_3^2 A_3 C_{113}] \\
&+ \tfrac{2}{7}a_3^2[- 2\Omega^2 A_3 + 4\Omega^2 C_{113} + (\tfrac{7}{2}\Omega^2 - 5A_1)(a_1^4 A_{113} + 2a_1^2 B_{113} - 3a_1^2 C_{1113}) \\
&- 5a_1^4(\tfrac{3}{2}B_{113} - A_{13})A_{11} + 5a_3^2 A_3(\tfrac{3}{2}C_{1133} - B_{113}) + \tfrac{5}{2}a_1^4 B_{113}A_{11} + a_1^2\Omega^2 B_{113}] \\
&+ \tfrac{12}{7}[\Omega^2(8a_1^2 C_{111} - 4a_1^2 B_{11} - 2a_1^4 A_{11} - \tfrac{1}{2}a_1^2\Omega^2) - 5a_1^6 A_{11}(\tfrac{5}{4}B_{111} - A_{11}) \\
&+ a_1^4(\tfrac{7}{2}\Omega^2 - 5A_1)(\tfrac{15}{4}a_1^2 B_{1111} - 2a_1^2 A_{111} - \tfrac{15}{4}B_{111} + 3A_{11} - \tfrac{1}{4}C_{1111}) \\
&+ \tfrac{5}{4}a_1^6 B_{111}A_{11} + \tfrac{5}{2}a_1^2 a_3^2 A_3 C_{1113} + a_1^4\Omega^2 B_{111}]
\end{aligned} \tag{96}$$

(Ω^2 is measured in the unit $\pi G\rho$ in eq. [96]) ,

where the new index symbol C_{ijk} , we have here introduced, is related to B_{ijk} by

$$C_{ijk} = B_{jk} - a_i^2 B_{ijk}\,, \tag{97}$$

and, like B_{ijk} , is symmetrical in its indices.

Under the same circumstances, equation (89) gives (without any ambiguity)

$$\delta\omega^2=\frac{2}{7a_1^2}\left[a_1^2(B_{11}-B_{13})+a_3^2(3B_{33}-B_{13})\right]\left[7a_1^2(S_1+S_2)+a_1^2S_4-a_3^2S_5\right]+E_{31}, \tag{98}$$

where it may be noted that E_{31} as defined in equations (41) agrees with the definition given in Paper II (eq. [29]) in the case $a_1 = a_2$. Also, by making use of the special relation ($a_3^2A_3 = a_1^4A_{11}$) that obtains at the point of bifurcation, equation (98) can be brought to the more familiar form (cf. Paper II, eq. [55])

$$\delta\omega^2=E_{31}-\tfrac{4}{7}a_1^2(a_1^2A_{11}-2a_3^2A_{13})\left[7(S_1+S_2)+S_4-\frac{a_3^2}{a_1^2}S_5\right]. \tag{99}$$

The remaining equations (83)–(87) are also unambiguous when $a_1 = a_2$; and they suffice to determine the constants S_3, S_4 ,and S_5 uniquely. As the equations for determining these constants, it is convenient to use in the present context the set

$$Q_{11}+Q_{22}-Q_{12}=0 \tag{100}$$

$$a_1^2(Q_{11}-Q_{22})-a_3^2(Q_{31}-Q_{23})=0\,, \tag{101}$$

and

$$a_3^4Q_{33}+a_1^4Q_{11}-a_3^2a_1^2Q_{31}=0\,. \tag{102}$$

We shall now show that for the case under consideration

$$S_4=-4S_3 \qquad \text{and} \qquad S_5=-\frac{17a_1^2}{3a_3^2}S_3\,. \tag{103}$$

We first verify that when the expressions for the $Q_{\mu\nu}$'s given in equations (81) are substituted in equations (100) and (101), the terms, besides the $P_{\mu\nu}$'s, cancel if the relations (103) are assumed; we shall therefore be left with

$$P_{11}+P_{22}-P_{12}=0 \tag{104}$$

and

$$a_1^2(P_{11}-P_{22})-a_3^2(P_{31}-P_{23})=0\,. \tag{105}$$

The expressions for the $P_{\mu\nu}$'s are given in equations (77); and in view of the equalities

$$\alpha_{11}=\alpha_{22}=\tfrac{1}{2}\alpha_{12} \qquad \text{and} \qquad \alpha_{31}=\alpha_{23} \tag{106}$$

which obtain in this case, the constant terms in equations (104) and (105) will cancel; the equations thus become homogeneous in S_3, S_4, and S_5 and the validity of the relations (103) will require that

$$\begin{aligned}&[u_{11}^{(3)}+u_{22}^{(3)}-u_{12}^{(3)}]-4[u_{11}^{(4)}+u_{22}^{(4)}-u_{12}^{(4)}]\\&-\frac{17a_1^2}{3a_3^2}[u_{11}^{(5)}+u_{22}^{(5)}-u_{12}^{(5)}]=0\end{aligned} \tag{107}$$

and

$$\begin{aligned}&\{a_1^2[u_{11}^{(3)}-u_{22}^{(3)}]-a_3^2[u_{31}^{(3)}-u_{23}^{(3)}]\}-4\{a_1^2[u_{11}^{(4)}-u_{22}^{(4)}]-a_3^2[u_{31}^{(4)}-u_{23}^{(4)}]\}\\&-\frac{17a_1^2}{3a_3^2}\{a_1^2[u_{11}^{(5)}-u_{22}^{(5)}]-a_3^2[u_{31}^{(5)}-u_{23}^{(5)}]\}=0\,.\end{aligned} \tag{108}$$

In Appendix II we verify that these equations are indeed satisfied.

Hence, the Lagrangian displacement, which deforms the Maclaurin spheroid at the point of bifurcation into its post-Newtonian figures, has the form

$$\xi = S_3\chi + S_1\xi^{(1)} + S_2\xi^{(2)}\,, \tag{109}$$

where

$$\chi = \xi^{(3)} - 4\xi^{(4)} - \frac{17a_1^2}{3a_3^2}\,\xi^{(5)}\,. \tag{110}$$

The value of the constant S_3, left undetermined by equations (100) and (101), follows from the remaining equation (102). It is found that (cf. the entries along the first line in Table 1)

$$S_3 = -0.0174266\,. \tag{111}$$

Returning to equations (94) and (98), we can write them, in view of the relations (103), in the forms

$$\delta\omega^2 = E_{12} + \tfrac{2}{7}a_1^2(a_3^2B_{113} - 3a_1^2B_{111})\,[7(S_1 + S_2) + \tfrac{5}{3}S_3] \tag{112}$$

and

$$\delta\omega^2 = E_{31} - \tfrac{4}{7}a_1^2(a_1^2A_{11} - 2a_3^2A_{13})\,[7(S_1 + S_2) + \tfrac{5}{3}S_3]\,. \tag{113}$$

Since these equations involve S_1 and S_2 only in the combination $(S_1 + S_2)$, it is clear that they will suffice to determine $\delta\omega^2$ and $(S_1 + S_2)$ uniquely; but they will not determine S_1 and S_2 separately. It is found that (cf. the entries along the first line in Table 2)

$$S_1 + S_2 = 0.201701 \tag{114}$$

and

$$\delta\omega^2 = 0.435318\, a_1^2\,. \tag{115}$$

Thus, *at the point of bifurcation an essential indeterminacy in the solution remains and a continuous range of equilibrium figures becomes compatible with the same value of* Ω^2. The origin and the meaning of this indeterminacy will become apparent once we have related the present solution with that obtained in Paper II with the restriction to axisymmetric figures.

As we have seen in Paper II, the axisymmetric figure which a Maclaurin spheroid assumes in the post-Newtonian approximation can be obtained by deforming the spheroid by a Lagrangian displacement which is a superposition of the displacements

$$\mathfrak{n}^{(1)} = (x_1, x_2, -2x_3)\,,$$

$$\mathfrak{n}^{(2)} = \frac{1}{a_1^2}\,[(x_1^2 + x_2^2)x_1, (x_1^2 + x_2^2)x_2, -4(x_1^2 + x_2^2)x_3]\,, \tag{116}$$

and

$$\mathfrak{n}^{(3)} = \frac{1}{a_1^2}\,(x_3^2x_1, x_3^2x_2, -\tfrac{2}{3}x_3^3)\,.$$

[These displacements were denoted by $\xi^{(1)}$, $\xi^{(2)}$, and $\xi^{(3)}$ in Paper II; we are denoting them here by $\mathfrak{n}^{(1)}$, $\mathfrak{n}^{(2)}$, and $\mathfrak{n}^{(3)}$ to distinguish them from what we have already denoted by $\xi^{(1)}$, $\xi^{(2)}$, and $\xi^{(3)}$.] We may also recall here that only two of these three displacements are linearly independent modulo the spheroid; thus (cf. Paper II, eq. [43])

$$\gamma\,\mathfrak{n}^{(2)} + \delta\,\frac{a_1^2}{a_3^2}\,\mathfrak{n}^{(3)} \equiv [\tfrac{1}{3}\delta\,\mathfrak{n}^{(1)} + (\gamma - \tfrac{1}{3}\delta)\,\mathfrak{n}^{(2)}](\mathrm{mod}\ S_{\mathrm{Mc}})\,, \tag{117}$$

where γ and δ are arbitrary constants.

We shall now express $\mathfrak{n}^{(2)}$ and $\mathfrak{n}^{(3)}$ in terms of the displacements $\xi^{(3)}, \ldots, \xi^{(8)}$ defined in this paper (eqs. [48] and [49]). From the definitions of the various displacements, it readily follows that

$$\mathfrak{n}^{(2)} = 3\xi^{(3)} + 3\xi^{(4)} - \xi^{(7)} + 4\xi^{(6)}$$

and (118)

$$\mathfrak{n}^{(3)} = -2\xi^{(5)} - \xi^{(8)} .$$

By making use of the relations (50)–(52) we can express the displacements $\xi^{(6)}$, $\xi^{(7)}$, and $\xi^{(8)}$, modulo the ellipsoid, in terms of $\xi^{(1)}, \ldots, \xi^{(5)}$; and we find that, in the special case $a_1 = a_2$ in which we are presently concerned,

$$\mathfrak{n}^{(2)} \equiv [-\tfrac{9}{13}\chi + \tfrac{16}{13}\xi^{(1)} + \tfrac{1}{13}\xi^{(2)}](\mathrm{mod}\ S_{\mathrm{Mc}})$$

and (119)

$$\frac{a_1^2}{a_3^2}\,\mathfrak{n}^{(3)} \equiv [+\tfrac{3}{13}\chi - \tfrac{1}{13}\xi^{(1)} + \tfrac{4}{13}\xi^{(2)}](\mathrm{mod}\ S_{\mathrm{Mc}}) ,$$

where χ is the same vector defined in equation (110). A linear combination of $\mathfrak{n}^{(2)}$ and $\mathfrak{n}^{(3)}$ can, therefore, be expressed in the form

$$\gamma\mathfrak{n}^{(2)} + \delta\,\frac{a_1^2}{a_3^2}\mathfrak{n}^{(3)}$$

(120)

$$\equiv [-\tfrac{3}{13}(3\gamma - \delta)\chi + \tfrac{1}{13}(16\gamma - \delta)\xi^{(1)} + \tfrac{1}{13}(\gamma + 4\delta)\xi^{(2)}](\mathrm{mod}\ S_{\mathrm{Mc}})$$

where γ and δ are arbitrary constants; or in view of the relation (117) we can also write

$$\alpha\mathfrak{n}^{(1)} + \beta\mathfrak{n}^{(2)} \equiv \left[-\frac{9\beta}{13}\chi + \left(\alpha + \frac{16\beta}{13}\right)\xi^{(1)} + \left(\alpha + \frac{\beta}{13}\right)\xi^{(2)}\right](\mathrm{mod}\ S_{\mathrm{Mc}}) , \quad (121)$$

where α and β are (different) arbitrary constants.

By comparing equations (109) and (121), we conclude that the general solution for ξ, given by equations (109), (111), and (114), includes an axisymmetric displacement of the form

$$\xi_{\mathrm{ax}} = \alpha\mathfrak{n}^{(1)} + \beta\mathfrak{n}^{(2)} , \quad (122)$$

if we identify

$$S_3 = -\frac{9\beta}{13} , \qquad S_1 = \alpha + \frac{16\beta}{13} , \qquad \text{and} \qquad S_2 = \alpha + \frac{\beta}{13} . \quad (123)$$

Since only $S_1 + S_2$ is determined in the general solution, the coefficients α and β in the axisymmetric solution (122) can be uniquely determined. Thus, letting

$$-\frac{9\beta}{13} = S_3 = -0.0174266 \qquad \text{and} \qquad 2\alpha + \frac{17\beta}{13} = S_1 + S_2 = 0.201701 , \quad (124)$$

we find that

$$\alpha = 0.084392 \qquad \text{and} \qquad \beta = 0.025172 . \quad (125)$$

The corresponding values of S_1 and S_2 are

$$S_{1,\mathrm{ax}} = 0.115372 \qquad \text{and} \qquad S_{2,\mathrm{ax}} = 0.086328 . \quad (126)$$

We observe that the value of β we have determined agrees with the coefficient of the displacement $\mathfrak{n}^{(2)}$ determined in Paper II (see the entry under S_2 opposite $e = e_J = 0.81267$ in Table 1 of Paper II).

It is of interest to verify that the value of $\delta\omega^2$ given by equation (113) agrees with what would be predicted by equation (55) of Paper II for the particular displacement (122) included in the general solution. By inserting for S_3 and $S_1 + S_2$, in accordance with the *definitions* (123), in equation (113), we obtain

$$\delta\omega^2 = E_{31} - \tfrac{8}{7}a_1^2(a_1^2 A_{11} - 2a_3^2 A_{13})(7\alpha + 4\beta) , \tag{127}$$

in agreement with equation (55) of Paper II.

The conclusion to be drawn from the foregoing analysis is that *the bifurcation of the non-axisymmetric sequence from the axisymmetric sequence occurs at a definite point where*

$$\Omega^2 = 0.37422\pi G\rho + 0.435318 \frac{(\pi G\rho)^2 a_1^2}{c^2} ; \tag{128}$$

and further that the unique axisymmetric post-Newtonian configuration at the point of bifurcation is obtained by deforming the Maclaurin spheroid at the Newtonian point of bifurcation by the Lagrangian displacement

$$\frac{\pi G\rho a_1^2}{c^2} \xi_{\text{ax}} = \frac{\pi G\rho a_1^2}{c^2} [0.084392\mathbf{n}^{(1)} + 0.025172\mathbf{n}^{(2)}] . \tag{129}$$

The solutions compatible with the post-Newtonian equations are more general than the axisymmetric solution (129) which arises from a particular choice of the constants S_1 and S_2 (namely, those given in eq. [126]). On the other hand, since $S_1 + S_2$ has to be the same for *all* solutions, it is clear that *the general solution is expressible in the form*

$$\xi = \xi_{\text{ax}} + \gamma[\xi^{(1)} - \xi^{(2)}] , \tag{130}$$

where ξ_{ax} is the uniquely determined axisymmetric solution and γ is an arbitrary constant.

Expressed in the form (130), the origin of the indeterminacy in the solution, through the occurrence of the term

$$\gamma[\xi^{(1)} - \xi^{(2)}] = \gamma(x_1, -x_2, 0) \tag{131}$$

is clear: it arises from the Maclaurin spheroid, at the point of bifurcation, being neutral to an infinitesimal deformation proportional to $(x_1, -x_2, 0)$ (cf. Chandrasekhar 1963, eq. [27]). *The continuous range of post-Newtonian equilibrium configurations which occur at the point of bifurcation is a general-relativistic manifestation of the Newtonian instability which sets in at this point if some dissipative mechanism is present.*

IX. THE ORIGIN OF THE SINGULARITY AT $\cos^{-1}a_3/a_1 = 75.^\circ 081$

The mathematical origin of the singularity in the solution that occurs at a determinate point ($\cos^{-1} a_3/a_1 = 75.^\circ 081$) along the Jacobian sequence is clear: at that point, the determinant of the homogeneous system, associated with the linear equations determining the constants S_3, S_4, and S_5 (say eqs. [83]–[85]), vanishes. The physical origin of the singularity, as we shall presently explain, is that at that point the Jacobi ellipsoid is neutral for deformation by an infinitesimal displacement of the form

$$\xi = \frac{\pi G\rho a_1^2}{c^2} \sum_{i=1}^{5} S_i \xi^{(i)} , \tag{132}$$

where the coefficients S_3, S_4, and S_5 of the cubic displacements, $\xi^{(3)}$, $\xi^{(4)}$, and $\xi^{(5)}$, are not all zero. (The factor $\pi G\rho a_1^2/c^2$ on the right-hand side of eq. [131] is not relevant for our present considerations; but it has been included in order that the results of §§ IV–VIII can be used with the minimum of alterations.)

Quite generally, the condition for the occurrence of a neutral point for a chosen deformation is that the conditions for hydrostatic equilibrium are unaffected by it. In the present context, the deformation of the Jacobi ellipsoid by the displacement (132) will change the gravitational potential U by an amount that will be given by equation (74). And the corresponding change in Ω^2 will be given by equations (66) with the terms in $E_{\mu\nu}$ on the right-hand sides suppressed. The analysis in § VII will continue to apply, and we shall eventually be led to formally the same set of equations (83)–(87) with the only difference that in the definitions of the P's given in equations (77) the "inhomogeneous" terms in the α's should be suppressed. And finally, the propositions, that S_1 and S_2 do not occur in equations (86) and (87) and that only three of the five equations (83)–(87) are linearly independent, continue to be valid in the absence of the terms in the α's. The condition, then, that the Jacobi ellipsoid is neutral to a non-trivial deformation, cubic in the coordinates and of the form (131), is that the determinant of the equations (83)–(85) (which are now homogeneous) vanishes; but this, as we have seen, is precisely the condition that the solution of the post-Newtonian equations has a singularity. The situation here is the same as that we encountered in Paper II at the point $e^* = 0.985226$ along the Maclaurin sequence. And as in Paper II, we may now say that *the Newtonian instability of the Jacobi ellipsoid, for a cubic deformation of the form (132) is excited by the post-Newtonian effects of general relativity.*

X. CONCLUDING REMARKS

The determination of the figures of equilibrium of uniformly rotating homogeneous masses in the post-Newtonian approximation has brought out two features of possible general interest: first, the instability of *Newtonian figures* to the *type* of deformation induced by general relativity makes their physical existence valid only in an asymptotic sense; and second, the neutrality of the Newtonian figures to the type of deformation induced by rotation makes for an indeterminacy in the relativistic solutions.

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APPENDIX I

Two principal facts concerning equations (83)–(87) that were stated in § VII without proofs are: (1) that there are no terms in S_1 and S_2 in equations (86) and (87) if we substitute for $\frac{1}{2}(a_1^2 - a_2^2)\delta\omega^2$ and $\frac{1}{2}a_1^2\delta\omega^2$, which occur in these equations via the terms $a_1^2Q_1 - a_2^2Q_2$ and $a_1^2Q_1$ (cf. eqs. [81]), the expressions (88) and (89), respectively; and (2) that equations (90) and (91) are satisfied and that in consequence only three of the five equations (83)–(87) are linearly independent. These two facts establish the sufficiency of equations (83)–(87) to determine the constants S_3, S_4, and S_5 uniquely. In this Appendix, we shall indicate, very briefly, how the truth of the two basic statements can be demonstrated

By substituting for Q_μ in accordance with equations (77) and (81) and for $\delta\omega^2$ in accordance with equation (88) (in the case of eq. [86]) or (89) (in the case of eq [87]), we find that the terms in S_1 and S_2 in the two equations are

$$\begin{aligned}\sum_{i=1}^{2} S_i a_1^2[a_1^2 u_1^{(i)} - a_2^2 u_2^{(i)}] - 2a_1^2a_3^2A_3S_1 + 2a_1^2a_3^2A_3S_2 \\ + S_1a_1^2[3a_1^2(B_{11} - B_{12}) - a_3^2(B_{13} - B_{23})] \\ - S_2a_1^2[3a_2^2(B_{22} - B_{12}) + a_3^2(B_{13} - B_{23})]\end{aligned} \tag{AI.1}$$

and

$$\sum_{i=1}^{2} S_i a_1^2 [a_3^2 u_3^{(i)} - a_1^2 u_1^{(i)}] + 2 a_1^2 a_3^2 A_3 (S_1 + S_2) + 2 a_1^2 a_3^2 A_3 S_1$$
$$- S_1 a_1^2 [a_1^2 (3B_{11} - B_{13} - 2B_{12}) + a_3^2 (3B_{33} - B_{13})] \tag{AI.2}$$
$$- S_2 a_1^2 [a_2^2 (B_{12} - B_{32}) + a_3^2 (3B_{33} - B_{13})] .$$

Substituting for the $u_\mu^{(i)}$'s from equations (72), we find that the coefficients of S_1 and S_2, in each of the two foregoing expressions, vanish if appropriate use is made of the elementary identities among the one- and the two-index symbols and of the particular relation

$$a_3^2 A_3 = a_1^2 a_2^2 A_{12} , \tag{AI.3}$$

which determines the geometry of the Jacobi ellipsoid.

Considering next equations (90) and (91), we first rewrite them in the forms

$$6a_1^4 Q_{11} - 6a_2^4 Q_{22} + a_3^2 a_1^2 Q_{31} - a_2^2 a_3^2 Q_{23} + 7a_1^2 Q_1 - 7a_2^2 Q_2 = 0 \tag{AI.4}$$

and

$$6a_3^4 Q_{33} - 6a_1^4 Q_{11} + a_2^2 a_3^2 Q_{23} - a_1^2 a_2^2 Q_{12} + 7a_3^2 Q_3 - 7a_1^2 Q_1 = 0 . \tag{AI.5}$$

Next, substituting for the Q's and for $\delta\omega^2$ (from eq. [88] in eq. [AI.4] and from eq. [89] in eq. [AI.5]), as before and writing out fully the expressions for the E's given in equation (41), we find that the terms in the α's cancel (in both equations) and we are left with

$$\sum_{i=3}^{5} S_i [6 a_1^4 u_{11}^{(i)} - 6 a_2^4 u_{22}^{(i)} + a_3^2 a_1^2 u_{31}^{(i)} - a_2^2 a_3^2 u_{23}^{(i)} + 7 a_1^2 u_1^{(i)} - 7 a_2^2 u_2^{(i)}]$$
$$+ S_3 [- 4a_1^2 a_3^2 A_3 + 3a_1^4 (B_{11} - B_{12}) + 3a_1^2 a_2^2 (B_{22} - B_{11})]$$
$$+ S_4 [+ 2a_2^2 a_3^2 A_3 - 3a_2^4 (B_{22} - B_{12}) - a_2^2 a_3^2 (B_{13} - B_{23})] \tag{AI.6}$$
$$+ S_5 [+ 2a_3^4 A_3 - 3a_1^2 a_3^2 (B_{11} - B_{12}) + a_3^4 (B_{13} - B_{23})] = 0$$

and

$$\sum_{i=3}^{5} S_i [6 a_3^4 u_{33}^{(i)} - 6 a_1^4 u_{11}^{(i)} + a_2^2 a_3^2 u_{23}^{(i)} - a_1^2 a_2^2 u_{12}^{(i)} + 7 a_3^2 u_3^{(i)} - 7 a_1^2 u_1^{(i)}]$$
$$+ S_3 [+ 2a_1^2 a_3^2 A_3 - a_1^4 (3B_{11} - B_{13} - 2B_{12}) + a_1^2 a_2^2 (B_{12} - B_{32})]$$
$$+ S_4 [+ 2a_2^2 a_3^2 A_3 - a_2^4 (B_{12} - B_{32}) - a_2^2 a_3^2 (3B_{33} - B_{13})] \tag{AI.7}$$
$$+ S_5 [- 4a_3^4 A_3 + a_1^2 a_3^2 (3B_{11} - B_{13} - 2B_{12}) + a_3^4 (3B_{33} - B_{13})] = 0 .$$

If we now substitute for the u's their known expressions (from eqs. [72] and [73]), we find that the coefficients of S_3, S_4, and S_5, in both equations, vanish though the reductions necessary to show that they do are considerable. However, the reductions present no special difficulties if, after expressing all the B-symbols in terms of the A-symbols with the aid of the formula

$$B_{ijk\ldots} = A_{jk\ldots} - a_i^2 A_{ijk\ldots} , \tag{AI.8}$$

we effect the reductions systematically with a view to utilizing the following relations between the symbols of successive orders:

$$
\begin{aligned}
&7A_{iiii}a_i^2 + A_{iiij}a_j^2 + A_{iiik}a_k^2 = 7A_{iii} , \\
&5A_{iiij}a_i^2 + 3A_{iijj}a_j^2 + A_{iijk}a_k^2 = 7A_{iij} , \\
&3A_{iijk}a_i^2 + 3A_{ijjk}a_j^2 + 3A_{ijkk}a_k^2 = 7A_{ijk} , \\
&5A_{iii}a_i^2 + A_{iij}a_j^2 + A_{iik}a_k^2 = 5A_{ii} , \\
&3A_{iij}a_i^2 + 3A_{ijj}a_j^2 + A_{ijk}a_k^2 = 5A_{ij} , \qquad (i \neq j \neq k) ;
\end{aligned}
\tag{AI.9}
$$

and no summation over repeated indices in eqs [AI 9]).

APPENDIX II

In this Appendix, we shall indicate how the validity of equations (107) and (108) can be demonstrated.

Considering first equation (107), we find by using the known expressions for the $u_{\mu\nu}$'s (from eq. [73]) that

$$u_{11}^{(3)} + u_{22}^{(3)} - u_{12}^{(3)} = \tfrac{8}{3} a_1^6 A_{1111} , \tag{AII.1}$$

$$u_{11}^{(4)} + u_{22}^{(4)} - u_{12}^{(4)} = \tfrac{2}{3} a_1^6 A_{1111} , \tag{AII.2}$$

and

$$u_{11}^{(5)} + u_{22}^{(5)} - u_{12}^{(5)} = 0 ; \tag{AII.3}$$

and equation (107) follows.

Considering next equation (108), we similaıly find

$$
\begin{aligned}
a_1^2 u_{11}^{(3)} - a_1^2 u_{22}^{(3)} - a_3^2 u_{31}^{(3)} + a_3^2 u_{23}^{(3)} &= \tfrac{8}{3} a_1^8 A_{1111} \\
&- 2a_1^6 a_3^2 A_{1113} - 2a_1^6 A_{111} + 2a_1^4 a_3^2 A_{113} ,
\end{aligned}
\tag{AII.4}
$$

$$
\begin{aligned}
a_1^2 u_{11}^{(4)} - a_1^2 u_{22}^{(4)} - a_3^2 u_{31}^{(4)} + a_3^2 u_{23}^{(4)} &= -\tfrac{8}{3} a_1^8 A_{1111} \\
&+ 3a_1^6 a_3^2 A_{1113} - 3a_1^4 a_3^4 A_{1133} + a_1^6 A_{111} - a_1^4 a_3^2 A_{113} ,
\end{aligned}
\tag{AII.5}
$$

and

$$
\begin{aligned}
a_1^2 u_{11}^{(5)} - a_1^2 u_{22}^{(5)} - a_3^2 u_{31}^{(5)} + a_3^2 u_{23}^{(5)} &= 3a_1^2 a_3^6 A_{1133} \\
&- a_1^4 a_3^4 A_{1113} + a_1^4 a_3^2 A_{111} - a_1^2 a_3^4 A_{113} .
\end{aligned}
\tag{AII.6}
$$

Inserting the foregoing relations on the left-hand side of equation (108) and reducing the resulting expression with the aid of equations (AI.9) we find that it vanishes as required.

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