THE EQUILIBRIUM AND THE STABILITY OF THE RIEMANN ELLIPSOIDS. II

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ABSTRACT

In this paper we consider ellipsoidal figures of equilibrium (of semi-axes a_1, a_2, and a_3) of homogeneous masses rotating uniformly with an angular velocity $\mathbf{\Omega}$ and with internal motions having a uniform vorticity $\boldsymbol{\zeta}$ (in the rotating frame) in the case that the directions of $\mathbf{\Omega}$ and $\boldsymbol{\zeta}$ do not coincide. Riemann's theorem, that in this case $\mathbf{\Omega}$ and $\boldsymbol{\zeta}$ must lie in a principal plane of the ellipsoid, is shown to follow from a consideration of the non-diagonal components of the second-order tensor-virial theorem. The conditions for equilibrium are also derived; and the domains of occupancy of these Riemann ellipsoids in the $(a_2/a_1, a_3/a_1)$-plane (on the assumptions, which entail no loss of generality, that $\mathbf{\Omega}$ and $\boldsymbol{\zeta}$ have no components in the x_1-direction and that $a_2 \geq a_3$) are explicitly specified.

It is shown that the equilibrium ellipsoids are of three types: ellipsoids of type I which occupy the domain $2a_1 \geq (a_2 + a_3)$ and $a_2 \geq a_1 \geq a_3$; ellipsoids of type II for which $a_2 \geq 2a_1$ and a_3/a_1 (≤ 1) are limited by a locus along which $\int p dx = 0$; and ellipsoids of type III which occupy the domain limited by $2a_1 \leq (a_2 - a_3)$ and a locus along which $\Omega_2 = \zeta_2 = 0$ and $a_3 \geq a_1$. And quite generally, it is shown that an ellipsoid, represented by a point in the allowed domain of occupancy, is a figure of equilibrium for two different states of motion $(\mathbf{\Omega}, \boldsymbol{\zeta})$ and $(\mathbf{\Omega}^\dagger, \boldsymbol{\zeta}^\dagger)$; and that the two resulting configurations are adjoints of one another in the sense of Dedekind's theorem.

Ellipsoids of type I may be considered as branching off from the Maclaurin sequence with an odd mode of oscillation neutralized at the point of bifurcation by the choice of Ω_3 and ζ_3 (Ω_2 and ζ_2 being zero). And ellipsoids of type III may be similarly considered as branching off from the ellipsoids of type S (for which the directions of $\mathbf{\Omega}$ and $\boldsymbol{\zeta}$ coincide with the x_3-axis) along the curve where they are marginally unstable.

The stability of the Riemann ellipsoids with respect to oscillations belonging to the second harmonics is also investigated. It is first shown that the characteristic frequencies of oscillation of an ellipsoid and its adjoint are the same; and further that $|\mathbf{\Omega}|$ and $|\mathbf{\Omega}^\dagger|$ are allowed proper frequencies. The loci along which instability sets in, in the different domains of occupancy, are determined. Of particular interest are the facts that *all* ellipsoids of type II are unstable; that along the curve where the ellipsoids of type III branch off from ellipsoids of type S, the stability passes from the latter to the former; and that among the ellipsoids of type I there are some very highly flattened ones that are stable.

Several statements of Riemann concerning the stability of these ellipsoids are not substantiated by the present detailed investigation. The origin of Riemann's errors is clarified in the paper by Lebovitz following this one.

I. INTRODUCTION

Pursuing earlier investigations of Dirichlet and Dedekind, Riemann (1860; see also Hicks 1882 and Basset 1888) proved that *the most general type of motion (linear in the coordinates) compatible with an ellipsoidal figure of equilibrium of a homogeneous mass consists of a superposition of a uniform rotation* $\mathbf{\Omega}$ *and internal motions of a uniform vorticity* $\boldsymbol{\zeta}$ *(in the rotating frame) about axes that lie in a principal plane of the ellipsoid.* More precisely, according to Riemann's theorem there are three distinct circumstances (and *only* three) under which ellipsoidal figures of equilibrium can arise. These are: (*a*) the case of uniform rotation $\mathbf{\Omega}$ about the least axis of the ellipsoid; (*b*) the case when the directions of $\mathbf{\Omega}$ and $\boldsymbol{\zeta}$ coincide with a principal axis of the ellipsoid; and (*c*) the case when the directions of $\mathbf{\Omega}$ and $\boldsymbol{\zeta}$ do not coincide but lie in a principal plane of the ellipsoid. Case (*a*) leads to the classical sequences of Maclaurin and Jacobi; case (*b*) leads to the various *Riemann sequences* considered in an earlier paper (Chandrasekhar 1965*b;* this paper will be referred to hereafter as "Paper I"); and case (*c*) will be considered in this paper.

II. THE EQUATIONS DETERMINING THE EQUILIBRIUM ELLIPSOIDS: RIEMANN'S THEOREM

We shall consider quite generally the conditions under which a homogeneous ellipsoid, with semi-axes a_1, a_2, and a_3, can be a figure of equilibrium when subject to a uniform rotation $\boldsymbol{\Omega}$ and internal motions (linear in the coordinates) with a uniform vorticity $\boldsymbol{\zeta}$ in the rotating frame.

We shall suppose that the coordinate axes are along the principal axes of the ellipsoid and, further, that $\boldsymbol{\Omega}$ and $\boldsymbol{\zeta}$, in the chosen coordinate system, have the components Ω_1, Ω_2, and Ω_3 and ζ_1, ζ_2, and ζ_3. The condition that the internal motion associated with $\boldsymbol{\zeta}$ preserve the ellipsoidal boundary requires that it be expressible in the form

$$u_1 = -\frac{a_1^2}{a_1^2+a_2^2}\zeta_3 x_2 + \frac{a_1^2}{a_1^2+a_3^2}\zeta_2 x_3 ,$$
$$u_2 = -\frac{a_2^2}{a_2^2+a_3^2}\zeta_1 x_3 + \frac{a_2^2}{a_2^2+a_1^2}\zeta_3 x_1 , \tag{1}$$
$$u_3 = -\frac{a_3^2}{a_3^2+a_1^2}\zeta_2 x_1 + \frac{a_3^2}{a_3^2+a_2^2}\zeta_1 x_2 .$$

To obtain the conditions that the ellipsoid is also in gravitational equilibrium, we shall make use of the second-order virial theorem. According to this theorem

$$\frac{d}{dt}\int_V \rho u_i x_j d\boldsymbol{x} = 2\mathfrak{T}_{ij} + \Omega^2 I_{ij} - \Omega_i\Omega_k I_{kj} + \mathfrak{W}_{ij} + \delta_{ij}\Pi + 2\epsilon_{ilm}\Omega_m\int_V \rho u_l x_j d\boldsymbol{x} , \tag{2}$$

where the various symbols have their usual meanings. Under conditions of a stationary state, equation (2) gives

$$2\mathfrak{T}_{ij} + \Omega^2 I_{ij} - \Omega_i\Omega_k I_{kj} + \mathfrak{W}_{ij} + 2\epsilon_{ilm}\Omega_m\int_V \rho u_l x_j d\boldsymbol{x} = -\delta_{ij}\Pi . \tag{3}$$

Consider first the non-diagonal components of equation (3). The (2,3)- and the (3,2)-components of equation (3) give, for example,

$$2\mathfrak{T}_{23} - \Omega_2\Omega_3 I_{33} - 2\Omega_3\int_V \rho u_1 x_3 d\boldsymbol{x} = 0 \tag{4}$$

and

$$2\mathfrak{T}_{32} - \Omega_3\Omega_2 I_{22} + 2\Omega_2\int_V \rho u_1 x_2 d\boldsymbol{x} = 0 , \tag{5}$$

since, in the chosen coordinate system, the tensors I_{ij} and $\mathfrak{W}_{ij}$ are diagonal and, moreover,

$$\int_V \rho u_i x_j d\boldsymbol{x} = 0 \qquad \text{if} \qquad i = j . \tag{6}$$

Adding and subtracting equations (4) and (5), we get

$$4\mathfrak{T}_{23} - \Omega_2\Omega_3(I_{22}+I_{33}) + 2\int_V \rho u_1(\Omega_2 x_2 - \Omega_3 x_3)d\boldsymbol{x} = 0$$

and

$$\Omega_2\Omega_3(I_{22}-I_{33}) - 2\int_V \rho u_1(\Omega_2 x_2 + \Omega_3 x_3)d\boldsymbol{x} = 0 . \tag{7}$$

For the motions specified in equations (1)

$$2\mathfrak{T}_{23}=-\frac{a_2^2a_3^2}{(a_1^2+a_2^2)(a_1^2+a_3^2)}\zeta_2\zeta_3 I_{11}, \tag{8}$$

$$\int_V \rho u_1 x_2 d\boldsymbol{x}=-\frac{a_1^2}{a_1^2+a_2^2}\zeta_3 I_{22}, \quad \text{and} \quad \int_V \rho u_1 x_3 d\boldsymbol{x}=+\frac{a_1^2}{a_1^2+a_3^2}\zeta_2 I_{33}. \tag{9}$$

Inserting the foregoing relations in equation (7) and substituting for I_{ij} its value in terms of the mass of the ellipsoid and its semi-axes, we find, after some rearrangements, the equations

$$a_2^2+a_3^2+\frac{2a_1^2a_3^2}{a_1^2+a_3^2}\frac{\zeta_2}{\Omega_2}+\frac{2a_1^2a_2^2}{a_1^2+a_2^2}\frac{\zeta_3}{\Omega_3}+\frac{2a_1^2a_2^2a_3^2}{(a_1^2+a_3^2)(a_1^2+a_2^2)}\frac{\zeta_2}{\Omega_2}\frac{\zeta_3}{\Omega_3}=0 \tag{10}$$

and

$$a_3^2+\frac{2a_1^2a_3^2}{a_1^2+a_3^2}\frac{\zeta_2}{\Omega_2}=a_2^2+\frac{2a_1^2a_2^2}{a_1^2+a_2^2}\frac{\zeta_3}{\Omega_3}, \tag{11}$$

where, in writing the equations in these forms, *we have supposed that* Ω_2 *and* Ω_3 *are different from zero.*

Now letting

$$\beta=-\frac{a_3^2}{a_1^2+a_3^2}\frac{\zeta_2}{\Omega_2} \quad \text{and} \quad \gamma=-\frac{a_2^2}{a_1^2+a_2^2}\frac{\zeta_3}{\Omega_3}, \tag{12}$$

we can rewrite equations (10) and (11) in the forms

$$\beta+\gamma-\beta\gamma=\frac{a_3^2+a_2^2}{2a_1^2} \tag{13}$$

and

$$\beta-\gamma=\frac{a_3^2-a_2^2}{2a_1^2}. \tag{14}$$

Equations (13) and (14) provide for β and γ the equations

$$\beta^2-\frac{4a_1^2+a_3^2-a_2^2}{2a_1^2}\beta+\frac{a_3^2}{a_1^2}=0 \tag{15}$$

and

$$\gamma^2-\frac{4a_1^2+a_2^2-a_3^2}{2a_1^2}\gamma+\frac{a_2^2}{a_1^2}=0. \tag{16}$$

The roots of these equations are

$$\beta=\frac{1}{4a_1^2}\{4a_1^2-a_2^2+a_3^2\pm\sqrt{[4a_1^2-(a_2+a_3)^2][4a_1^2-(a_2-a_3)^2]}\} \tag{17}$$

and

$$\gamma=\frac{1}{4a_1^2}\{4a_1^2+a_2^2-a_3^2\pm\sqrt{[4a_1^2-(a_2+a_3)^2][4a_1^2-(a_2-a_3)^2]}\}. \tag{18}$$

Thus, if Ω_2 and Ω_3 are assumed to be different from zero, the ratios ζ_2/Ω_2 and ζ_3/Ω_3 are determined by the foregoing equations. In particular, equation (15), expressed in terms of ζ_2/Ω_2, is

$$\left(\frac{\zeta_2}{\Omega_2}\right)^2+(4a_1^2+a_3^2-a_2^2)\frac{a_1^2+a_3^2}{2a_1^2a_3^2}\left(\frac{\zeta_2}{\Omega_2}\right)+\frac{(a_1^2+a_3^2)^2}{a_1^2a_3^2}=0. \tag{19}$$

On the other hand, if Ω_1 is also different from zero, then the (1,2)- and the (2,1)-components of equation (3) would have led to the equation

$$\left(\frac{\zeta_2}{\Omega_2}\right)^2+(4a_3^2+a_1^2-a_2^2)\frac{a_1^2+a_3^2}{2a_1^2a_3^2}\left(\frac{\zeta_2}{\Omega_2}\right)+\frac{(a_1^2+a_3^2)^2}{a_1^2a_3^2}=0\,. \tag{20}$$

Equations (19) and (20) are clearly incompatible unless $a_1 = a_3$; and the consideration of the equations governing ζ_3/Ω_3 would have similarly required that $a_1 = a_2$. It therefore follows that *non-trivial solutions are obtained only if no more than two of the three pairs of components* (ζ_1,Ω_1), (ζ_2,Ω_2), *and* (ζ_3,Ω_3) *are different from zero.* This is Riemann's theorem.

If we assume that (ζ_2,Ω_2) and (ζ_3,Ω_3) are different from zero, while (ζ_1,Ω_1) is zero, then, the (1,2)-, (2,1)-, (1,3)-, and (3,1)-components of equation (3) will be trivially satisfied and the only non-trivial relations are those that follow from the (2,3)- and the (3,2)-components; and these relations, as we have seen, determine the ratios ζ_2/Ω_2 and ζ_3/Ω_3. On the other hand, if two of the three components of $\mathbf{\Omega}$ and $\boldsymbol{\zeta}$, say (Ω_1,ζ_1) and (Ω_2,ζ_2), are assumed to vanish, then *all* the non-diagonal components of equation (3) will be trivially satisfied and the problem reduces to the one already considered in detail in Paper I.

In our further considerations, we shall suppose that Ω_2 and Ω_3 are different from zero while Ω_1 and ζ_1 are zero. Then, the internal motions specified in equation (1) become

$$\begin{aligned} u_1&=-\frac{a_1^2}{a_1^2+a_2^2}\,\zeta_3x_2+\frac{a_1^2}{a_1^2+a_3^2}\,\zeta_2x_3=\Omega_3\gamma\,\frac{a_1^2}{a_2^2}\,x_2-\Omega_2\beta\,\frac{a_1^2}{a_3^2}\,x_3\,,\\ u_2&=+\frac{a_2^2}{a_2^2+a_1^2}\,\zeta_3x_1=-\Omega_3\gamma x_1\,,\\ u_3&=-\frac{a_3^2}{a_3^2+a_1^2}\,\zeta_2x_1=+\Omega_2\beta x_1\,; \end{aligned} \tag{21}$$

and the ratios ζ_2/Ω_2 and ζ_3/Ω_3 are determined by the solutions for β and γ expressed in equations (17) and (18).

It remains to determine the values of Ω_2 and Ω_3 that are to be associated with the ellipsoid. The necessary additional relations follow from a consideration of the diagonal components of equation (3). And under the present circumstances ($\Omega_1 = \zeta_1 = 0$) equation (3) gives

$$2\mathfrak{T}_{11}+(\Omega_2^2+\Omega_3^2)I_{11}+\mathfrak{W}_{11}+2\int_V\rho x_1(\Omega_3u_2-\Omega_2u_3)\,dx=-\Pi\,, \tag{22}$$

$$2\mathfrak{T}_{22}+\Omega_3^2I_{22}+\mathfrak{W}_{22}-2\Omega_3\int_V\rho u_1x_2dx=-\Pi\,, \tag{23}$$

and

$$2\mathfrak{T}_{33}+\Omega_2^2I_{33}+\mathfrak{W}_{33}+2\Omega_2\int_V\rho u_1x_3dx=-\Pi\,. \tag{24}$$

On evaluating the components of the kinetic-energy tensor and the moments of the velocities that occur in the foregoing equations, in accordance with equations (21), we find

$$\left[\Omega_3^2\left(1-2\gamma+\frac{a_1^2}{a_2^2}\,\gamma^2\right)+\Omega_2^2\left(1-2\beta+\frac{a_1^2}{a_3^2}\,\beta^2\right)\right]I_{11}+\mathfrak{W}_{11}=-\Pi\,, \tag{25}$$

$$\Omega_3^2\left(\gamma^2-2\gamma+\frac{a_2^2}{a_1^2}\right)I_{11}+\mathfrak{W}_{22}=-\Pi\,, \tag{26}$$

and

$$\Omega_2^2\left(\beta^2-2\beta+\frac{a_3^2}{a_1^2}\right)I_{11}+\mathfrak{W}_{33}=-\Pi\,, \tag{27}$$

where, by equations (15) and (16),

$$\beta^2-2\beta+\frac{a_3^2}{a_1^2}=\frac{a_3^2-a_2^2}{2a_1^2}\beta\,;\qquad \gamma^2-2\gamma+\frac{a_2^2}{a_1^2}=\frac{a_2^2-a_3^2}{2a_1^2}\gamma\,; \tag{28}$$

$$1-2\beta+\frac{a_1^2}{a_3^2}\beta^2=\frac{4a_1^2-a_2^2-3a_3^2}{2a_3^2}\beta\,; \tag{29}$$

and

$$1-2\gamma+\frac{a_1^2}{a_2^2}\gamma^2=\frac{4a_1^2-a_3^2-3a_2^2}{2a_2^2}\gamma\,. \tag{30}$$

Eliminating Π between equations (26) and (27) and making use of the relations (28), we find

$$\beta\Omega_2^2+\gamma\Omega_3^2=\frac{2a_1^2}{a_3^2-a_2^2}\,\frac{\mathfrak{W}_{22}-\mathfrak{W}_{33}}{I_{11}}\,. \tag{31}$$

The expressions for the components of the moment of inertia and the potential-energy tensors of homogeneous ellipsoids have been given in an earlier paper (Chandrasekhar and Lebovitz 1962, eqs. [57] and [58]); with their aid, equation (31) gives

$$\beta\Omega_2^2+\gamma\Omega_3^2=4\,\frac{A_3a_3^2-A_2a_2^2}{a_3^2-a_2^2}=4B_{23}\,, \tag{32}$$

where the index symbols A_i, A_{ij}, and B_{ij} are so normalized that $\Sigma A_i=2$ and Ω^2 and ζ^2 are measured in the unit $\pi G\rho$ (see eqs. [44] and [45] below).

Next, eliminating Π between equations (25) and (26), we have (on making use of eqs. [28])

$$\Omega_3^2\left(1-\frac{a_2^2}{a_1^2}\right)\left(1+\frac{a_1^2}{a_2^2}\gamma^2\right)+\Omega_2^2\left(1-2\beta+\frac{a_1^2}{a_3^2}\beta^2\right)=\frac{2}{a_1^2}(A_1a_1^2-A_2a_2^2)\,, \tag{33}$$

or, alternatively (cf. eq. [29]),

$$2B_{12}-\Omega_3^2\left(1+\frac{a_1^2}{a_2^2}\gamma^2\right)=\frac{a_1^2}{a_2^2-a_1^2}\,\frac{3a_3^2-4a_1^2+a_2^2}{2a_3^2}\,\Omega_2^2\beta\,. \tag{34}$$

Similarly, by eliminating Π between equations (25) and (27), we obtain

$$2B_{13}-\Omega_2^2\left(1+\frac{a_1^2}{a_3^2}\beta^2\right)=\frac{a_1^2}{a_3^2-a_1^2}\,\frac{3a_2^2-4a_1^2+a_3^2}{2a_2^2}\,\Omega_3^2\gamma\,. \tag{35}$$

Making use of the readily verified relation (cf. eq. [16])

$$\frac{1}{\gamma}\left(1+\frac{a_1^2}{a_2^2}\gamma^2\right)=\frac{4a_1^2+a_2^2-a_3^2}{2a_2^2}\,, \tag{36}$$

and eliminating Ω_3^2 between equations (32) and (34), we obtain

$$\Omega_2^2\beta=\frac{4a_3^2(a_2^2-a_1^2)}{a_2^2-a_3^2}\,\frac{(4a_1^2+a_2^2-a_3^2)B_{23}-a_2^2B_{12}}{4a_1^4-a_1^2(a_2^2+a_3^2)+a_2^2a_3^2}\,. \tag{37}$$

Similarly, by eliminating Ω_2^2 between equations (32) and (35), we obtain

$$\Omega_3^2\gamma = \frac{4a_2^2(a_3^2-a_1^2)}{a_3^2-a_2^2}\,\frac{(4a_1^2+a_3^2-a_2^2)B_{23}-a_3^2B_{13}}{4a_1^4-a_1^2(a_2^2+a_3^2)+a_2^2a_3^2}. \tag{38}$$

Equations (37) and (38) together with equations (17) and (18) determined the angular velocities and the vorticities that are to be associated with an ellipsoid with semi-axes a_1, a_2, and a_3.

To complete the solution, we must determine Π. First, we observe that, by making use of equations (28)–(30), we can rewrite equations (25)–(27) in the forms

$$(\Omega_2^2\beta+\Omega_3^2\gamma)-\tfrac{1}{2}(4a_1^2-a_2^2-a_3^2)\left(\frac{\Omega_2^2\beta}{a_3^2}+\frac{\Omega_3^2\gamma}{a_2^2}\right)+2A_1=\frac{5\Pi}{Ma_1^2}, \tag{39}$$

$$\frac{a_3^2-a_2^2}{2a_2^2}\,\Omega_3^2\gamma+2A_2=\frac{5\Pi}{Ma_2^2}, \tag{40}$$

and

$$\frac{a_2^2-a_3^2}{2a_3^2}\,\Omega_2^2\beta+2A_3=\frac{5\Pi}{Ma_3^2}, \tag{41}$$

where M denotes the mass of the ellipsoid. From equations (40) and (41) we obtain

$$\tfrac{1}{2}\left(\frac{\Omega_2^2\beta}{a_3^2}+\frac{\Omega_3^2\gamma}{a_2^2}\right)+2A_{23}=\frac{5\Pi}{Ma_2^2a_3^2}, \tag{42}$$

where we have made use of the relation $(A_2-A_3)/(a_3^2-a_2^2)=A_{23}$. Now combining equations (32), (39), and (42), we obtain

$$\frac{5\Pi}{2Ma_1^2a_2^2a_3^2}=\frac{2B_{23}+(4a_1^2-a_2^2-a_3^2)A_{23}+A_1}{4a_1^4-a_1^2(a_2^2+a_3^2)+a_2^2a_3^2}. \tag{43}$$

In the chosen normalization $(\Sigma A_i=2)$ the index symbols have the values

$$A_i=a_1a_2a_3\int_0^\infty\frac{du}{(a_i^2+u)\Delta},\qquad A_{ij}=a_1a_2a_3\int_0^\infty\frac{du}{(a_i^2+u)(a_j^2+u)\Delta},$$

and

$$B_{ij}=a_1a_2a_3\int_0^\infty\frac{u\,du}{(a_i^2+u)(a_j^2+u)\Delta}=A_i-a_j^2A_{ij}=A_j-a_i^2A_{ij}, \tag{44}$$

where

$$\Delta^2=(a_1^2+u)(a_2^2+u)(a_3^2+u). \tag{45}$$

Inserting for the index symbols that appear in equations (37), (38), and (43), in accordance with the foregoing definitions, we find

$$\Omega_2^2\beta=4a_1a_2a_3\,\frac{a_3^2(a_2^2-a_1^2)}{(a_2^2-a_3^2)D}$$
$$\times\int_0^\infty[(4a_1^2-a_3^2)u+a_1^2(4a_1^2+a_2^2-a_3^2)-a_2^2a_3^2]\frac{u\,du}{\Delta^3}, \tag{46}$$

$$\Omega_3^2\gamma=4a_1a_2a_3\,\frac{a_2^2(a_3^2-a_1^2)}{(a_3^2-a_2^2)D}$$
$$\times\int_0^\infty[(4a_1^2-a_2^2)u+a_1^2(4a_1^2+a_3^2-a_2^2)-a_2^2a_3^2]\frac{u\,du}{\Delta^3}, \tag{47}$$

and

$$\frac{5\Pi}{2a_1^3a_2^3a_3^3M}=\frac{1}{D}\int_0^\infty (3u^2+6a_1^2u+D)\frac{du}{\Delta^3}, \tag{48}$$

where

$$D = 4a_1^4 - a_1^2(a_2^2+a_3^2)+a_2^2a_3^2 . \tag{49}$$

The equations in essentially these forms are due to Riemann.

III. THE DOMAIN OF OCCUPANCY IN THE $(a_2/a_1,\ a_3/a_1)$-PLANE

It is clear that any set of values (a_1, a_2, a_3) that is consistent with equations (17), (18), (37), (38), and (43) (or, equivalently, eqs. [17], [18], and [46]–[49]) and leads to realizable values for the various physical parameters provides an admissible solution. As we shall presently see in some detail, the physical requirements that β, γ, Ω_2, and Ω_3 are real and that $\Pi \geq 0$ limit the domain of occupancy of these ellipsoidal figures of equilibrium in the $(a_2/a_1, a_3/a_1)$-plane. In determining the nature of these limits, we shall follow Riemann's original discussion (see also Basset 1888). But we shall arrange the arguments somewhat differently; and shall, moreover, specify the domain of occupancy explicitly.

First, we observe that since all the equations are symmetric in the indices 2 and 3, the domain of occupancy in the $(a_2/a_1, a_3/a_1)$-plane must be symmetrically situated about the 45°-line, $a_2 = a_3$. Therefore, without loss of generality, we may restrict ourselves to the part of the plane

$$a_2 \geq a_3 . \tag{50}$$

Next, we observe that the reality of β and γ requires that

$$\textit{either } 2a_1 \geq (a_2+a_3) \textit{ or } 2a_1 \leq |a_2-a_3| = a_2-a_3 ; \tag{51}$$

and these two cases must be considered separately.

a) *Case I:* $2a_1 \geq (a_2+a_3)$ *and* $a_2 \geq a_3$

Under the restrictions of this case

$$4a_1^2 \pm (a_3^2-a_2^2) \geq (a_2+a_3)^2 \pm (a_3^2-a_2^2) > 0 . \tag{52}$$

In view of this inequality, it follows from equations (15) and (16) that

$$\beta > 0 \qquad \text{and} \qquad \gamma > 0 . \tag{53}$$

The reality of Ω_2 and Ω_3 now requires that the quantities on the right-hand sides of equations (46) and (47) are positive. Clearly,

$$D \geq a_1^2(a_2+a_3)^2 - a_1^2(a_2^2+a_3^2)+a_2^2a_3^2 > 0 . \tag{54}$$

Also, making use of the inequality (52), we have

$$a_1^2[4a_1^2 \pm (a_2^2-a_3^2)] - a_2^2a_3^2 \geq \tfrac{1}{4}(a_2+a_3)^2[(a_2+a_3)^2 \pm (a_2^2-a_3^2)] - a_2^2a_3^2 . \tag{55}$$

The right-hand side of this inequality is

$$\tfrac{1}{2}a_2[(a_2+a_3)^3 - 2a_2a_3^2] \qquad \text{or} \qquad \tfrac{1}{2}a_3[(a_3+a_2)^3 - 2a_3a_2^2] ; \tag{56}$$

and in either case it is positive. Since $4a_1^2 - a_3^2$ and $4a_1^2 - a_2^2$ are also positive, the integrands on the right-hand sides of equations (46) and (47) are positive definite; the integrals are, therefore, positive. As D and β have already been shown to be positive, it follows that the reality of Ω_2 requires that

$$a_2 \geq a_1 \,. \tag{57}$$

Hence, we are, in this case, limited to the domain

$$2a_1 \geq (a_2 + a_3) \qquad \text{and} \qquad a_2 \geq a_1 \geq a_3 \,. \tag{58}$$

Under these circumstances the reality of Ω_3 is also assured. Moreover, since $D > 0$ the quantity on the right-hand side of equation (48) is manifestly positive definite and assures that $\Pi > 0$. All ellipsoids represented in the triangle $SMcR_{\mathrm{I}}$ in Figure 1, therefore, are allowed figures of equilibrium; we shall call them *Riemann ellipsoids of type I.*

b) The Case $2a_1 < (a_2 - a_3)$ and $a_2 \geq a_3$

In this case

$$4a_1^2 \leq a_2^2 - a_3^2 \,, \tag{59}$$

since $2a_1$ is necessarily less than $a_2 + a_3$. From equations (15) and (16) it now follows that

$$\beta < 0 \qquad \text{and} \qquad \gamma > 0 \,. \tag{60}$$

Also, under the circumstances of this case, the integrand appearing on the right-hand side of equation (47) defining $\Omega_3^2\gamma$ is clearly negative. The integral is accordingly negative; and since γ has been shown to be positive and $a_2 \geq a_3$ (by definition), the reality of Ω_3 requires

$$\frac{a_3^2 - a_1^2}{D} \geq 0 \,. \tag{61}$$

Hence

$$\textit{either } a_3 < a_1 \qquad \text{and} \qquad D < 0 \,,$$

$$\textit{or } a_3 > a_1 \qquad \text{and} \qquad D > 0 \,; \tag{62}$$

and these two cases must be considered separately.

c) Case II: $2a_1 \leq (a_2 - a_3)$ and $a_3 \leq a_1$

In this case we must require (cf. eq. [62])

$$D = 4a_1^4 - a_1^2(a_2^2 + a_3^2) + a_2^2 a_3^2 \leq 0 \,. \tag{63}$$

This restriction on D implies that

$$\frac{a_3}{a_1} \leq \left(\frac{a_2^2 - 4a_1^2}{a_2^2 - a_1^2}\right)^{1/2} . \tag{64}$$

It can be verified that the further restriction

$$\frac{a_3}{a_1} \leq \frac{a_2}{a_1} - 2 \tag{65}$$

assures that inequality (64) is satisfied so long as (see Fig. 2)

$$2 \leq \frac{a_2}{a_1} \leq 1 + \sqrt{3} \,. \tag{66}$$

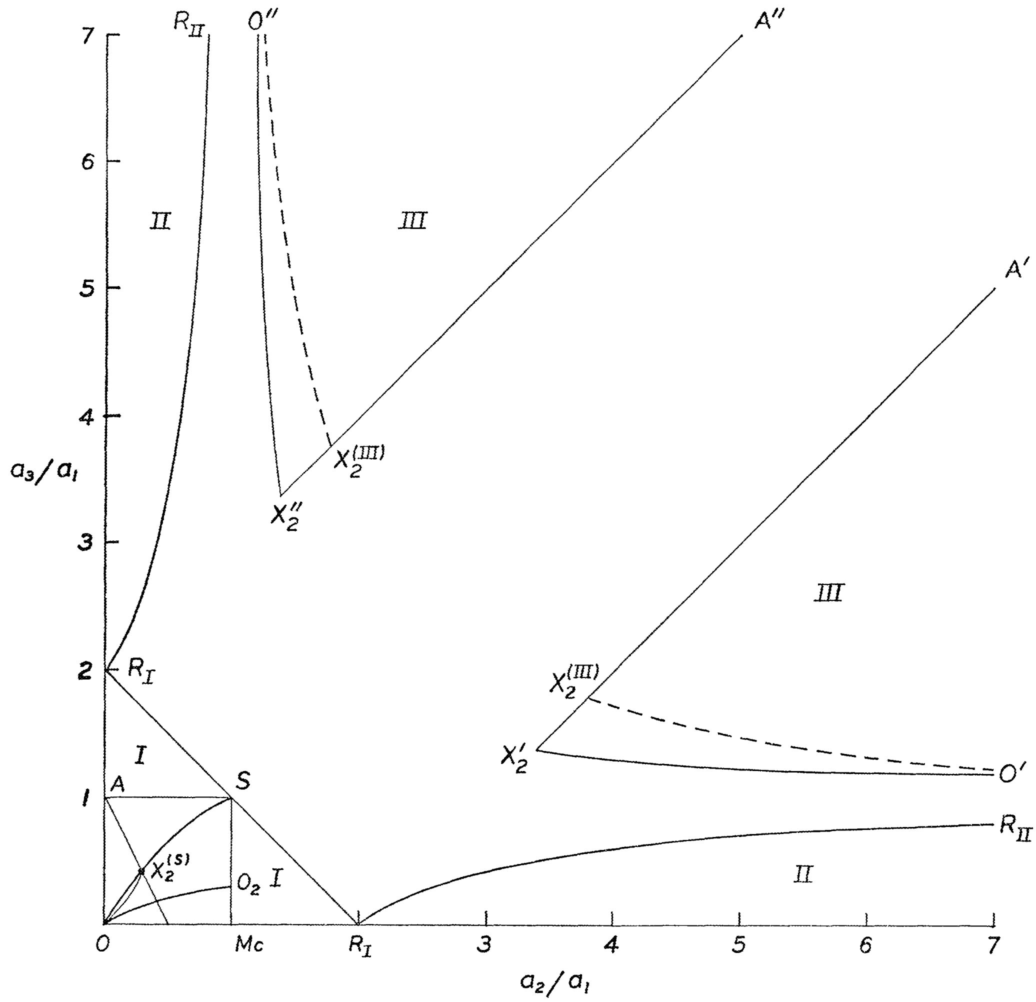

FIG. 1.—The domain of occupancy of the Riemann ellipsoids in the $(a_2/a_1, a_3/a_1)$-plane. The stable part of the Maclaurin sequence is represented by the segment O_2S on the line $a_2 = a_1$. At O_2 the Maclaurin spheroid becomes unstable by overstable oscillations.

The Riemann ellipsoids of type S (for which the directions of rotation and vorticity coincide with the a_3-axis) are included between the self-adjoint sequences represented by SO and O_2O. Along the arc $X_2^{(S)}O$ the Riemann ellipsoids of type S become unstable by an odd mode of oscillation belonging to the second harmonics.

The Riemann ellipsoids, in which the directions of rotation and vorticity do not coincide but lie in the (a_2,a_3)-plane, are of three types—I, II, and III—with the domains of occupancy shown. Type I ellipsoids adjoin the Maclaurin sequence and are bounded on one side (SR_I) by a self-adjoint sequence. Along the locus R_IR_{II}, which limits the domain of occupancy of the type II ellipsoids, the pressure is zero. And along the loci $X_2'O'$ and $X_2''O''$, limiting the domain of occupancy of the type III ellipsoids, the directions of Ω and ζ coincide with one of the principal axes (the a_3-axis in the case $a_2 > a_3$ and the a_2-axis in the case $a_2 < a_3$). The locus $X_2'O'$ (for the case $a_2 > a_3$) is transformed into $X_2^{(S)}O$ if the roles of a_1 and a_2 are interchanged; and simultaneously the domain of occupancy $A'X_2'O'$ similarly becomes transformed into the domain $AX_2^{(S)}O$. The dotted curve $X_2^{(III)}O'$ defines the locus of configurations, among the type III ellipsoids, that are marginally overstable by a mode of oscillation belonging to the second harmonics.

ence, we must require (65) so long as $2 \leq a_2/a_1 \leq 1 + \sqrt{3}$ and (64) for $a_2/a_1 \geq 1 +$ 3. With a_3/a_1 so restricted, the reality of Ω_3 is assured.

Turning next to equation (37) defining $\Omega_2^2\beta$ and rewriting it in the manner

$$\Omega_2^2\beta = 4a_1a_2a_3 \frac{a_3^2(a_2^2 - a_1^2)}{(a_2^2 - a_3^2)D} \int_0^\infty \left(\frac{4a_1^2 + a_2^2 - a_3^2}{a_3^2 + u} - \frac{a_2^2}{a_1^2 + u}\right) \frac{u\,du}{(a_2^2 + u)\Delta}, \tag{67}$$

e observe that the integrand is positive, since

$$\frac{4a_1^2 - a_3^2}{a_3^2 + u} + a_2^2\left(\frac{1}{a_3^2 + u} - \frac{1}{a_1^2 + u}\right) > 0. \tag{68}$$

ie integral on the right-hand side of equation (67) is therefore positive and since, rther, $a_2 \geq a_1 \geq a_3$, $D < 0$, and $\beta < 0$, the reality of Ω_2 is assured. On the other hand,

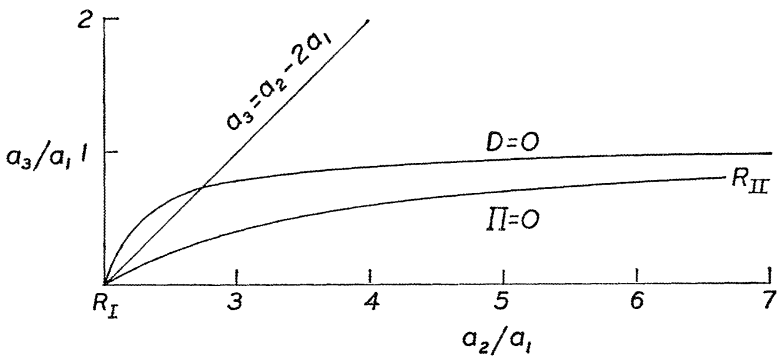

FIG. 2.—The loci $\Pi = 0$, $D = 0$, and $a_3 = a_2 - 2a_1$ which are used to determine the domain of cupancy of the type II ellipsoids.

TABLE 1

THE LOCUS ALONG WHICH $A_{23}(a_2^2 + a_3^2 - 2a_1^2) = 1$

a_2/a_1	a_3/a_1	a_2/a_1	a_3/a_1
2	0	5	0.70215
20/7	.37591	$6\frac{2}{3}$	0 79853
10/3	.49063	10	0.88676
4	0 59938	∞	1.00000

nce $D < 0$, the positive definiteness of Π is not manifest from equation (48). Indeed e requirement (cf. the alternative expression [43] for Π),

$$2B_{23} + (4a_1^2 - a_2^2 - a_3^2)A_{23} + A_1 \leq 0, \tag{69}$$

rovides a limit on a_3/a_1, for an assigned a_2/a_1 (≥ 2), which, it will appear, is more ringent than either (64) or (65).

The condition (69) can be expressed more conveniently by making use of the relations

$$B_{23} = A_2 - a_3^2A_{23} = A_3 - a_2^2A_{23} \qquad \text{and} \qquad A_1 + A_2 + A_3 = 2; \tag{70}$$

ius,

$$A_{23}(a_2^2 + a_3^2 - 2a_1^2) \geq 1. \tag{71}$$

The locus in the $(a_2/a_1, a_3/a_1)$-plane along which the inequality (71) becomes an quality has been determined (see Table 1). And the curves, labeled $R_{\mathrm{I}}R_{\mathrm{II}}$ in Figures

1 and 2, define this locus. It is apparent from Figure 2 that it is the requirement $\Pi > 0$ that limits the domain of occupancy in this case.

We shall call the ellipsoids, represented in the domain limited by the a_2-axis and the inequality (71), *Riemann ellipsoids of type II*.

d) Case III: $2a_1 \leq (a_2 - a_3)$ *and* $a_2 \geq a_3 > a_1$

In this case a_1 is the least axis; and

$$D = 3a_1^4 + (a_2^2 - a_1^2)(a_3^2 - a_1^2) > 0 , \tag{72}$$

as required by (62). And since $D > 0$, Π is manifestly positive. The reality of Ω_3 has already been assured. It remains to insure the reality of Ω_2. Since β is negative, the reality of Ω_2 requires (cf. eq. [37])

$$(4a_1^2 + a_2^2 - a_3^2)B_{23} - a_2^2 B_{12} \leq 0 . \tag{73}$$

TABLE 2

THE LOCUS DETERMINED BY EQUATION (74)

a_2/a_1	a_3/a_1
3.3746	1 3746
4 1739	1 3
5 8677	1 2
∞	1 0

And this inequality requires that, for a given a_2 (≥ 3.3746, as we shall see presently), a_3 ($\leq a_2$) exceeds a certain lower limit. The limit is determined by the condition

$$(4a_1^2 + a_2^2 - a_3^2)B_{23} - a_2^2 B_{12} = 0 . \tag{74}$$

A few pairs of values (a_2/a_1, a_3/a_1) along the locus defined by equation (74) are listed in Table 2; and the locus (labeled $X_2'O'$) is delineated in Figure 1.

We shall call the ellipsoids, limited by the locus (74) and the line $2a_1 = a_2 - a_3$, *Riemann ellipsoids of type III*.

It is to be particularly noted that, along the locus (74), $\Omega_2 = \zeta_2 = 0$ so that Ω_3 and ζ_3 are the only non-vanishing components of $\boldsymbol{\Omega}$ and $\boldsymbol{\zeta}$. Accordingly, the ellipsoids along this locus "belong" among the ellipsoids considered in Paper I; as to precisely "where" they belong, we shall return in § VI.

To distinguish the ellipsoids considered in Paper I from the ones which we have now designated as of types I, II, and III, we shall call them *Riemann ellipsoids of type S*.

IV. THE ADJOINT RIEMANN ELLIPSOIDS AND DEDEKIND'S THEOREM

For any pair of values (a_2/a_1, a_3/a_1), which represents a point in the permitted domain of occupancy of the Riemann ellipsoids of types I, II, and III, there are two states of motion, compatible with equilibrium, corresponding to the two roots of β and γ given by equations (17) and (18). It is clear on general grounds that the two physically distinct configurations which one obtains in this way must be adjoints of one another in the sense of Dedekind's theorem (cf. Paper I, § I). It is of interest to verify that this is the case.

Let β and $\beta^\dagger$ and γ and $\gamma^\dagger$ be the two roots of equations (15) and (16). Clearly,

$$\beta\beta^\dagger = a_3^2/a_1^2 \qquad \text{and} \qquad \gamma\gamma^\dagger = a_2^2/a_1^2 \,. \tag{75}$$

Also, since the right-hand sides of equations (37) and (38) depend only on the geometry of the ellipsoid, it is clear that

$$\Omega_2^2\beta = \Omega_2^{\dagger 2}\beta^\dagger \qquad \text{and} \qquad \Omega_3^2\gamma = \Omega_3^{\dagger 2}\gamma^\dagger \,. \tag{76}$$

From the foregoing relations it follows, for example, that

$$(\Omega_3\gamma a_1)\left(\Omega_3^\dagger\gamma^\dagger \frac{a_1^2}{a_2}\right) = (\Omega_3 a_1)(\Omega_3^\dagger a_2)$$

and

$$(\Omega_3\gamma a_1)(\Omega_3 a_1) = \left(\Omega_3^\dagger\gamma^\dagger \frac{a_1^2}{a_2}\right)(\Omega_3^\dagger a_2) \tag{77}$$

Hence

$$\Omega_3\gamma a_1 = \Omega_3^\dagger a_2 \qquad \text{and} \qquad \Omega_3 a_1 = \Omega_3^\dagger\gamma^\dagger \frac{a_1^2}{a_2}. \tag{78}$$

Equations (78) will continue to be valid if each quantity is replaced by its adjoint; thus

$$\Omega_3^\dagger\gamma^\dagger a_1 = \Omega_3 a_2 \qquad \text{and} \qquad \Omega_3^\dagger a_1 = \Omega_3\gamma \frac{a_1^2}{a_2}. \tag{79}$$

In exactly the same way,

$$\Omega_2\beta a_1 = \Omega_2^\dagger a_3 \,, \qquad \Omega_2 a_1 = \Omega_2^\dagger\beta^\dagger \frac{a_1^2}{a_3} \,,$$

$$\Omega_2^\dagger\beta^\dagger a_1 = \Omega_2 a_3 \,, \qquad \text{and} \qquad \Omega_2^\dagger a_1 = \Omega_2\beta \frac{a_1^2}{a_3}. \tag{80}$$

Now the motion $(\boldsymbol{u})$ in the frame of reference rotating with the angular velocity $\boldsymbol{\Omega}$ is given by equations (21). The motion, $\boldsymbol{u}^{(0)}$, in the inertial frame follows from the equation

$$\boldsymbol{u}^{(0)} = \boldsymbol{u} + \boldsymbol{\Omega} \times \boldsymbol{x} \,. \tag{81}$$

Expressing $\boldsymbol{u}^{(0)}$ in the manner required in the enunciation of Dedekind's theorem, we have

$$\boldsymbol{u}^{(0)} = \begin{vmatrix} 0 & -\Omega_3(1-\gamma a_1^2/a_2^2)a_2 & +\Omega_2(1-\beta a_1^2/a_3^2)a_3 \\ +\Omega_3(1-\gamma)a_1 & 0 & 0 \\ -\Omega_2(1-\beta)a_1 & 0 & 0 \end{vmatrix} \begin{vmatrix} x_1/a_1 \\ x_2/a_2 \\ x_3/a_3 \end{vmatrix} . \tag{82}$$

And the motion $\boldsymbol{u}^{(0)\dagger}$ in the configuration derived from $\beta^\dagger$ and $\gamma^\dagger$ is, similarly, given by

$$\boldsymbol{u}^{(0)\dagger} = \begin{vmatrix} 0 & -\Omega_3^\dagger(1-\gamma^\dagger a_1^2/a_2^2)a_2 & +\Omega_2^\dagger(1-\beta^\dagger a_1^2/a_3^2)a_3 \\ +\Omega_3^\dagger(1-\gamma^\dagger)a_1 & 0 & 0 \\ -\Omega_2^\dagger(1-\beta^\dagger)a_1 & 0 & 0 \end{vmatrix} \begin{vmatrix} x_1/a_1 \\ x_2/a_2 \\ x_3/a_3 \end{vmatrix} . \tag{83}$$

And Dedekind's theorem requires that the two matrices expressing $\boldsymbol{u}^{(0)}$ and $\boldsymbol{u}^{(0)\dagger}$ are the transposed of one another. In other words, we must have

$$\begin{aligned} \Omega_3^\dagger(1-\gamma^\dagger)a_1 &= -\Omega_3(1-\gamma a_1^2/a_2^2)a_2\,, \\ \Omega_2^\dagger(1-\beta^\dagger)a_1 &= -\Omega_2(1-\beta a_1^2/a_3^2)a_3\,, \\ \Omega_3(1-\gamma)a_1 &= -\Omega_3^\dagger(1-\gamma^\dagger a_1^2/a_2^2)a_2\,, \end{aligned} \tag{84}$$

and

$$\Omega_2(1-\beta)a_1 = -\Omega_2^\dagger(1-\beta^\dagger a_1^2/a_3^2)a_3\,;$$

and these relations are clearly valid in virtue of equations (78)–(80). *The configurations derived from the two roots for β and γ are, therefore, adjoints of one another in the sense of Dedekind's theorem.*

If $\beta = \beta^\dagger$ and $\gamma = \gamma^\dagger$, then the configuration is *self*-adjoint. Such self-adjoint configurations occur on the lines

$$2a_1 = a_2 + a_3 \qquad \text{and} \qquad 2a_1 = |a_2 - a_3|\,; \tag{85}$$

i.e., on one of the boundaries that limit the domains of occupancy of the ellipsoids of types I and III.

V. THE MACLAURIN SPHEROIDS AS LIMITING FORMS OF THE RIEMANN ELLIPSOIDS OF TYPE I

In this section we shall show how the Maclaurin spheroids may be considered as limiting forms of the Riemann ellipsoids of type I.

Let $a_2/a_1 \to 1$ while a_3 remains finite. From equations (17) and (18), we find that in this limit

$$\beta = \frac{1}{4a_1^2}[\,3a_1^2 + a_3^2 \pm \sqrt{(9a_1^2 - a_3^2)(a_1^2 - a_3^2)}\,],$$

and

$$\gamma = \frac{1}{4a_1^2}[\,5a_1^2 - a_3^2 \pm \sqrt{(9a_1^2 - a_3^2)(a_1^2 - a_3^2)}\,] \qquad (a_1 = a_2). \tag{86}$$

At the same time it follows from equations (37) and (38) that

$$\Omega_2^2\beta = 0 \qquad \text{and} \qquad \Omega_3^2\gamma = 4B_{13}\,. \tag{87}$$

From equation (12) we may now conclude that

$$\zeta_2 = 0 \qquad \text{and} \qquad \zeta_3 = -2\gamma\Omega_3\,. \tag{88}$$

Hence, on the line $a_2 = a_1$, the Riemann ellipsoids of type I become spheroids and are attributed the parameters,

$$\Omega_3^2 = 4B_{13}/\gamma\,, \qquad \zeta_3 = -2\gamma\Omega_3\,, \qquad \text{and} \qquad \zeta_2 = \Omega_2 = 0 \qquad (a_2 \to a_1)\,, \tag{89}$$

while ζ_2/Ω_2 tends to a finite value.

The relations (89) give

$$(\Omega_3 + \tfrac{1}{2}\zeta_3)^2 = \Omega_3^2(1-\gamma)^2 = 4B_{13}\,\frac{(1-\gamma)^2}{\gamma}. \tag{90}$$

Now, when $a_1 \rightarrow a_2$, the equation governing γ becomes

$$\gamma^2 - \left[2 + \tfrac{1}{2}\left(1 - \frac{a_3^2}{a_1^2}\right)\right]\gamma + 1 = 0 \,, \tag{91}$$

or

$$(\gamma - 1)^2 = \tfrac{1}{2}\gamma\left(1 - \frac{a_3^2}{a_1^2}\right). \tag{92}$$

In view of this last relation, we can rewrite equation (90) in the form

$$(\Omega_3 + \tfrac{1}{2}\zeta_3)^2 = 2B_{13}\left(1 - \frac{a_3^2}{a_1^2}\right). \tag{93}$$

But it is known that the angular velocity of rotation Ω_{Mc}, associated with a Maclaurin spheroid of axes $a_1 = a_2$ and a_3, is given by

$$\Omega_{\mathrm{Mc}}^2 = 2B_{13}\left(1 - \frac{a_3^2}{a_1^2}\right). \tag{94}$$

We have, therefore, the relation

$$\Omega_3 + \tfrac{1}{2}\zeta_3 = \Omega_{\mathrm{Mc}} \,; \tag{95}$$

and this is exactly the relation which must be satisfied if what we are viewing is a Maclaurin spheroid, rotating uniformly with an angular velocity Ω_{Mc} (in an inertial frame), from a frame of reference rotating with an angular velocity Ω_3 different from Ω_{Mc}. The Riemann ellipsoids of type I, therefore, degenerate to Maclaurin spheroids when $a_2 \rightarrow a_1$ (from the right); but they are viewed from a frame of reference in which they are attributed internal motions with a vorticity ζ_3. We can arrive at this same conclusion, somewhat differently, by arguing along the lines of § IX of Paper I.

When a Maclaurin spheroid is viewed from a frame of reference rotating with an angular velocity Ω ($\neq \Omega_{\mathrm{Mc}}$), it will be attributed internal motions having the components (cf. Paper I, eq. [140])

$$u_1 = -(\Omega_{\mathrm{Mc}} - \Omega)x_2 = Q_1 x_2$$

and

$$u_2 = +(\Omega_{\mathrm{Mc}} - \Omega)x_1 = Q_2 x_1 \,. \tag{96}$$

(Note that $Q_1 = -Q_2$.) We now ask: can we deform the spheroid quasi-statically, without in any way affecting its equilibrium (*as viewed* from the chosen frame of reference), by a non-trivial (odd) Lagrangian displacement of the form

$$\xi_1 = \alpha_1 x_3 \,, \qquad \xi_2 = \alpha_2 x_3 \,, \qquad \text{and} \qquad \xi_3 = \alpha_3 x_1 + \alpha_4 x_2 \,, \tag{97}$$

where $\alpha_1, \ldots, \alpha_4$ are constants? We shall show that such a quasi-static deformation is possible if Ω is chosen to be equal to Ω_3 given by equations (86) and (89).

We require that, in the chosen frame of reference, the displacement (97) have the properties requisite for a neutral mode of oscillation; and the conditions for this to be the case can be written down from the equations derived in Paper I, § VII.

For the deformation specified in (97), the only non-vanishing virials are those that are odd in the index 3:

$$V_{1;3} \,, \qquad V_{3;1} \,, \qquad V_{2;3} \,, \qquad \text{and} \qquad V_{3;2} \,. \tag{98}$$

Under these conditions, the equations governing the virials even in the index 3 (namely, Paper I, eqs. [101] and [102]) are trivially satisfied. Next, setting $\lambda = 0$ (as required for a neutral mode) in Paper I, equations (106)–(109), and remembering that in the case ($a_1 = a_2$) we are presently considering $B_{13} = B_{23}$, we find that the condition, that the virials listed in (97) do not vanish identically, is (cf. Paper I, eq. [112])

$$\begin{bmatrix} 2B_{13} - \Omega^2 & 2B_{13} - \Omega^2 + Q_1Q_2 - 2\Omega Q_2 \\ 2B_{13} & 2B_{13} + Q_1Q_2 \end{bmatrix} = 0 . \tag{99}$$

On simplification equation (99) becomes

$$\begin{bmatrix} \Omega & \Omega + 2Q_2 \\ 2B_{13} & 2B_{13} + Q_1Q_2 \end{bmatrix} = 0 ; \tag{100}$$

and on expanding the determinant, we are left with

$$\Omega Q_1 - 4B_{13} = 0 . \tag{101}$$

And inserting the value of Q_1 ($= \Omega - \Omega_{\text{Mc}}$) in equation (101), we finally obtain

$$\Omega^2 - \Omega\Omega_{\text{Mc}} - 4B_{13} = 0 . \tag{102}$$

We observe that equation (102) for Ω is identical with the equation determining the characteristic frequencies of the odd modes of oscillation of the Maclaurin spheroid in the frame of reference rotating with the angular velocity Ω_{Mc} (Lebovitz 1961, eq. [169]; also Chandrasekhar 1964, p. 69, eq. [30]).

It remains to verify that the values of Ω which are given by equation (102) are the same as those that follow from equations (86) and (89). To verify this fact, set

$$\Omega^2 = 4B_{13}/z \tag{103}$$

in equation (102). On further substituting for Ω_{Mc} its value given by equation (94), we obtain

$$\frac{4B_{13}}{z} - \left[\frac{8B_{13}^2}{z}\left(1 - \frac{a_3^2}{a_1^2}\right)\right]^{1/2} - 4B_{13} = 0 . \tag{104}$$

On simplification, equation (104) becomes

$$z^2 - \left[2 + \tfrac{1}{2}\left(1 - \frac{a_3^2}{a_1^2}\right)\right] z + 1 = 0 ; \tag{105}$$

and this equation is identical with equation (91) for γ. Hence $z = \gamma$; and this completes the proof that Ω_3 determined by equation (102) agrees with Ω_3 appropriate for the Riemann ellipsoid when $a_2 \to a_1$.

An alternative, but equivalent, way of arriving at the relation (102) is to recall that the frequencies of the odd modes of oscillation of a Maclaurin spheroid, in a frame of reference rotating with an angular velocity Ω (different from Ω_{Mc}), is given by (cf. Paper I, n. 7, eq. [ii])

$$2\sigma_o = 2\Omega - \Omega_{\text{Mc}} \pm \sqrt{(16B_{13} + \Omega_{\text{Mc}}^2)} ; \tag{106}$$

accordingly, these modes can be "neutralized" by the choice

$$\Omega = \tfrac{1}{2}[\Omega_{\text{Mc}} \pm \sqrt{(16B_{13} + \Omega_{\text{Mc}}^2)}] ; \tag{107}$$

and these values of Ω are the same as those which follow from equation (102).

The parameters that are to be associated with the Maclaurin spheroids when considered as the first members of the Riemann ellipsoids of type I are listed in Table 3.

VI. THE ELLIPSOIDS OF TYPE III AS BRANCHING OFF FROM THE ELLIPSOIDS OF TYPE S ALONG A CURVE OF BIFURCATION

As we have already remarked in § III*d*, along the locus (74), which limits the domain of the Riemann ellipsoids of type III,

$$\Omega_2 = \zeta_2 = 0 . \tag{108}$$

Accordingly, for these ellipsoids the only non-vanishing components of $\mathbf{\Omega}$ and $\boldsymbol{\zeta}$ are Ω_3 and ζ_3 along the x_3-axis. These ellipsoids are, therefore, of the type S considered in Paper I. But to be in agreement with the convention adopted in that paper, namely,

TABLE 3

THE PARAMETERS TO BE ASSOCIATED WITH THE MACLAURIN SPHEROIDS WHEN CONSIDERED AS THE FIRST MEMBERS OF THE RIEMANN ELLIPSOIDS OF TYPE I

e	Ω_3	ζ_3	Ω_3†	ζ_3†
0	1 03280	−2 06559	1 03280	−2 06559
.20	0 96509	−2 22313	1.11156	−1 93018
.30	0 93498	−2 31093	1 15547	−1.86996
.40	0 90713	−2 40508	1.20254	−1.81426
.50	0 88124	−2 50544	1.25272	−1 76249
.60	0 85682	−2 61109	1 30555	−1 71365
.70	0 83274	−2 71875	1.35937	−1 66548
.80	0 80578	−2 81682	1 40841	−1.61156
.82	0 79932	−2 83247	1 41624	−1 59864
.84	0.79215	−2 84542	1 42271	−1 58430
.86	0.78398	−2 85449	1 42724	−1 56797
.88	0 77438	−2.85787	1 42894	−1.54875
90	0 76261	−2.85266	1.42633	−1 52521
.92	0 74742	−2 83373	1 41687	−1 49483
94	0 72636	−2.79117	1 39558	−1 45273
96	0 69381	−2 70205	1 35102	−1 38762
0 98	0 63158	−2 49282	1 24641	−1 26316

that a_1 is the longest axis, we must interchange the roles of the indices 1 and 2 since, by our present convention, a_2 is the longest axis for the ellipsoids of type III. With the indices 1 and 2 interchanged, the loci limiting the domain of the ellipsoids become (see Fig. 1)

$$a_2 = 0 , \qquad 2a_2 + a_3 = a_1 , \tag{109}$$

and (cf. eq. [74])

$$(4a_2^2 + a_1^2 - a_3^2)B_{13} - a_1^2 B_{12} = 0 . \tag{110}$$

Now, it is clear on general grounds that, *along the locus* (*110*), *the Riemann ellipsoids must be characterized by a neutral mode of oscillation and, further, that stability must pass from the ellipsoids of type S to the ellipsoids of type III; in other words, that the locus* (*110*) *is a curve of bifurcation.*

As we have shown in Paper I, instability by an odd mode of oscillation sets in along the Riemann sequences for $f < -2$. And according to Paper I, equation (114), instability occurs when (cf. Paper I, eq. [37])

$$4B_{13} = \Omega Q_1 = -2B_{12}\frac{a_1}{a_2}\frac{1}{x+1/x}. \tag{111}$$

But by Paper I, equation (38),

$$\begin{aligned} x+\frac{1}{x} &= -\frac{2a_1a_2(A_1a_1^2 - A_2a_2^2)}{(a_1^2-a_2^2)a_3^2A_3 + a_1^2a_2^2(A_1-A_2)} \\ &= -\frac{2a_1a_2B_{12}}{a_3^2A_3 - a_1^2a_2^2A_{12}}. \end{aligned} \tag{112}$$

TABLE 4

THE PROPERTIES OF A FEW RIEMANN ELLIPSOIDS OF TYPE I

a_2/a_1 . .	1 05263	1.25000	1.44065	1.66667	1 36444	1.69351	1.52303	1.78590
a_3/a_1 . . .	0.41667	0.50000	0 49273	0 33333	0 09518	0.11813	0 05315	0 06233
Ω_2 . .	+0 14834	+0.39259	+0 57179	+0 71251	+0 05632	+0 15764	+0 03311	+0.08952
Ω_3	+0.73257	+0 66536	+0 59896	+0 52815	+0 40707	+0.38504	+0 29600	+0 28558
ζ_2	−1 41355	−2 19983	−2 24560	−2.37502	−6 68275	−6 27092	−9.85239	−9 19424
ζ_3 . .	−2 61578	−1 93895	−1 49425	−1.19714	−1 24612	−1 02536	−0 84580	−0 74657
$\Omega_2^\dagger$ . .	+0 50185	+0 87993	+0.89032	+0 71251	+0 63035	+0.73061	+0 52221	+0 57083
$\Omega_3^\dagger$	+1 30617	+0 94583	+0 69996	+0 52815	+0 59414	+0 44893	+0.38805	+0.31825
$\zeta_2^\dagger$.	−0.41783	−0 98148	−1 44219	−2.37502	−0.59714	−1 35309	−0 62474	−1 4418
$\zeta_3^\dagger$ . .	−1 46707	−1 36398	−1.27866	−1 19714	−0.85376	−0 87944	−0.64518	−0 66992

Inserting from this last equation in equation (111), we obtain for the locus of marginal stability the equation

$$4a_2^2B_{13} - a_3^2A_3 + a_1^2a_2^2A_{12} = 0 \tag{113}$$

or, alternatively,

$$0 = 4a_2^2B_{13} - a_3^2A_3 + a_1^2(A_1 - B_{12}) = (4a_2^2 + a_1^2 - a_3^2)B_{13} - a_1^2B_{12}\,; \tag{114}$$

and this is the same as equation (110).

We have already verified in Paper I (n. 6 on page 916) that the point, on the self-adjoint sequence $x = -1$ (which limits on the side $a_3 > a_2$ the domain of the Riemann ellipsoids of type S), at which instability sets in satisfies the condition $2a_2 + a_3 = a_1$. This completes the demonstration that *the curve along which the Riemann ellipsoids of type S become marginally unstable is also the curve along which the Riemann ellipsoids of type III branch off*. And, finally, in § XI we shall show that along the curve of bifurcation stability passes from the ellipsoids of type S to the ellipsoids of type III.

VII. SOME NUMERICAL EXAMPLES

The properties of the Riemann ellipsoids of the three types, in their respective domains of occupancy, have been determined, with the aid of the formulae of § II, for a large number of cases. In Tables 4 and 5 we list them for a few typical cases. More extensive tables will be published elsewhere.

VIII. THE SECOND-ORDER VIRIAL EQUATIONS GOVERNING SMALL OSCILLATIONS ABOUT EQUILIBRIUM: THE CHARACTERISTIC EQUATION

Suppose that an equilibrium ellipsoid determined consistently with respect to the equations derived in § III is slightly perturbed. Let the ensuing motions be described in terms of a Lagrangian displacement of the form

$$\xi(x)e^{\lambda t}, \tag{115}$$

where λ is a parameter whose characteristic values are to be determined. Then proceeding exactly as in Paper I, § VI, we find that the linearized form of the virial equation (2) gives (cf. Paper I, eq. [84])

$$\begin{aligned}\lambda^2 V_{i;j} - 2\lambda Q_{jl}V_{i;l} - 2\lambda\epsilon_{ilm}\Omega_m V_{l;j} + Q_{jl}{}^2 V_{i;l} + Q_{il}{}^2 V_{j;l}\\ + 2\epsilon_{ilm}\Omega_m(Q_{jk}V_{l;k} - Q_{lk}V_{j;k}) - \Omega^2 V_{ij} + \Omega_i\Omega_k V_{kj} - \delta\mathfrak{W}_{ij} = \delta_{ij}\delta\Pi\,,\end{aligned} \tag{116}$$

TABLE 5

THE PROPERTIES OF A FEW RIEMANN ELLIPSOIDS OF TYPES II AND III

	ELLIPSOIDS OF TYPE II			ELLIPSOIDS OF TYPE III			
a_2/a_1 . . .	3 05590	4 31608	6 24270	5.00000	4 00000	3.75000	4 66667
a_3/a_1. . . .	0 10665	0 45115	0.21787	3.00000	2.00000	1.50000	1.40000
Ω_2	+0 70172	+1 13288	+0 83636	+0 44981	+0 40641	+0 36030	+0 45130
Ω_3	+0 16788	+0 17511	+0 06655	+0.43847	+0.43856	+0 32561	+0 20871
ζ_2. . .	+0 26688	+0 18979	+0 05016	+1 49937	+1 01603	+0 36548	+0.17456
ζ_3 . . .	−0 86595	−1 69461	−1.32826	−2.28003	−1 86387	−1 81498	−2 10718
Ω_2†.	+0.02814	+0 07114	+0 01044	+0 44981	+0 40641	+0 16868	+0 08256
Ω_3† . . .	+0 25596	+0.37263	+0.20745	+0 43847	+0 43856	+0.45186	+0.43172
ζ_2† .	+6 65457	+3 02219	+4.02055	+1.49937	+1 01603	+0 78066	+0 95417
ζ_3† . . .	−0 56795	−0 79638	−0 42612	−2 28003	−1 86387	−1 30787	−1 01872

where the various symbols have their usual meanings and the assumption has been made that the internal motion in the equilibrium configuration is given by

$$u_j = Q_{jl}x_l\,, \tag{117}$$

where the Q_{jl}'s are certain constants. For the case of the Riemann ellipsoids, presently considered, the matrices $\boldsymbol{Q}$ and $\boldsymbol{Q}^2$ are of the forms (cf. eq. [21])

$$\boldsymbol{Q} = \begin{vmatrix} 0 & Q_{12} & Q_{13} \\ Q_{21} & 0 & 0 \\ Q_{31} & 0 & 0 \end{vmatrix} \tag{118}$$

and

$$\boldsymbol{Q}^2 = \begin{vmatrix} Q_{12}Q_{21} + Q_{13}Q_{31} & 0 & 0 \\ 0 & Q_{21}Q_{12} & Q_{21}Q_{13} \\ 0 & Q_{31}Q_{12} & Q_{31}Q_{13} \end{vmatrix}, \tag{119}$$

where

$$Q_{12} = +\Omega_3 \frac{a_1^2}{a_2^2}\gamma , \qquad Q_{21} = -\Omega_3\gamma ,$$
$$Q_{13} = -\Omega_2 \frac{a_1^2}{a_3^2}\beta , \quad \text{and} \quad Q_{31} = +\Omega_2\beta . \tag{120}$$

We shall now write down the explicit forms which the different components of equation (116) take in view of the special forms of the matrices $\boldsymbol{Q}$ and $\boldsymbol{Q}^2$. The three diagonal components of equation (116) are

$$\begin{aligned}\lambda^2 V_{1;1} &- 2\lambda(Q_{12}V_{1;2} + Q_{13}V_{1;3}) - 2\lambda(\Omega_3 V_{2;1} - \Omega_2 V_{3;1}) \\ &+ 2(Q_{12}Q_{21} + Q_{13}Q_{31})V_{1;1} + 2\Omega_3(Q_{12}V_{2;2} - Q_{21}V_{1;1} + Q_{13}V_{2;3}) \\ &- 2\Omega_2(Q_{13}V_{3;3} - Q_{31}V_{1;1} + Q_{12}V_{3;2}) - (\Omega_2^2 + \Omega_3^2)V_{11} - \delta\mathfrak{W}_{11} = \delta\Pi ,\end{aligned} \tag{121}$$

$$\begin{aligned}\lambda^2 V_{2;2} &- 2\lambda Q_{21}V_{2;1} + 2\lambda\Omega_3 V_{1;2} + 2(Q_{21}Q_{12}V_{2;2} + Q_{21}Q_{13}V_{2;3}) \\ &+ 2\Omega_3(Q_{12}V_{2;2} - Q_{21}V_{1;1} + Q_{13}V_{2;3}) - \Omega_3^2 V_{22} + \Omega_2\Omega_3 V_{32} - \delta\mathfrak{W}_{22} = \delta\Pi ,\end{aligned} \tag{122}$$

and

$$\begin{aligned}\lambda^2 V_{3;3} &- 2\lambda Q_{31}V_{3;1} - 2\lambda\Omega_2 V_{1;3} + 2(Q_{31}Q_{12}V_{3;2} + Q_{31}Q_{13}V_{3;3}) \\ &- 2\Omega_2(Q_{13}V_{3;3} - Q_{31}V_{1;1} + Q_{12}V_{3;2}) - \Omega_2^2 V_{33} + \Omega_2\Omega_3 V_{23} - \delta\mathfrak{W}_{33} = \delta\Pi .\end{aligned} \tag{123}$$

Eliminating $\delta\Pi$ from the foregoing equations, we obtain the pair of equations

$$\begin{aligned}&[\lambda^2 + 2(Q_{12}Q_{21} + Q_{13}Q_{31}) + 2\Omega_2 Q_{31} - 2(\Omega_2^2 + \Omega_3^2)]V_{1;1} - (\lambda^2 + 2Q_{12}Q_{21} - 2\Omega_3^2)V_{2;2} \\ &\quad - 2\Omega_2 Q_{13}V_{3;3} - 2\lambda(Q_{12} + \Omega_3)V_{1;2} - 2\lambda(\Omega_3 - Q_{21})V_{2;1} - 2\lambda Q_{13}V_{1;3} + 2\lambda\Omega_2 V_{3;1} \\ &\quad - (2Q_{21}Q_{13} + \Omega_2\Omega_3)V_{2;3} - (2\Omega_2 Q_{12} + \Omega_2\Omega_3)V_{3,2} - \delta\mathfrak{W}_{11} + \delta\mathfrak{W}_{22} = 0 ,\end{aligned} \tag{124}$$

and

$$\begin{aligned}&[\lambda^2 + 2(Q_{12}Q_{21} + Q_{13}Q_{31}) - 2\Omega_3 Q_{21} - 2(\Omega_2^2 + \Omega_3^2)]V_{1;1} - (\lambda^2 + 2Q_{13}Q_{31} - 2\Omega_2^2)V_{3;3} \\ &\quad + 2\Omega_3 Q_{12}V_{2;2} - 2\lambda(Q_{13} - \Omega_2)V_{1;3} + 2\lambda(\Omega_2 + Q_{31})V_{3;1} - 2\lambda Q_{12}V_{1;2} - 2\lambda\Omega_3 V_{2;1} \\ &\quad - (2Q_{31}Q_{12} + \Omega_2\Omega_3)V_{3;2} + (2\Omega_3 Q_{13} - \Omega_2\Omega_3)V_{2;3} - \delta\mathfrak{W}_{11} + \delta\mathfrak{W}_{33} = 0 .\end{aligned} \tag{125}$$

The remaining six non-diagonal components of equation (116) are

$$\begin{aligned}\lambda^2 V_{2;1} &+ 2\lambda\Omega_3 V_{1;1} - 2\lambda(Q_{12}V_{2,2} + Q_{13}V_{2;3}) + (Q_{12}Q_{21} + Q_{13}Q_{31})V_{2;1} + Q_{21}Q_{12}V_{1;2} \\ &+ Q_{21}Q_{13}V_{1;3} - \Omega_3^2 V_{21} + \Omega_2\Omega_3 V_{31} - \delta\mathfrak{W}_{12} = 0 ,\end{aligned} \tag{126}$$

$$\lambda^2 V_{1;2} - 2\lambda Q_{21} V_{1;1} - 2\lambda(\Omega_3 V_{2;2} - \Omega_2 V_{3;2}) + (Q_{12}Q_{21} + Q_{13}Q_{31}) V_{2;1} + Q_{12}Q_{21} V_{1;2}$$
$$+ Q_{21}Q_{13} V_{1;3} - 2\Omega_2(Q_{21} V_{3;1} - Q_{31} V_{2;1}) - (\Omega_2^2 + \Omega_3^2) V_{12} - \delta\mathfrak{W}_{12} = 0 , \tag{127}$$

$$\lambda^2 V_{3;1} - 2\lambda\Omega_2 V_{1;1} - 2\lambda(Q_{12} V_{3;2} + Q_{13} V_{3;3}) + (Q_{12}Q_{21} + Q_{13}Q_{31}) V_{3;1} + Q_{13}Q_{31} V_{1;3}$$
$$+ Q_{31}Q_{12} V_{1;2} - \Omega_2^2 V_{31} + \Omega_2\Omega_3 V_{21} - \delta\mathfrak{W}_{13} = 0 , \tag{128}$$

$$\lambda^2 V_{1;3} - 2\lambda Q_{31} V_{1;1} - 2\lambda(\Omega_3 V_{2;3} - \Omega_2 V_{3;3}) + (Q_{12}Q_{21} + Q_{13}Q_{31}) V_{3;1} + Q_{13}Q_{31} V_{1;3}$$
$$+ Q_{31}Q_{12} V_{1;2} - 2\Omega_3(Q_{21} V_{3;1} - Q_{31} V_{2;1}) - (\Omega_2^2 + \Omega_3^2) V_{13} - \delta\mathfrak{W}_{13} = 0 , \tag{129}$$

$$\lambda^2 V_{3;2} - 2\lambda Q_{21} V_{3;1} - 2\lambda\Omega_2 V_{1;2} + Q_{31}Q_{12} V_{2;2} + Q_{21}Q_{13} V_{3;3} + Q_{31}Q_{13} V_{2;3} + Q_{21}Q_{12} V_{3;2}$$
$$- 2\Omega_2(Q_{12} V_{2;2} - Q_{21} V_{1;1} + Q_{13} V_{2;3}) - \Omega_2^2 V_{32} + \Omega_2\Omega_3 V_{22} - \delta\mathfrak{W}_{32} = 0 , \tag{130}$$

and

$$\lambda^2 V_{2;3} - 2\lambda Q_{31} V_{2;1} + 2\lambda\Omega_3 V_{1;3} + Q_{31}Q_{12} V_{2;2} + Q_{21}Q_{13} V_{3;3} + Q_{31}Q_{13} V_{2;3} + Q_{21}Q_{12} V_{3;2}$$
$$+ 2\Omega_3(Q_{13} V_{3;3} - Q_{31} V_{1;1} + Q_{12} V_{3;2}) - \Omega_3^2 V_{32} + \Omega_2\Omega_3 V_{33} - \delta\mathfrak{W}_{23} = 0 . \tag{131}$$

The eight equations (124)–(131) must be supplemented by the condition

$$\frac{V_{1;1}}{a_1^2} + \frac{V_{2;2}}{a_2^2} + \frac{V_{3;3}}{a_3^2} = 0 \tag{132}$$

required by the solenoidal character of ξ.

On inserting the values of Q_{12}, etc., in accordance with equations (120), we find that the system of equations (124)–(132) can be written in matrix notation in the form given on pages 862–863, where we have substituted for the $\delta\mathfrak{W}_{ij}$'s their known values (cf., for example, Paper I, eqs. [87] and [88]).

The required characteristic equation follows from setting the determinant of the matrix on the left-hand side of equation (133) equal to zero.

IX. THE EQUALITY OF THE CHARACTERISTIC FREQUENCIES OF AN ELLIPSOID AND ITS ADJOINT AND OTHER THEOREMS

Eliminating $V_{1;1}$ from the system of equations (133) and multiplying the different rows and columns of the resulting secular matrix by suitable factors, we find that the secular determinant can be brought to the form given on pages 864–865.

$$
\left|\begin{array}{ccccc}
\lambda^2-2(\boldsymbol{\Omega}^2+\boldsymbol{\Omega}^{\dagger 2})+2\Omega_2^2\beta & -\lambda^2+2\Omega_3^{\dagger 2}+2\Omega_3^2 & 2\dfrac{a_1^2}{a_3^2}\Omega_2^2\beta+2B_{13}-2B_{23} & -2\lambda\Omega_3\left(1+\dfrac{a_1^2}{a_2^2}\gamma\right) & -2\lambda\Omega_3(1+\gamma) \\
\quad +6B_{11}-2B_{12} & \quad -6B_{22}+2B_{12} & & & \\
\lambda^2-2(\boldsymbol{\Omega}^2+\boldsymbol{\Omega}^{\dagger 2})+2\Omega_3^2\gamma & 2\dfrac{a_1^2}{a_2^2}\Omega_3^2\gamma+2B_{12}-2B_{23} & -\lambda^2+2\Omega_2^{\dagger 2}+2\Omega_2^2 & -2\lambda\dfrac{a_1^2}{a_2^2}\Omega_3\gamma & -2\lambda\Omega_3 \\
\quad +6B_{11}-2B_{13} & & \quad -6B_{33}+2B_{13} & & \\
2\lambda\Omega_3 & -2\lambda\Omega_3\dfrac{a_1^2}{a_2^2}\gamma & 0 & -(\Omega_3^2+\Omega_3^{\dagger 2})+2B_{12} & +\lambda^2-\boldsymbol{\Omega}^{\dagger 2}-\Omega_3^2+2B_{12} \\
2\lambda\Omega_3\gamma & -2\lambda\Omega_3 & 0 & +\lambda^2-\boldsymbol{\Omega}^2-\Omega_3^{\dagger 2}+2B_{12} & -(\boldsymbol{\Omega}^2+\boldsymbol{\Omega}^{\dagger 2})+2\Omega_2^2\beta+2B_{12} \\
-2\lambda\Omega_2 & 0 & 2\lambda\Omega_2\dfrac{a_1^2}{a_3^2}\beta & \Omega_2\Omega_3\left(1+\dfrac{a_1^2}{a_2^2}\beta\gamma\right) & \Omega_2\Omega_3 \\
-2\lambda\Omega_2\beta & 0 & 2\lambda\Omega_2 & \Omega_2\Omega_3\dfrac{a_1^2}{a_2^2}\beta\gamma & 2\Omega_2\Omega_3\beta \\
-2\Omega_2\Omega_3\gamma & \Omega_2\Omega_3 & \Omega_2\Omega_3\dfrac{a_1^2}{a_3^2}\beta\gamma & -2\lambda\Omega_2 & 0 \\
-2\Omega_2\Omega_3\beta & \Omega_2\Omega_3\dfrac{a_1^2}{a_2^2}\beta\gamma & \Omega_2\Omega_3 & 0 & -2\lambda\Omega_2\beta \\
\dfrac{1}{a_1^2} & \dfrac{1}{a_2^2} & \dfrac{1}{a_3^2} & 0 & 0
\end{array}\right.
$$

$$
\begin{array}{cccc}
2\lambda\Omega_2 \dfrac{a_1^2}{a_3^2}\beta & 2\lambda\Omega_2 & -\Omega_2\Omega_3\left(1+2\dfrac{a_1^2}{a_3^2}\beta\gamma\right) & -\Omega_2\Omega_3\left(1+2\dfrac{a_1^2}{a_2^2}\gamma\right) \\
2\lambda\Omega_2\left(1+\dfrac{a_1^2}{a_3^2}\beta\right) & 2\lambda\Omega_2(1+\beta) & -\Omega_2\Omega_3\left(1+2\dfrac{a_1^2}{a_3^2}\beta\right) & -\Omega_2\Omega_3\left(1+2\dfrac{a_1^2}{a_2^2}\beta\gamma\right) \\
\Omega_2\Omega_3\left(1+\dfrac{a_1^2}{a_3^2}\beta\gamma\right) & \Omega_2\Omega_3 & 2\lambda\Omega_2\dfrac{a_1^2}{a_3^2}\beta & 0 \\
\Omega_2\Omega_3\dfrac{a_1^2}{a_3^2}\beta\gamma & 2\Omega_2\Omega_3\gamma & 0 & 2\lambda\Omega_2 \\
-\Omega_2^2-\Omega_2^{\dagger 2}+2B_{13} & \lambda^2-\boldsymbol{\Omega}^{\dagger 2}-\Omega_2^2+2B_{13} & 0 & -2\lambda\Omega_3\dfrac{a_1^2}{a_2^2}\gamma \\
\lambda^2-\boldsymbol{\Omega}^2-\Omega_2^{\dagger 2}+2B_{13} & -(\boldsymbol{\Omega}^2+\boldsymbol{\Omega}^{\dagger 2})+2\Omega_3^2\gamma+2B_{13} & -2\lambda\Omega_3 & 0 \\
0 & 2\lambda\Omega_3\gamma & -\Omega_2^2-\Omega_2^{\dagger 2}+2\Omega_2^2\dfrac{a_1^2}{a_3^2}\beta+2B_{23} & \lambda^2-\Omega_2^2-\Omega_3^{\dagger 2}+2B_{23} \\
2\lambda\Omega_3 & 0 & \lambda^2-\Omega_3^2-\Omega_2^{\dagger 2}+2B_{23} & -\Omega_3^2-\Omega_3^{\dagger 2}+2\Omega_3^2\dfrac{a_1^2}{a_2^2}\gamma+2B_{23} \\
0 & 0 & 0 & 0
\end{array}
\left\|
\begin{array}{c}
V_{1;1} \\ V_{2;2} \\ V_{3;3} \\ V_{1;2} \\ V_{2;1} \\ V_{1;3} \\ V_{3;1} \\ V_{2;3} \\ V_{3;2}
\end{array}
\right| = 0\,. \quad (133)
$$

$$
\left[\begin{array}{cccc}
(1,1) & (1,2) & -2\lambda\left(\Omega_3+\frac{a_1}{a_2}\,\Omega_3{}^{\dagger}\right) & -2\lambda\left(\Omega_3+\frac{a_2}{a_1}\,\Omega_3{}^{\dagger}\right) \\
(2,1) & (2,2) & -2\lambda\Omega_3{}^{\dagger}\,\frac{a_1}{a_2} & -2\lambda\Omega_3 \\
-2\lambda\left(\Omega_3{}^{\dagger}+\frac{a_1}{a_2}\,\Omega_3\right) & -2\lambda\Omega_3\,\frac{a_1}{a_2} & -(\Omega_3{}^2+\Omega_3{}^{\dagger 2})+2B_{12} & \lambda^2-\boldsymbol{\Omega}^{\dagger 2}-\Omega_3{}^2+2B_{12} \\
-2\lambda\left(\Omega_3{}^{\dagger}+\frac{a_2}{a_1}\,\Omega_3\right) & -2\lambda\Omega_3{}^{\dagger} & \lambda^2-\boldsymbol{\Omega}^2-\Omega_3{}^{\dagger 2}+2B_{12} & -(\boldsymbol{\Omega}^2+\boldsymbol{\Omega}^{\dagger 2})+2\Omega_2{}^2\beta+2B_{12} \\
2\lambda\Omega_2\,\frac{a_1}{a_3} & 2\lambda\left(\Omega_2{}^{\dagger}+\frac{a_1}{a_3}\,\Omega_2\right) & 2\Omega_2\Omega_3{}^{\dagger}\,\frac{a_1}{a_3} & \Omega_2\Omega_3\,\frac{a_2}{a_3} \\
2\lambda\Omega_2{}^{\dagger} & 2\lambda\left(\Omega_2{}^{\dagger}+\frac{a_3}{a_1}\,\Omega_2\right) & \Omega_2{}^{\dagger}\Omega_3{}^{\dagger} & 2\Omega_2{}^{\dagger}\Omega_3\,\frac{a_2}{a_1} \\
2\Omega_2\Omega_3\,\frac{a_2}{a_3}+\Omega_2{}^{\dagger}\Omega_3{}^{\dagger} & 2\Omega_2{}^{\dagger}\Omega_3{}^{\dagger}+\frac{a_2}{a_3}\,\Omega_2\Omega_3 & -2\lambda\Omega_2\,\frac{a_1}{a_3} & 0 \\
2\Omega_2{}^{\dagger}\Omega_3{}^{\dagger}+\frac{a_3}{a_2}\,\Omega_2\Omega_3 & \Omega_2{}^{\dagger}\Omega_3{}^{\dagger}+2\,\frac{a_3}{a_2}\,\Omega_2\Omega_3 & 0 & -2\lambda\Omega_2{}^{\dagger}
\end{array}\right.
$$

$$
\left.\begin{array}{cccc}
2\lambda\Omega_2^\dagger \dfrac{a_1}{a_3} & 2\lambda\Omega_2 & \Omega_2\Omega_3 + 2\dfrac{a_2}{a_3}\Omega_2^\dagger\Omega_3^\dagger & 2\Omega_2\Omega_3 + \dfrac{a_3}{a_2}\Omega_2^\dagger\Omega_3^\dagger \\
2\lambda\left(\Omega_2 + \dfrac{a_1}{a_3}\Omega_2^\dagger\right) & 2\lambda\left(\Omega_2 + \dfrac{a_3}{a_1}\Omega_2^\dagger\right) & 2\Omega_2\Omega_3 + \dfrac{a_2}{a_3}\Omega_2^\dagger\Omega_3^\dagger & \Omega_2\Omega_3 + 2\dfrac{a_3}{a_2}\Omega_2^\dagger\Omega_3^\dagger \\
2\Omega_2^\dagger\Omega_3 \dfrac{a_1}{a_3} & \Omega_2\Omega_3 & -2\lambda\Omega_2^\dagger\dfrac{a_1}{a_3} & 0 \\
\Omega_2^\dagger\Omega_3^\dagger \dfrac{a_2}{a_3} & 2\Omega_2\Omega_3^\dagger \dfrac{a_2}{a_1} & 0 & -2\lambda\Omega_2 \\
\dfrac{a_2}{a_3}(-\Omega_2^2 - \Omega_2^{\dagger 2} + 2B_{13}) & \dfrac{a_2}{a_3}(\lambda^2 - \mathbf{\Omega}^{\dagger 2} - \Omega_2^2 + 2B_{13}) & 0 & 2\lambda\Omega_3^\dagger\dfrac{a_1}{a_3} \\
\dfrac{a_2}{a_3}(\lambda^2 - \mathbf{\Omega}^2 - \Omega_2^{\dagger 2} + 2B_{13}) & \dfrac{a_2}{a_3}(-\mathbf{\Omega}^2 - \mathbf{\Omega}^{\dagger 2} + 2\Omega_3^2\gamma + 2B_{13}) & +2\lambda\Omega_3\dfrac{a_2}{a_3} & 0 \\
0 & 2\lambda\Omega_3^\dagger\dfrac{a_2}{a_3} & \dfrac{a_1}{a_3}\left(\Omega_2^2 + \Omega_2^{\dagger 2} - 2\Omega_2^2\dfrac{a_1^2}{a_3^2}\beta - 2B_{23}\right) & \dfrac{a_1}{a_3}(-\lambda^2 + \Omega_2^2 + \Omega_3^{\dagger 2} - 2B_{23}) \\
2\lambda\Omega_3\dfrac{a_1}{a_3} & 0 & \dfrac{a_1}{a_3}(-\lambda^2 + \Omega_2^{\dagger 2} + \Omega_3^2 - 2B_{23}) & \dfrac{a_1}{a_3}\left(+\Omega_3^2 + \Omega_3^{\dagger 2} - 2\Omega_3^2\dfrac{a_1^2}{a_2^2}\gamma - 2B_{23}\right)
\end{array}\right]
$$

$$= 0\,, \tag{134}$$

where

$$
\begin{aligned}
(1,1) &= -\frac{a_1}{a_2}(\lambda^2 - 2\mathbf{\Omega}^2 - 2\mathbf{\Omega}^{\dagger 2} + 2\Omega_2^2\beta + 6B_{11} - 2B_{12}) - \frac{a_2}{a_1}(\lambda^2 - 2\Omega_3^2 - 2\Omega_3^{\dagger 2} + 6B_{22} - 2B_{12})\,, \\
(2,2) &= -\frac{a_1}{a_2}(\lambda^2 - 2\mathbf{\Omega}^2 - 2\mathbf{\Omega}^{\dagger 2} + 2\Omega_3^2\gamma + 6B_{11} - 2B_{13}) - \frac{a_3^2}{a_1 a_2}(\lambda^2 - 2\Omega_2^2 - 2\Omega_2^{\dagger 2} + 6B_{33} - 2B_{13})\,,
\end{aligned} \tag{135}
$$

and

$$(1,2) = (2,1) = -\frac{a_1}{a_2}\left\{\lambda^2 - 2\mathbf{\Omega}^2 - 2\mathbf{\Omega}^{\dagger 2} + 6B_{11} - 2\left[\left(\frac{a_2^2 a_3^2}{a_1^2} - a_1^2\right)B_{123} + B_{23}\right]\right\}.$$

In reducing the secular determinant to the form (134) use must be made of the relations (cf. eqs. [78] and [80])

$$\Omega_3^\dagger = \Omega_3 \frac{a_1}{a_2}\gamma \qquad \text{and} \qquad \Omega_2^\dagger = \Omega_2 \frac{a_1}{a_3}\beta . \tag{136}$$

From the form to which the secular equation has been reduced, it is manifest that if in each element of the secular matrix we replace the components of $\boldsymbol{\Omega}$ and $\boldsymbol{\Omega}^\dagger$, which occur, by their respective adjoints we obtain the transposed matrix. It, therefore, follows that *the characteristic frequencies of oscillation, belonging to these "second harmonics," of an ellipsoid and its adjoint are the same.* This theorem, first established in the contexts of the Dedekind and the Jacobian sequences (Chandrasekhar 1965*a*) and then generalized to the Riemann sequences (Paper I), is now seen to be an entirely general property.

We shall now prove that $|\boldsymbol{\Omega}|$ *and* $|\boldsymbol{\Omega}^\dagger|$ *are characteristic frequencies provided as roots of the characteristic equation.* We shall prove this theorem by showing that the following set of three equations, included as simple linear combinations in the system of equations (133), are linearly dependent if λ^2 is set equal to $-|\boldsymbol{\Omega}|^2 = -(\Omega_2^2 + \Omega_3^2)$:

$$\begin{aligned}\lambda^2(V_{1;2} - V_{2;1}) - 2\lambda(Q_{21}V_{1;1} - Q_{12}V_{2;2} - Q_{13}V_{2;3}) - 2\lambda(\Omega_3 V_{2;2} - \Omega_2 V_{3;2} + \Omega_3 V_{1;1}) \\ - 2\Omega_2(Q_{21}V_{3;1} - Q_{31}V_{2;1}) - \Omega_2^2 V_{12} - \Omega_2\Omega_3 V_{13} = 0 ,\end{aligned} \tag{137}$$

$$\begin{aligned}\lambda^2(V_{1;3} - V_{3;1}) - 2\lambda(Q_{31}V_{1;1} - Q_{13}V_{3;3} - Q_{12}V_{3;2}) - 2\lambda(\Omega_3 V_{2;3} - \Omega_2 V_{3;3} - \Omega_2 V_{1;1}) \\ - 2\Omega_3(Q_{21}V_{3;1} - Q_{31}V_{2;1}) - \Omega_3^2 V_{13} - \Omega_2\Omega_3 V_{12} = 0 ,\end{aligned} \tag{138}$$

and

$$\begin{aligned}\lambda^2(V_{2;3} - V_{3;2}) - 2\lambda(Q_{31}V_{2;1} - Q_{21}V_{3;1}) + 2\lambda(\Omega_3 V_{1;3} + \Omega_2 V_{1;2}) \\ + 2\Omega_3(Q_{13}V_{3;3} - Q_{31}V_{1;1} + Q_{12}V_{3;2}) + 2\Omega_2(Q_{12}V_{2;2} - Q_{21}V_{1;1} + Q_{13}V_{2;3}) \\ + (\Omega_2^2 - \Omega_3^2)V_{23} + \Omega_2\Omega_3(V_{33} - V_{22}) = 0 .\end{aligned} \tag{139}$$

These equations are obtained by subtracting equations (126), (128), and (130) from equations (127), (129), and (131), respectively. Eliminating $V_{1;1}$ from these equations with the aid of the divergence condition (132), we obtain

$$\begin{aligned}(\lambda^2 - \Omega_2^2)V_{1;2} - (\lambda^2 + \Omega_2^2 - 2\Omega_2 Q_{31})V_{2;1} - \Omega_2\Omega_3 V_{1;3} \\ - \Omega_2(2Q_{21} + \Omega_3)V_{3;1} + 2\lambda Q_{13}V_{2;3} + 2\lambda\Omega_2 V_{3;2} \\ + 2\lambda\left[Q_{12} - \Omega_3 + \frac{a_1^2}{a_2^2}(Q_{21} + \Omega_3)\right]V_{2;2} + 2\lambda\frac{a_1^2}{a_3^2}(Q_{21} + \Omega_3)V_{3;3} = 0 ,\end{aligned} \tag{140}$$

$$\begin{aligned}- \Omega_2\Omega_3 V_{1;2} + \Omega_3(2Q_{31} - \Omega_2)V_{2;1} + (\lambda^2 - \Omega_3^2)V_{1;3} \\ - (\lambda^2 + \Omega_3^2 + 2\Omega_3 Q_{21})V_{3;1} - 2\lambda\Omega_3 V_{2;3} + 2\lambda Q_{12}V_{3;2} \\ + 2\lambda\frac{a_1^2}{a_2^2}(Q_{31} - \Omega_2)V_{2;2} + 2\lambda\left[Q_{13} + \Omega_2 + \frac{a_1^2}{a_3^2}(Q_{31} - \Omega_2)\right]V_{3;3} = 0 ,\end{aligned} \tag{141}$$

and

$$\begin{aligned}2\lambda\Omega_2 V_{1;2} - 2\lambda Q_{31}V_{2;1} + 2\lambda\Omega_3 V_{1;3} + 2\lambda Q_{21}V_{3;1} + (\lambda^2 + 2\Omega_2 Q_{13} + \Omega_2^2 - \Omega_3^2)V_{2;3} \\ - (\lambda^2 - 2\Omega_3 Q_{12} - \Omega_2^2 + \Omega_3^2)V_{3;2} + 2\left[\Omega_2(Q_{12} - \Omega_3) + \frac{a_1^2}{a_2^2}(\Omega_3 Q_{31} + \Omega_2 Q_{21})\right]V_{2;2} \\ + 2\left[\Omega_3(Q_{13} + \Omega_2) + \frac{a_1^2}{a_3^2}(\Omega_3 Q_{31} + \Omega_2 Q_{21})\right]V_{3;3} = 0 .\end{aligned} \tag{142}$$

It can now be verified directly that all the three-rowed determinants of the 3×8 matrix representing the foregoing system of equations vanish identically if λ^2 is set equal to $-(\Omega_2^2 + \Omega_3^2)$. The equations are therefore linearly dependent in the case considered. The linear dependence, under the same circumstances, of the equations of the system (133) follows a fortiori. This establishment of the linear dependence proves that $|\boldsymbol{\Omega}|$ is a characteristic frequency; that $|\boldsymbol{\Omega}^\dagger|$ is also a characteristic frequency follows from the theorem proved earlier that for an ellipsoid and its adjoint the characteristic equation (134) gives the same frequencies.

Finally, we shall show that $\lambda^2 = 0$ is a non-trivial double root of the characteristic equation. More precisely, we shall show that for the proper solutions belonging to the zero root the only non-vanishing virials are $V_{1;2}$, $V_{2;1}$, $V_{1;3}$, and $V_{3;1}$.

By setting

$$\lambda = 0 \qquad \text{and} \qquad V_{1;1} = V_{2;2} = V_{3;3} = V_{2;3} = V_{3;2} = 0\,, \tag{143}$$

we satisfy trivially equations (124), (125), (130), (131), and (132). Considering (127), (129), (137), and (138) as the remaining four equations, we find that under the circumstances specified (namely, [143]) equations (127) and (129) give

$$\begin{gathered}(Q_{12}Q_{21} + Q_{13}Q_{31})V_{2;1} + Q_{21}Q_{12}V_{1;2} + Q_{21}Q_{13}V_{1;3} \\ -2\Omega_2(Q_{21}V_{3;1} - Q_{31}V_{2;1}) - (\Omega_2^2 + \Omega_3^2 - 2B_{12})V_{12} = 0\end{gathered} \tag{144}$$

and

$$\begin{gathered}(Q_{12}Q_{21} + Q_{13}Q_{31})V_{3;1} + Q_{31}Q_{13}V_{1;3} + Q_{31}Q_{12}V_{1;2} \\ - 2\Omega_3(Q_{21}V_{3;1} - Q_{31}V_{2;1}) - (\Omega_2^2 + \Omega_3^2 - 2B_{13})V_{13} = 0\,,\end{gathered} \tag{145}$$

while equations (137) and (138) provide the *single* equation

$$2(Q_{21}V_{3;1} - Q_{31}V_{2;1}) + \Omega_2 V_{12} + \Omega_3 V_{13} = 0\,. \tag{146}$$

Equations (144) and (145) can be rewritten in the forms

$$[Q_{12}Q_{21} - (\Omega_2^2 + \Omega_3^2) + 2B_{12}]V_{12} + Q_{21}Q_{13}V_{13} - (2\Omega_2 + Q_{13})(Q_{21}V_{3;1} - Q_{31}V_{2;1}) = 0 \tag{147}$$

and

$$[Q_{13}Q_{31} - (\Omega_2^2 + \Omega_3^2) + 2B_{13}]V_{13} + Q_{31}Q_{12}V_{12} - (2\Omega_3 - Q_{12})(Q_{21}V_{3;1} - Q_{31}V_{2;1}) = 0\,. \tag{148}$$

We observe that equations (146)–(148) are linear and homogeneous in the virials

$$V_{12},\ V_{13}\,, \qquad \text{and} \qquad Q_{21}V_{3;1} - Q_{31}V_{2;1}\,; \tag{149}$$

and if these virials do not vanish identically, it must be true that

$$\begin{vmatrix} 2B_{12} - \boldsymbol{\Omega}^2 - \dfrac{a_1^2}{a_2^2}\Omega_3^2\gamma^2 & \dfrac{a_1^2}{a_3^2}\Omega_2\Omega_3\beta\gamma & -\Omega_2\left(2 - \dfrac{a_1^2}{a_3^2}\beta\right) \\ \dfrac{a_1^2}{a_2^2}\Omega_2\Omega_3\beta\gamma & 2B_{13} - \boldsymbol{\Omega}^2 - \dfrac{a_1^2}{a_3^2}\Omega_2^2\beta^2 & -\Omega_3\left(2 - \dfrac{a_1^2}{a_2^2}\gamma\right) \\ \Omega_2 & \Omega_3 & 2 \end{vmatrix} = 0\,. \tag{150}$$

By making use of the relations (34) and (35), the requirement (150) can be reduced to the form

$$\begin{bmatrix} -\dfrac{a_1^2(3a_3^2-4a_1^2+a_2^2)\beta}{2a_3^2(a_1^2-a_2^2)}-1 & \dfrac{a_1^2}{a_3^2}\beta\gamma & -2+\dfrac{a_1^2}{a_3^2}\beta \\ \dfrac{a_1^2}{a_2^2}\beta\gamma & -\dfrac{a_1^2(3a_2^2-4a_1^2+a_3^2)\gamma}{2a_2^2(a_1^2-a_3^2)}-1 & -2+\dfrac{a_1^2}{a_2^2}\gamma \\ 1 & 1 & 2 \end{bmatrix}=0\,, \tag{151}$$

and by direct evaluation it can be verified that the determinant is in fact zero. Equations (124)–(132) can, therefore, be satisfied non-trivially for $\lambda^2 = 0$ with non-vanishing values for the virials listed in (149). Moreover, it is clear that there are two linearly independent solutions; and since the original characteristic equation (134) is even in λ^2 (of degree *eight*), $\lambda^2 = 0$ is a double root.

Excluding the roots $-|\boldsymbol{\Omega}|^2$, $-|\boldsymbol{\Omega}^\dagger|^2$, and zero (of multiplicity two) we have four roots of the characteristic equation yet to determine. These remaining roots have been determined numerically for a number of ellipsoids of types I, II, and III and are considered in § XI below.

X. THE ASYMPTOTIC PROPERTIES OF THE DISKLIKE RIEMANN ELLIPSOIDS ON THE a_2-AXIS

As $a_3 \to 0$, the ellipsoids of types I and II become disklike and their asymptotic properties are of interest.

It can be readily verified that, in the limit

$$\epsilon = a_3/a_1 \to 0\,, \tag{152}$$

the index symbols A_i (cf. eq. [44]) have the behavior

$$A_1 = \alpha_1\epsilon\,, \qquad A_2 = \alpha_2\epsilon\,, \qquad \text{and} \qquad A_3 = 2\,, \tag{153}$$

where α_1 and α_2 are certain constants expressible in terms of the *complete* elliptic integrals

$$E(\theta) = \int_0^{\pi/2} d\phi(1-\sin^2\theta\,\sin^2\phi)^{1/2}$$

and (154)

$$F(\theta) = \int_0^{\pi/2} d\phi(1-\sin^2\theta\,\sin^2\phi)^{-1/2}\,,$$

with the argument

$$\theta = \sec^{-1}(a_2/a_1)\,. \tag{155}$$

Thus,

$$\alpha_1 = \frac{2}{\sin^2\theta}[E(\theta)-F(\theta)\cos^2\theta] \qquad \text{and} \qquad \alpha_2 = \frac{2}{\tan^2\theta}[F(\theta)-E(\theta)]\,. \tag{156}$$

The corresponding asymptotic forms of the two-index symbols are

$$B_{11} = \beta_{11}\epsilon\,, \qquad B_{22} = \beta_{22}\epsilon\,, \qquad B_{33} = \tfrac{4}{3}\,,$$

(157)

$$B_{12} = \beta_{12}\epsilon\,, \qquad B_{23} = \beta_{23}\epsilon\,, \qquad \text{and} \qquad B_{31} = \beta_{31}\epsilon\,,$$

where

$$\beta_{11}=\frac{(\alpha_1-\alpha_2)a_2^2}{3(a_2^2-a_1^2)},\qquad \beta_{22}=\frac{(\alpha_1-\alpha_2)a_1^2}{3(a_2^2-a_1^2)},$$

$$\beta_{12}=\frac{\alpha_2a_2^2-\alpha_1a_1^2}{a_2^2-a_1^2},\qquad \beta_{23}=\alpha_2,\qquad\text{and}\qquad \beta_{31}=\alpha_1. \tag{158}$$

Similarly, from equations (17), (18), and (75), we conclude that in this same limit,

$$\beta=\frac{4a_1^2-a_2^2}{2a_1^2},\quad \gamma=2,\quad \beta^{\dagger}=\frac{2a_1^2}{4a_1^2-a_2^2}\epsilon^2,\qquad\text{and}\qquad \gamma^{\dagger}=\frac{a_2^2}{2a_1^2}. \tag{159}$$

Inserting the foregoing asymptotic forms of the various constants in equations (37), (38), and (136), we find

$$\Omega_2=\omega_2\epsilon^{3/2},\qquad \Omega_3=\omega_3\epsilon^{1/2},\qquad \Omega_2^{\dagger}=\omega_2^{\dagger}\epsilon^{1/2},\qquad\text{and}\qquad \Omega_3^{\dagger}=\omega_3^{\dagger}\epsilon^{1/2}, \tag{160}$$

where

$$\omega_2=\frac{2a_1^2}{a_2(4a_1^2-a_2^2)}[2(3\alpha_2+\alpha_1)a_2^2-8\alpha_2a_1^2]^{1/2},\qquad \omega_3=(2\alpha_2)^{1/2},$$

$$\omega_2^{\dagger}=\omega_2\beta=\frac{1}{a_2}[2(3\alpha_2+\alpha_1)a_2^2-8\alpha_2a_1^2]^{1/2},\qquad\text{and}\qquad \omega_3^{\dagger}=\frac{a_1}{a_2}(8\alpha_2)^{1/2}. \tag{161}$$

The corresponding asymptotic forms of the components of the vorticity are

$$\zeta_2=z_2\epsilon^{-1/2},\qquad \zeta_3=z_3\epsilon^{1/2},\qquad \zeta_2^{\dagger}=z_2^{\dagger}\epsilon^{1/2},\qquad\text{and}\qquad \zeta_3^{\dagger}=z_3^{\dagger}\epsilon^{1/2}, \tag{162}$$

where

$$z_2=-\frac{4a_1^2-a_2^2}{2a_1^2}\omega_2,\qquad z_3=-2\frac{a_1^2+a_2^2}{a_2^2}\omega_3,$$

$$z_2^{\dagger}=-\frac{2a_1^2}{4a_1^2-a_2^2}\omega_2^{\dagger},\qquad z_3^{\dagger}=-\frac{a_1^2+a_2^2}{2a_1^2}\omega_3^{\dagger}. \tag{163}$$

The properties of the disklike ellipsoids of type I, determined with the aid of the foregoing formulae, are included in Table 10, § XI below.

a) The Asymptotic Form of the Characteristic Equation

Turning next to the stability of the disklike objects, we find that in the limit considered

$$\lambda=x\epsilon^{1/2}, \tag{164}$$

where the constant of proportionality x is determined by the appropriate limiting form of the characteristic equation (134).

On inserting the asymptotic forms of the various constants in the different elements of the secular matrix (134), we find that, while all the elements in the first four columns occur with a factor ϵ, the remaining four columns tend to finite limits. After the removal of the factor ϵ^4 the secular determinant takes a finite form that is, moreover, manifestly the product of the two determinants

$$\begin{bmatrix} x^2-3\omega_3^2 & -2x\omega_3^{\dagger} \\ +2x\omega_3^{\dagger} & x^2+\omega_3^2-(\omega_3^{\dagger})^2 \end{bmatrix}=0 \tag{165}$$

and

$$\begin{vmatrix} (1,1) & (1,2) & -2x\left(\omega_3+\frac{a_1}{a_2}\,\omega_3^\dagger\right) & -2x\left(\omega_3+\frac{a_2}{a_1}\,\omega_3^\dagger\right) & 2x\omega_2^\dagger & -2\omega_2^\dagger\omega_3^\dagger\frac{a_2}{a_1} \\ (2,1) & (2,2) & -2x\omega_3^\dagger\frac{a_1}{a_2} & -2x\omega_3 & 2x\omega_2^\dagger & -\omega_2^\dagger\omega_3^\dagger\frac{a_2}{a_1} \\ -2x\left(\omega_3^\dagger+\frac{a_1}{a_2}\,\omega_3\right) & -2x\omega_3\frac{a_1}{a_2} & -\omega_3^2-(\omega_3^\dagger)^2+2\beta_{12} & x^2-\omega_3^2-(\omega_2^\dagger)^2-(\omega_3^\dagger)^2+2\beta_{12} & 2\omega_2^\dagger\omega_3 & 2x\omega_2^\dagger \\ -2x\left(\omega_3^\dagger+\frac{a_2}{a_1}\,\omega_3\right) & -2x\omega_3^\dagger & x^2-\omega_3^2-(\omega_3^\dagger)^2+2\beta_{12} & -\omega_3^2-(\omega_2^\dagger)^2-(\omega_3^\dagger)^2+2\beta_{12} & \omega_2^\dagger\omega_3^\dagger\frac{a_2}{a_1} & 0 \\ 2x\omega_2^\dagger & 2x\omega_2^\dagger & \omega_2^\dagger\omega_3^\dagger & 2\omega_2^\dagger\omega_3\frac{a_2}{a_1} & \frac{a_2}{a_1}[x^2-\omega_3^2-(\omega_2^\dagger)^2+2\beta_{13}] & -2x\omega_3\frac{a_2}{a_1} \\ 2\omega_2^\dagger\omega_3^\dagger & \omega_2^\dagger\omega_3^\dagger & 0 & -2x\omega_2^\dagger & 2x\omega_3 & x^2-(\omega_2^\dagger)^2 \end{vmatrix} = 0\,. \tag{166}$$

where

$$(1,1) = -\left(\frac{a_2}{a_1}+\frac{a_1}{a_2}\right)[x^2-2\omega_3^2-2(\omega_3^\dagger)^2-2\beta_{12}]+\frac{a_1}{a_2}[2(\omega_2^\dagger)^2-12\beta_{11}]\,,$$

$$(1,2) = (2,1) = -\frac{a_1}{a_2}[x^2-2\omega_3^2-2(\omega_2^\dagger)^2-2(\omega_3^\dagger)^2+6\beta_{11}-2\beta_{12}]\,, \tag{167}$$

and

$$(2,2) = -\frac{a_1}{a_2}[x^2+2\omega_3^2-2(\omega_2^\dagger)^2-2(\omega_3^\dagger)^2+6\beta_{11}-2\beta_{13}]\,.$$

On expanding the determinant (165) and making use of the special forms of ω_3^2 and $(\omega_3^\dagger)^2$ (see eqs. [161]), we find that equation (165) provides the two characteristic roots

$$x^2 = -\omega_3^2 \qquad \text{and} \qquad x^2 = -\frac{3}{a_2^2}(4a_1^2 - a_2^2)\omega_3^2 . \tag{168}$$

Accordingly, the second of these two roots makes all the disklike ellipsoids of type II (for which $a_2 \geq 2a_1$) unstable. But the determination of the stability of the analogous ellipsoids of type I requires a consideration of the roots of equation (166) (see § XI below).

XI. THE DOMAINS OF STABILITY WITH RESPECT TO THE OSCILLATIONS BELONGING TO THE SECOND HARMONICS IN THE $(a_2/a_1,\ a_3/a_1)$-PLANE

The characteristic equation (134) has been solved for its roots for some hundred ellipsoids in their domains of occupancy; and by interpolation among the roots so obtained, the loci of the marginally stable configurations in the $(a_2/a_1,\ a_3/a_1)$-plane were determined. The results of the calculations, as they pertain to the loci of marginal stability (rather *marginal overstability*, as it happens), are summarized in Tables 6 and 7; and in Tables 8 and 9 we enumerate the characteristic frequencies of oscillation of the ellipsoids whose properties have been listed in § VII.

TABLE 6*a*

THE PROPERTIES OF THE MARGINALLY OVERSTABLE RIEMANN ELLIPSOIDS OF TYPE I*

(Along the Locus $O_2X_2^{(\mathrm{II})}$)

a_2/a_1	1 0000	1 0526	1 1111	1.1765	1 2500	1 3333	1.4286	1.5385	1 6722
a_3/a_1	0 3033	0 3712	0.4230	0 4560	0 4703	0 4676	0 4474	0 4053	0 3278
Ω_2	0	+0.1283	+0 2153	+0 2942	+0.3639	+0.4269	+0 4877	+0 5550	+0.7107
Ω_3	+0.7073	+0 7176	+0 7098	+0 6901	+0 6633	+0 6329	+0.5999	+0 5635	+0 5142
ζ_2	0	−1 5014	−1 8984	−2 1276	−2 2794	−2 3842	−2.4626	−2 5307	−2 4011
ζ_3	−2 7417	−2 5977	−2.3978	−2 1787	−1 9637	−1 7621	−1.5752	−1 3984	−1 1673
$\Omega_2^\dagger$	0	+0 4898	+0 6812	+0 8032	+0 8778	+0 9150	+0.9178	+0 8807	+0 7107
$\Omega_3^\dagger$	+1.3708	+1 2972	+1 1922	+1 0751	+0 9579	+0 8458	+0 7400	+0 6390	+0.5142
$\zeta_2^\dagger$	0	−0.3931	−0.6000	−0 7794	−0 9450	−1.1125	−1 3082	−1 5937	−2.4011
$\zeta_3^\dagger$	−1 4147	−1 4371	−1.4275	−1.3984	−1 3599	−1 3186	−1.2768	−1 2330	−1 1673

* The angular velocities and the vorticities are expressed in the unit $(\pi G\rho)^{1/2}$.

TABLE 6*b*

THE PROPERTIES OF THE MARGINALLY OVERSTABLE RIEMANN ELLIPSOIDS OF TYPE I*

(Along the Locus D_2Q)

a_2/a_1	1 1582	1.1846	1 2124	1 2418	1 2727	1 3050	1 3707
a_3/a_1	0 1411	0 1238	0 1057	0 0866	0 0666	0 0455	ϵ
Ω_2	+0 0618	+0 0558	+0 0480	+0 0389	+0 0286	+0 0176	+2 1492 $\epsilon^{3/2}$
Ω_3	+0 5209	+0 4903	+0 4554	+0 4146	+0 3658	+0 3044	+1 4485 $\epsilon^{1/2}$
ζ_2	−4 1927	−4 7796	−5 4901	−6.4045	−7 6880	−9 7572	−2 2795 $\epsilon^{-1/2}$
ζ_3	−1 8047	−1 6695	−1.5236	−1 3629	−1 1810	−0 9654	−4 4390 $\epsilon^{1/2}$
$\Omega_2^\dagger$	+0 5802	+0 5829	+0 5737	+0 5506	+0 5098	+0 4436	+2.2795 $\epsilon^{1/2}$
$\Omega_3^\dagger$	+0 8927	+0 8229	+0 7479	+0 6658	+0 5737	+0 4661	+2 1135 $\epsilon^{1/2}$
$\zeta_2^\dagger$	−0 4469	−0 4573	−0 4598	−0 4523	−0.4308	−0 3873	−2 1492 $\epsilon^{1/2}$
$\zeta_3^\dagger$	−1 0530	−0 9947	−0 9277	−0 8488	−0 7529	−0 6305	−3.0422 $\epsilon^{1/2}$

* The angular velocities and the vorticities are expressed in the unit $(\pi G\rho)^{1/2}$.

TABLE 6c

THE PROPERTIES OF THE MARGINALLY OVERSTABLE RIEMANN ELLIPSOIDS OF TYPE I*
(Along the Locus QR_{I})

a_2/a_1	1 2907	1 4954	1 6417	1 7679	1 8651
a_3/a_1	0 1573	0 1563	0.1431	0 1233	0 0976
Ω_2	+0 0979	+0.1427	+0 1769	+0 2211	+0 2856
Ω_3	+0.5082	+0 4633	+0 4219	+0 3784	+0 3310
ζ_2	−4 6947	−5.1812	−5 5580	−6 0132	−6.7416
ζ_3	−1.6093	−1 3221	−1 1381	−0 9802	−0 8341
$\Omega_2\dagger$	+0 7206	+0 7908	+0 7788	+0 7280	+0 6299
$\Omega_3\dagger$	+0 7791	+0 6109	+0 5056	+0 4200	+0 3471
$\zeta_2\dagger$	−0.6376	−0.9355	−1.2612	−1.8137	−2 8546
$\zeta_3\dagger$	−1 0498	−1.0027	−0 9496	−0 8829	−0 7942

* The angular velocities and the vorticities are expressed in the unit $(\pi G\rho)^{1/2}$.

TABLE 7

THE PROPERTIES OF THE MARGINALLY OVERSTABLE RIEMANN ELLIPSOIDS OF TYPE III*
(Along the Locus $X_2^{(\mathrm{III})}O'$)

a_2/a_1	4.0000	4 4141	4.9777	5.3909
a_3/a_1	1.7210	1.6000	1.4933	1.4000
Ω_2	+0.4966	+0.5410	+0.5567	+0.5497
Ω_3	+0.3281	+0 2553	+0 1968	+0.1657
ζ_2	+0.5283	+0.3190	+0.1992	+0 1430
ζ_3	−1 9954	−2 1577	−2.2582	−2.2936
$\Omega_2\dagger$	+0.2198	+0 1440	+0 0914	+0 0676
$\Omega_3\dagger$	+0 4787	+0 4641	+0.4359	+0 4113
$\zeta_2\dagger$	+1 0946	+1 2088	+1.1951	+1 1610
$\zeta_3\dagger$	−1.4219	−1 1826	−1 0190	−0 9242

* The angular velocities and the vorticities are expressed in the unit $(\pi G\rho)^{1/2}$.

TABLE 8

THE SQUARES OF THE CHARACTERISTIC FREQUENCIES OF OSCILLATIONS OF SOME TYPICAL RIEMANN ELLIPSOIDS OF TYPE I*

$\Omega_2^2+\Omega_3^2$	0 55867	0 59683	0 68570	0 78661	0 16888	0 17311	0 08871	0 08956
$\Omega_2\dagger^2+\Omega_3\dagger^2$	1 95795	1 66888	1 28261	0 78661	0 75034	0 73533	0 42328	0 42713
σ_1^2	3 95684	2 90595	2 42382	1 30300	1 98060	0 76906	1 11840	1 24447
σ_2^2	3 10667	2 14020	1 41447	1 08486	1 13927	0 48044	0 59519	0 49385
σ_3^2	1 41497	1 06182	0 79359	0.60901	0.31730	0 18666	0 14508	0 06994
σ_4^2	6 75010	7.99131	8 22190	6 47555	0.62319	2 36704	0.20346	0 17370

* The squares of the characteristic frequencies are expressed in the unit $\pi G\rho$.

The loci of the marginally overstable configurations are delineated in Figures 1 and 3; and the properties of these configurations are further exhibited in Figures 4 and 5.

It will be observed that, among the ellipsoids of type I, there are two disconnected domains of stability with respect to the oscillations considered. The existence of the stable domain, adjoining the stable Maclaurin spheroids along SO_2 is, of course, to be expected. But the existence of the second domain, bounded by the segment D_2R_{I} of the a_2-axis, is unexpected. The point (see Table 6*b*),

$$a_2/a_1 = 1.3707\,, \tag{169}$$

limiting the stable disklike ellipsoids of type I was determined with the aid of equation (166). In Table 10 we list the asymptotic forms of the characteristic frequencies of oscillation, together with some of the other properties, of these disklike ellipsoids.

The calculations show that *all ellipsoids of type II are unstable.* As we have already remarked in § X, their instability along the a_2-axis follows directly from equations (168).

TABLE 9

THE SQUARES OF THE CHARACTERISTIC FREQUENCIES OF OSCILLATIONS OF SOME TYPICAL RIEMANN ELLIPSOIDS OF TYPES II AND III*

	Ellipsoids of Type II			Ellipsoids of Type III			
$\Omega_2^2+\Omega_3^2$	+0 52061	+1 31413	+0 70393	+0 39458	+0 35751	+0 23651	+0 24718
$\Omega_2^{\dagger 2}+\Omega_3^{\dagger 2}$	+0 06631	+0 14392	+0 04312	+0.39458	+0 35751	+0 23192	+0 19297
σ_1^2	+0 11530	+0 21425	+0 07331	+0 50590	+0 46352	+0 70381	+0 55228
σ_2^2	−0 10399	−0 07905	−0 09417	±0 28039*i*	±0 20708*i*	+0 15463	+0 20340
σ_3^2	+0 51325	+1 25723	+0 68147	+1.27177	+1 79119	+1 74081	+1 18118
σ_4^2	+1 44145	+3 61524	+2 04314	+1.39306	+1 87486	+2 65271	+3 06709

* The squares of the characteristic frequencies are expressed in the unit $\pi G\rho$.

TABLE 10

THE ASYMPTOTIC PROPERTIES AND THE SQUARES OF THE CHARACTERISTIC FREQUENCIES OF OSCILLATION OF THE DISKLIKE ELLIPSOIDS OF TYPE I

$\theta=\sec^{-1}(a_2/a_1)$	42°	43°	44°	45°	50°	58°
a_2/a_1	1 34563	1 36733	1 39016	1.41421	1 55572	1 88708
ω_2	+ 2.05429	+ 2 13629	+ 2.22576	+ 2 32434	+ 3 05912	+11 27045
ω_3	+ 1.46610	+ 1 45084	+ 1 43516	+ 1 41906	+ 1 33207	+ 1 16871
z_2	− 2.24870	− 2 27560	− 2 30081	− 2 32434	− 2 41628	− 2 47347
z_3	− 4 55156	− 4 45371	− 4 35556	− 4.25717	− 3.76491	− 2 99380
$\omega_2^\dagger$	+ 2 24870	+ 2.27560	+ 2.30081	+ 2 32434	+ 2 41628	+ 2.47347
$\omega_3^\dagger$	+ 2 17905	+ 2 12215	+ 2 06473	+ 2 00685	+ 1.71248	+ 1 23864
$z_2^\dagger$	− 2.05429	− 2 13629	− 2.22576	− 2 32434	− 3 05912	−11 27045
$z_3^\dagger$	− 3 06236	− 3 04484	− 3.02747	− 3 01027	− 2 92858	− 2.82477
ω_3^2	2.14945	2 10493	2.05968	2 01372	1 77442	1 36588
$(\omega_2^\dagger)^2+(\omega_3^\dagger)^2$	9 80493	9 68185	9 55686	9 43002	8.77100	7 65228
$3(4a_1^2-a_2^2)\omega_3^2/a_2^2$	7 79644	7 19577	6.61032	6.04117	3 47451	0 50507
σ_1^2	19 86078	19 55939	17.53812	16 45406	12.98259	9 39418
	±	±				
σ_2^2	2 05833*i*	0 73302*i*	20 96929	21 43377	21 70035	19 90914
σ_3^2	2 58994	2 55633	2 52123	2 48458	2 27657	1.84879

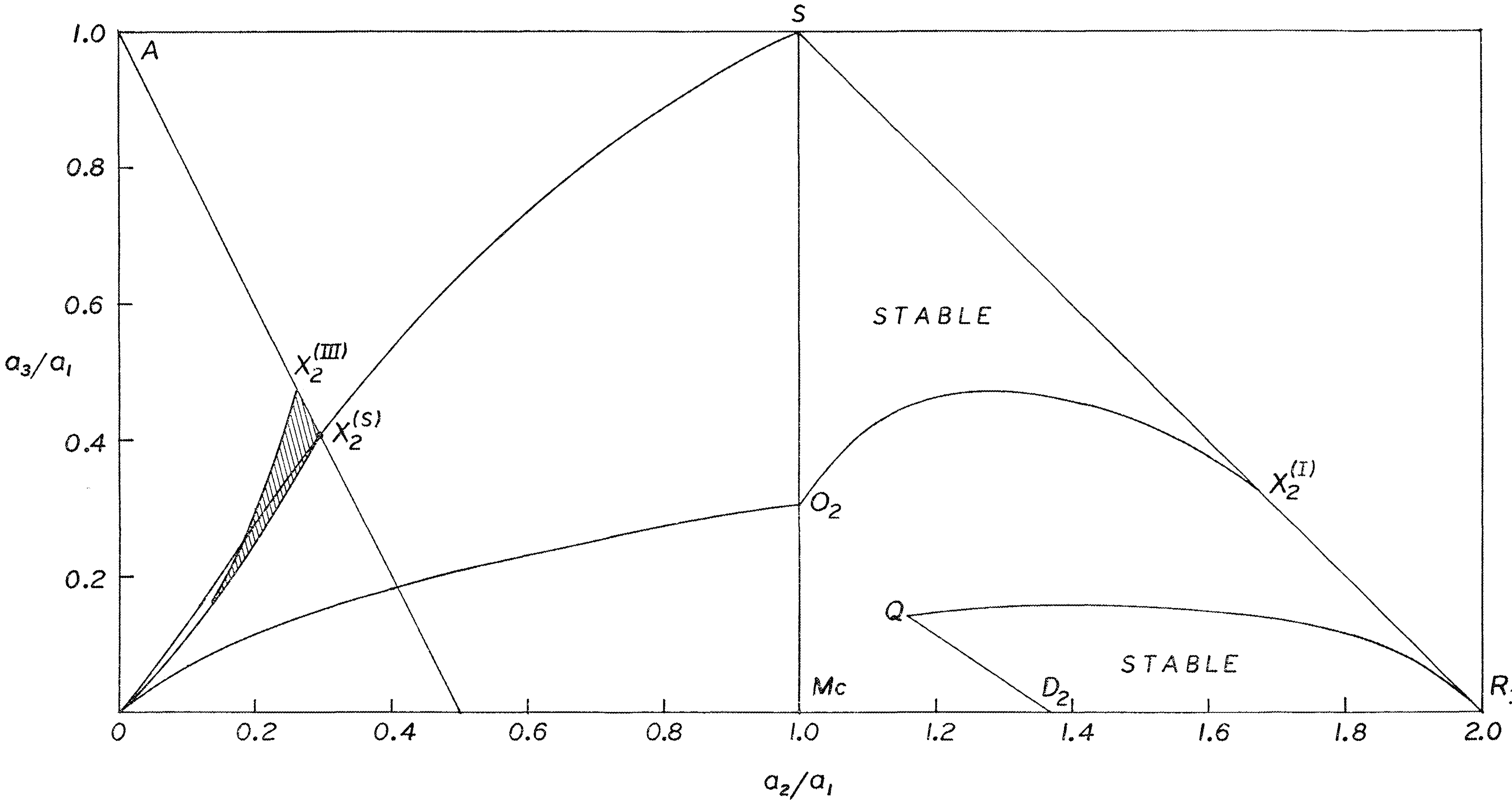

FIG. 3.—The loci of marginally stable configurations in the $(a_2/a_1, a_3/a_1)$-plane. The type S ellipsoids are bounded by two self-adjoint sequences (SO and O_2O) and the stable part of the Maclaurin sequence represented by SO_2. Along the arc $X_2^{(S)}O$ the type S ellipsoids become unstable by a mode of oscillation belonging to the second hormonics; and along this same arc the stability passes to the type III ellipsoids whose domain of occupancy is $AX_2^{(S)}O$. The shaded region included between $X_2^{(III)}O$ and $X_2^{(S)}O$ represents the domain of stability for type III ellipsoids with respect to oscillations belonging to the second harmonics.

The type I ellipsoids occupy the triangle SM_cR_I; and the region of the stable members is included in the two domains marked "stable." The domain $SO_2X_2^{(I)}$ of stable ellipsoids adjoining the stable Maclaurin spheroids is to be expected; but the domain D_2QR_I including disklike ellipsoids along D_2R_I is unexpected.

Among the ellipsoids of type III, there is a fringe of stable configurations (stable, that is, with respect to the oscillations considered) bordering on the boundary $X_2'O'$ of their domain of occupancy. As we have shown in § VI, by interchanging the roles of the indices 1 and 2 (so that a_1 becomes the longest axis, as it is among the ellipsoids of type S) the locus $X_2'O'$ is transformed to the locus X_2O of the marginally unstable ellipsoids of type S (see Fig. 3). One should expect under these circumstances that the stability passes from the ellipsoids of type S to the ellipsoids of type III along their common curve of bifurcation; and this is exactly what happens. However, since the ellipsoids of type S become

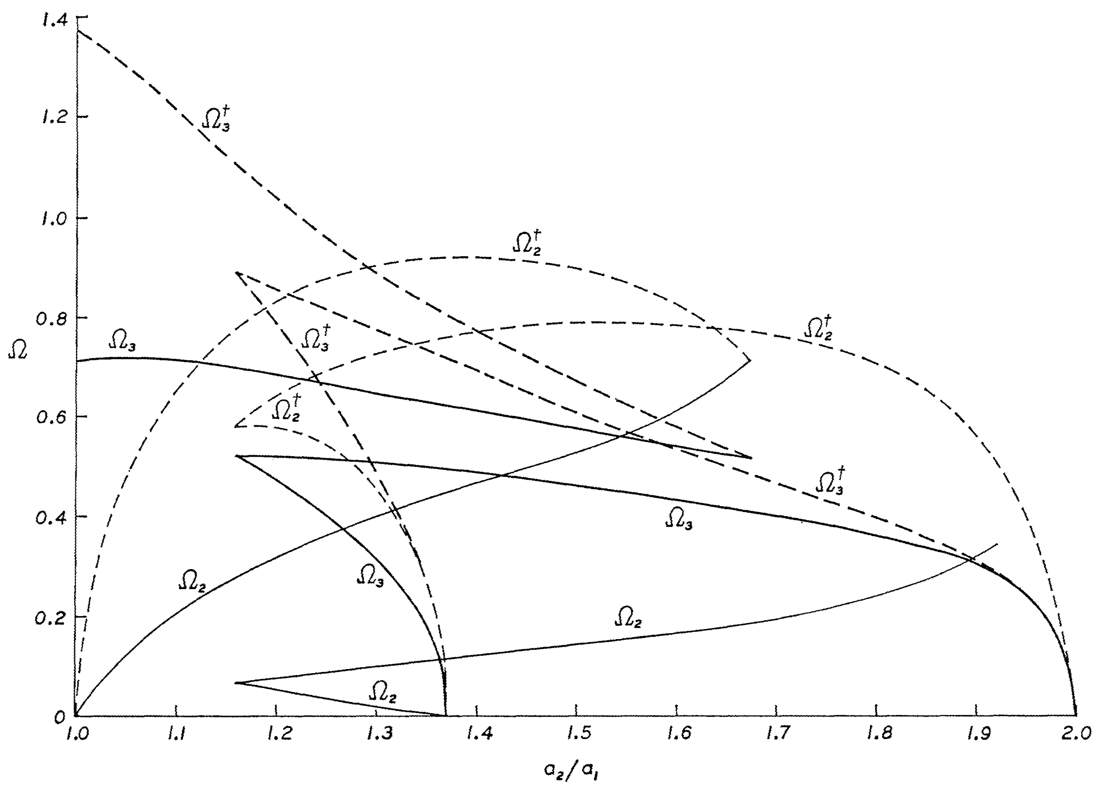

FIG. 4.—The variation of the components of the angular velocity along the marginally overstable ellipsoids delineated in Fig. 3. The curves distinguished by Ω and $\Omega\dagger$ are appropriate for the adjoint configurations having the same figure.

unstable with respect to an oscillation belonging to the *third* harmonics, *prior* to the onset of instability by an odd mode of oscillation belonging to the second harmonics, *it is very likely that ellipsoids of type III are all unstable with respect to a third-harmonic oscillation.*

From the foregoing account it would appear that only among the ellipsoids of types I and S do stable ones occur.

In his paper, Riemann considers the stability of his ellipsoidal figures by an energy criterion. But most of the conclusions he derives from his criterion (with the notable exception of those pertaining to the Maclaurin spheroid) are false. His criterion is clearly in error; the origin of this error (which mars an otherwise most remarkable paper) is clarified by Lebovitz (1966) in the paper following this one.

XII. CONCLUDING REMARKS

The present paper completes the series of investigations initiated some six years ago with a view toward completing and consolidating the classical work on the ellipsoidal figures of equilibrium of homogeneous masses. As the investigations proceeded, several misconceptions in the earlier work (e.g., the Roche ellipsoids become "unstable" at the Roche limit, or that the bifurcation of the Jacobian from the Maclaurin sequence is a "unique" phenomenon) became apparent; and these have been eliminated.

In many ways, the most curious aspect of the subject has been the almost total neglect of the fundamental papers of Dirichlet, Dedekind, and Riemann (all published in 1860). Nevertheless, the completion of Riemann's work has been essential to a comprehensive view of the subject. The fruitful exploration of these classical avenues is by

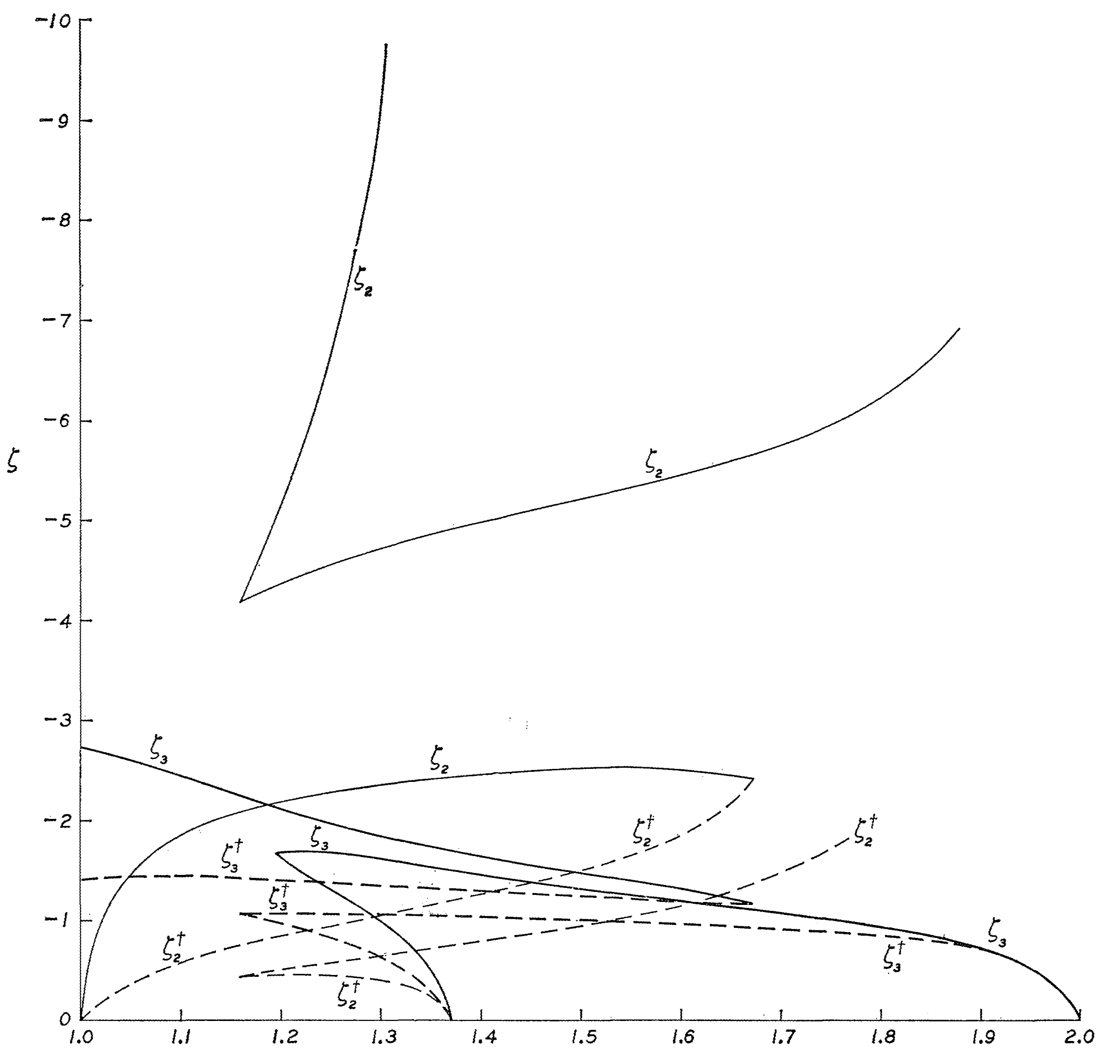

FIG. 5.—The variation of the components of the vorticity along the marginally overstable ellipsoids delineated in Fig. 3. The curves distinguished by ζ and $\zeta\dagger$ are appropriate for the adjoint configurations having the same figure.

no means ended: the continuation of Dirichlet's work on the non-linear finite amplitude oscillations of ellipsoidal figures appears to hold rich promise. These further areas of research are, however, beyond the scope of the present series.

I am grateful to Dr. M. Clement whose careful scrutiny of the analysis helped the elimination of a number of oversights and obscurities; he also generously programmed for machine calculations,[1] the characteristic equation (both in its finite and asymptotic forms) for the determination of its roots. I am equally grateful to Dr. N. R. Lebovitz for many clarifying discussions and particularly for his examination of Riemann's criterion for determining the stability of his figures and demonstration of the place where he erred. I am also indebted to Miss Donna Elbert for her continued patience in assisting with these investigations; in this particular instance it was most essential.

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REFERENCES

Basset, A. B. 1888, *A Treatise on Hydrodynamics* (Cambridge, Eng.: Deighton Bell & Co.; reprinted 1961, Dover Publications, New York), Vol. **2**.
Chandrasekhar, S. 1964, *Lectures in Theoretical Physics*, ed. W. E. Brittin and W. R. Chappell (Boulder: University of Colorado Press), p. 1.
———. 1965*a*, *Ap. J.*, **141**, 1043.
———. 1965*b*, *ibid.*, **142**, 890.
Chandrasekhar, S., and Lebovitz, N. R. 1962, *Ap. J.*, **136**, 1037.
Hicks, W. M. 1882, *Reports to the British Association*, pp. 57–61.
Lebovitz, N. R. 1961, *Ap. J.*, **134**, 500.
———. 1966, *ibid.*, **145**, 878.
Riemann, B. 1860, *Abh. d. Königl. Gesell. der Wiss. zur Göttingen*, **9**, 3; see also 1892, *Gesammelte mathematische Werke* (Leipzig: Verlag von B. G. Teubner; reprinted 1953, Dover Publications, New York), p. 182.

[1] The machine calculations were performed with an IBM 1620 computer at the Yerkes Observatory.