THE STABILITY OF GASEOUS MASSES FOR RADIAL AND NON-RADIAL OSCILLATIONS IN THE POST-NEWTONIAN APPROXIMATION OF GENERAL RELATIVITY

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ABSTRACT

The stability of gaseous masses with respect to radial as well as non-radial oscillations is considered in the framework of the post-Newtonian equations of hydrodynamics The onset of dynamical instability at a radius R determined by a formula of the type

$$R=\frac{2GM}{c^2}\frac{K}{\gamma-\frac{4}{3}}$$

(where K is a constant) is confirmed in case the "ratio of the specific heats" $\gamma = (\partial \log p/\partial \log \rho)_s$ (where the subscript s denotes that the derivative is with respect to constant entropy) is a constant. An expression for K is derived which does not involve any knowledge of the equilibrium configuration beyond the Newtonian framework; and the values of K appropriate to the polytropes are also listed. With respect to the onset of instability for non-radial oscillations, it is shown that the classical criterion of Schwarzschild based on the discriminant

$$S(r)=\frac{dp}{dr}-\gamma\frac{p}{\rho}\frac{d\rho}{dr}$$

is replaced by one based on the discriminant

$$\mathfrak{S}(r)=S(r)+\frac{\Pi}{c^2}\frac{dp}{dr}\left(\Gamma-\gamma+\frac{1}{\Gamma-1}\frac{d\Gamma/dr}{d\rho/\rho dr}\right),$$

where Π is the internal energy (per unit volume) and Γ is a ratio defined by the relation $\rho\Pi = p/(\Gamma-1)$. An alternative form for $\mathfrak{S}(r)$, namely,

$$\mathfrak{S}(r)=S(r)\left[1+\frac{\Pi}{c^2}\frac{\Gamma_3-\Gamma}{\Gamma_3-1}\frac{d(\log p)/dr}{d(\log \rho)/dr}\right],$$

where $\Gamma_3 = 1 + (\partial \log T/\partial \log \rho)_s$, shows that the condition for the occurrence of convective instability is unaltered in the post-Newtonian approximation.

I. INTRODUCTION

The post-Newtonian equations of hydrodynamics derived elsewhere in this issue (Chandrasekhar 1965; this paper will be referred to hereafter as "Paper I") enable a systematic investigation of the initial effects of general relativity on a wide variety of problems. In many ways, the most interesting question to which one should like to find an answer with the aid of these equations concerns the stability of gaseous masses with respect to *non-radial oscillations:* because it is known (Chandrasekhar 1964*a*, *b*) that post-Newtonian effects do induce instability with respect to *radial oscillations*. However, we shall find that no new instabilities are predicted.

In treating the problem of non-radial oscillations and convective instability in the framework of the post-Newtonian equations, we shall use the methods which have been developed recently for treating the same problems in the framework of the Newtonian equations (Chandrasekhar 1964*c;* Chandrasekhar and Lebovitz 1964; and Lebovitz 1965*a*, *b*).

II. THE EULERIAN AND THE LAGRANGIAN FORMS OF THE EQUATIONS OF MOTION GOVERNING SMALL OSCILLATION ABOUT EQUILIBRIUM

Consider a spherically symmetric distribution of matter in hydrostatic equilibrium. The equations governing the equilibrium are (Paper I, eqs. [40], [41], and [68])

$$\frac{d}{dr}\left[\left(1+\frac{2U}{c^2}\right)p\right]=\rho\frac{dU}{dr}+\frac{2}{c^2}\rho\left(\phi\frac{dU}{dr}+\frac{d\Phi}{dr}\right), \tag{1}$$

where

$$\phi=U+\tfrac{1}{2}\Pi+\tfrac{3}{2}\frac{p}{\rho}, \tag{2}$$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dU}{dr}\right)=-4\pi G\rho\,,\qquad\text{and}\qquad \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Phi}{dr}\right)=-4\pi G\rho\phi\,. \tag{3}$$

Also, in the foregoing equations, ρ is the density, p is the pressure, and Π is the internal energy per unit mass. It should be further noted in this connection that a basic assumption which underlies the entire development of the present post-Newtonian theory is that the energy-density ϵ is related to ρ and Π by the relation

$$\epsilon=\rho c^2+\rho\Pi\,. \tag{4}$$

Now let the equilibrium configuration considered be slightly perturbed; and let $\delta\rho$, δp, δU, etc., denote the resulting *Eulerian changes* in the respective quantities. The linearized form of the equation of motion governing the perturbation is (Paper I, eqs. [45], [63], [74], and [80])

$$\begin{aligned}\frac{\partial}{\partial t}(\sigma v_\alpha)&+\frac{1}{2c^2}\rho\frac{\partial}{\partial t}(U_\alpha-U_{\mu;\alpha\mu})+\frac{4}{c^2}\rho\frac{\partial}{\partial t}(v_\alpha U-U_\alpha)\\
&=-\frac{\partial}{\partial x_\alpha}\left[\left(1+\frac{2U}{c^2}\right)\delta p+\frac{2}{c^2}p\delta U\right]+\frac{\delta\rho}{\rho}\frac{\partial}{\partial x_\alpha}\left[\left(1+\frac{2U}{c^2}\right)p\right]\\
&\quad+\rho\frac{\partial}{\partial x_\alpha}\delta U+\frac{2}{c^2}\rho\left(\delta\phi\frac{\partial U}{\partial x_\alpha}+\phi\frac{\partial}{\partial x_\alpha}\delta U+\frac{\partial}{\partial x_\alpha}\delta\Phi\right),\end{aligned} \tag{5}$$

where

$$\sigma=\rho\left[1+\frac{1}{c^2}\left(2U+\Pi+\frac{p}{\rho}\right)\right], \tag{6}$$

$$U_\alpha(x)=G\int_V\frac{\rho(x')v_\alpha(x')}{|x-x'|}dx', \tag{7}$$

and

$$U_{\mu;\alpha\mu}(x)=G\int_V\rho(x')v_\mu(x')(x_\mu-x_\mu')(x_\alpha-x_\alpha')\frac{dx'}{|x-x'|^3}; \tag{8}$$

also the integrations in equations (7) and (8) are effected over the volume V occupied by the fluid.

In addition to equation (5), we have the equation of continuity (Paper I, eqs. [117] and [118])

$$\frac{\partial}{\partial t}\delta\rho^*+\frac{\partial}{\partial x_\alpha}(\rho^* v_\alpha)=0\,, \tag{9}$$

where

$$\rho^* = \rho\left(1+\frac{3}{c^2}U\right). \tag{10}$$

Let the perturbed state be described by a Lagrangian displacement of the form

$$\xi(x)e^{i\omega t}, \tag{11}$$

where ω is a characteristic frequency to be determined. Equation (5) then becomes

$$\begin{aligned}\omega^2\left[\sigma\xi_\alpha+\frac{1}{2c^2}\rho(U_\alpha-U_{\mu\,\alpha\mu})+\frac{4}{c^2}\rho(\xi_\alpha U-U_\alpha)\right]\\ =\frac{\partial}{\partial x_\alpha}\left[\left(1+\frac{2U}{c^2}\right)\delta p+\frac{2}{c^2}p\delta U\right]-\frac{\delta\rho}{\rho}\frac{\partial}{\partial x_\alpha}\left[\left(1+\frac{2U}{c^2}\right)p\right]\\ -\rho\frac{\partial}{\partial x_\alpha}\delta U-\frac{2}{c^2}\rho\left(\delta\phi\frac{\partial U}{\partial x_\alpha}+\phi\frac{\partial}{\partial x_\alpha}\delta U+\frac{\partial}{\partial x_\alpha}\delta\Phi\right),\end{aligned} \tag{12}$$

where $\delta\rho$, δp, δU, and $\delta\Phi$ now denote the Eulerian changes in the respective quantities caused by the spatial displacement ξ and U_α and $U_{\mu;\alpha\mu}$ are defined as in equations (7) and (8) but with $\xi_\alpha(x')$ in place of $v_\alpha(x')$. And equation (9) gives

$$\delta\rho^* = -\operatorname{div}(\rho^*\xi). \tag{13}$$

Equations (12) and (13) govern the Eulerian changes in the various quantities caused by the displacement ξ. We shall find, as in the corresponding treatment of the Newtonian problem (cf. Chandrasekhar and Lebovitz 1964; see particularly the Appendix to this paper), that it is more convenient to work with the *Lagrangian changes* than with the Eulerian changes: the boundary conditions are more directly applied to the Lagrangian variables and the treatment, moreover, does not require that ρ vanish on the boundary or, indeed, that it be continuous in the interior.

Let the Lagrangian change in a quantity $F(x)$ be denoted by $\Delta F(x)$ when the corresponding Eulerian change is $\delta F(x)$; the two changes are related by

$$\Delta F(x) = \delta F(x) + \xi_j\frac{\partial F}{\partial x_j}. \tag{14}$$

We, also, have the relation

$$\Delta\left(\frac{\partial F}{\partial x_i}\right)=\frac{\partial\Delta F}{\partial x_i}-\frac{\partial F}{\partial x_k}\frac{\partial\xi_k}{\partial x_i}. \tag{15}$$

Making use of these relations, we readily verify that equations (12) and (13) in the Lagrangian variables become

$$\begin{aligned}\omega^2\left[\sigma\xi_\alpha+\frac{1}{2c^2}\rho(U_\alpha-U_{\mu;\alpha\mu})+\frac{4}{c^2}\rho(\xi_\alpha U-U_\alpha)\right]\\ =\frac{\partial}{\partial x_\alpha}\left[\left(1+\frac{2U}{c^2}\right)\Delta p+\frac{2}{c^2}p\Delta U\right]-\frac{\Delta\rho}{\rho}\frac{\partial}{\partial x_\alpha}\left[\left(1+\frac{2U}{c^2}\right)p\right]\\ -\rho\frac{\partial\Delta U}{\partial x_\alpha}-\frac{2}{c^2}\rho\left(\Delta\phi\frac{\partial U}{\partial x_\alpha}+\phi\frac{\partial\Delta U}{\partial x_\alpha}+\frac{\partial\Delta\Phi}{\partial x_\alpha}\right),\end{aligned} \tag{16}$$

and

$$\Delta\rho^* = -\rho^*\operatorname{div}\xi. \tag{17}$$

It remains to express the Lagrangian changes in the various quantities explicitly in terms of ξ.

First we observe that by combining equations (10) and (17), we have

$$\Delta \rho^* = \left(1 + \frac{3}{c^2} U\right) \Delta \rho + \frac{3}{c^2} \rho \Delta U = -\rho \left(1 + \frac{3}{c^2} U\right) \operatorname{div} \xi . \tag{18}$$

Therefore to $O(1/c^2)$

$$\Delta \rho = -\rho \left(\operatorname{div} \xi + \frac{3}{c^2} \Delta U\right). \tag{19}$$

The corresponding Lagrangian changes in p and Π follow directly from the relations

$$\Delta p = \gamma \frac{p}{\rho} \Delta \rho \qquad \text{and} \qquad \rho \Delta \Pi = \frac{p}{\rho} \Delta \rho . \tag{20}$$

The first of these relations essentially serves as a *definition* of the "*ratio of the specific heats*" γ and the second is an immediate consequence of the exact differential relation

$$d\Pi = \frac{p}{\rho^2} d\rho \tag{21}$$

which obtains in this theory (cf. Paper I, eq. [113]; see also Fock 1964, p. 104). With $\Delta\rho$ given by equation (19), the relations (20) give

$$\Delta p = -\gamma p \left(\operatorname{div} \xi + \frac{3}{c^2} \Delta U\right) \tag{22}$$

and

$$\rho \Delta \Pi = -p \left(\operatorname{div} \xi + \frac{3}{c^2} \Delta U\right). \tag{23}$$

Next from the definition of ϕ (eq. [2]), we have

$$\Delta \phi = \Delta U + \tfrac{1}{2} \Delta \Pi + \tfrac{3}{2} \Delta \left(\frac{p}{\rho}\right); \tag{24}$$

and we find with the aid of the relations (19), (22), and (23) that

$$\Delta \phi = \Delta U - \tfrac{1}{2} \frac{p}{\rho} (3\gamma - 2) \left(\operatorname{div} \xi + \frac{3}{c^2} \Delta U\right). \tag{25}$$

And finally, the expressions for ΔU and $\Delta\Phi$ can be written down from the knowledge that the corresponding Eulerian changes are to be determined as solutions of the equations

$$\nabla^2 \delta U = -4\pi G \delta\rho \qquad \text{and} \qquad \nabla^2 \delta\Phi = -4\pi G \delta(\rho\phi) ; \tag{26}$$

thus

$$\delta U = G \int_V \rho(x') \xi_\alpha(x') \frac{\partial}{\partial x_\alpha'} \frac{1}{|x - x'|} dx' - \frac{3}{c^2} G \int_V \frac{\rho(x') \Delta U(x')}{|x - x'|} dx' \tag{27}$$

and

$$\delta \Phi = G \int_V \rho(x') \phi(x') \xi_\alpha(x') \frac{\partial}{\partial x_\alpha'} \frac{1}{|x - x'|} dx' + G \int_V \frac{\rho(x') \Delta\phi(x')}{|x - x'|} dx' + O(c^{-2}), \tag{28}$$

where it should be noted that in the second term in equation (28) it is the *Lagrangian change* $\Delta\phi$ that occurs. With the Eulerian changes δU and $\delta\Phi$ determined by equations (27) and (28), the corresponding Lagrangian changes follow from the equations

$$\Delta U = \delta U + \xi_j \frac{\partial U}{\partial x_j} = \delta U + \xi_r \frac{dU}{dr}$$

and (29)

$$\Delta\Phi = \delta\Phi + \xi_j \frac{\partial \Phi}{\partial x_j} = \delta\Phi + \xi_r \frac{d\Phi}{dr},$$

where ξ_r is the component of $\boldsymbol{\xi}$ in the radial direction.

With the changes (Lagrangian or Eulerian) experienced by all the quantities now expressed in terms of the displacement, equations (12) and (16) become explicitly equations governing $\boldsymbol{\xi}$.

III. THE VARIATIONAL PRINCIPLE

We have seen in § II how equation (12) (or [16]), when supplemented by the expressions for the various changes (Eulerian or Lagrangian) in terms of ξ, becomes explicitly an equation for ξ. Solutions of the equation ([12] or [16]) must be sought which satisfy the boundary conditions that

$$\Delta p = 0 \text{ on the boundary of the configuration at } r = R \tag{30}$$

and that none of the quantities has a singularity at the origin. Equation (12) (or [16]) together with the boundary conditions at $r = R$ and at $r = 0$ constitute a characteristic value problem for ω^2. It is shown in the Appendix that the problem is a self-adjoint one. A variational base for determining ω^2 is therefore obtained by multiplying *either* equation by ξ_α, contracting, and integrating over the volume of the configuration. While the final expressions one obtains by the procedure described, from equations (12) and (16), must be equivalent, we shall work with the latter equation in the Lagrangian variables as it has the advantages that we have stated earlier.

Multiplying, then, equation (16) by ξ_α, contracting, and integrating over the volume, we readily find that the terms on the left-hand side of the equation give

$$\begin{aligned}\omega^2 \Big\{ \int_V \sigma |\boldsymbol{\xi}|^2 dx + \frac{1}{c^2}\Big[\tfrac{1}{2}G \int_V\int_V \rho(x)\rho(x') \frac{\boldsymbol{\xi}(x)\cdot\boldsymbol{\xi}(x')}{|x-x'|}\,dx\,dx' \\ - \tfrac{1}{2}G\int_V\int_V \rho(x)\rho(x') \frac{[\boldsymbol{\xi}(x)\cdot(x-x')][\boldsymbol{\xi}(x')\cdot(x-x')]}{|x-x'|^3}\,dx\,dx' \\ + 2G\int_V\int_V \rho(x)\rho(x') \frac{|\boldsymbol{\xi}(x)-\boldsymbol{\xi}(x')|^2}{|x-x'|}\,dx\,dx'\Big]\Big\} = Q\omega^2 \text{ (say)}.\end{aligned} \tag{31}$$

As defined, Q is a positive definite quantity.

Carrying out the same procedure on the terms on the right-hand side of equation (16), we obtain

$$\begin{aligned}&\int_V \operatorname{div}\boldsymbol{\xi}\Big[\gamma p\Big(1+\frac{2U}{c^2}\Big)\Big(\operatorname{div}\boldsymbol{\xi}+\frac{3}{c^2}\Delta U\Big)-\frac{2}{c^2}p\Delta U\Big]dx \\ &+\int_V\Big(\operatorname{div}\boldsymbol{\xi}+\frac{3}{c^2}\Delta U\Big)\xi_\alpha\frac{\partial}{\partial x_\alpha}\Big[\Big(1+\frac{2U}{c^2}\Big)p\Big]dx-\int_V\rho\,\xi_\alpha\frac{\partial\Delta U}{\partial x_\alpha}dx \\ &-\frac{2}{c^2}\int_V\rho\Big(\Delta\phi\,\xi_\alpha\frac{\partial U}{\partial x_\alpha}+\phi\,\xi_\alpha\frac{\partial\Delta U}{\partial x_\alpha}+\xi_\alpha\frac{\partial\Delta\Phi}{\partial x_\alpha}\Big)dx,\end{aligned} \tag{32}$$

where an integration by parts has been performed with respect to the first term and the expressions (19) and (22) have been substituted for $\Delta\rho$ and Δp. Next, substituting for $\Delta\phi$ (in the last line of [32]) in accordance with equation (25) (but retaining, as clearly sufficient, only the zero-order terms), regrouping the terms, and finally combining with the earlier result (31), we obtain

$$\begin{aligned} Q\omega^2 = & \underset{\text{(I)}}{\int_V \gamma p \left(1+\frac{2U}{c^2}\right)(\operatorname{div} \xi)^2 dx} + \underset{\text{(II)}}{\int_V \operatorname{div} \xi \; \xi_\alpha \frac{\partial}{\partial x_\alpha}\left[\left(1+\frac{2U}{c^2}\right) p\right] dx} \\ & + \underset{\text{(III)}}{\frac{1}{c^2}\int_V (3\gamma-2) p \Delta U \operatorname{div} \xi dx} + \underset{\text{(IV)}}{\frac{1}{c^2}\int_V \Delta U \xi_\alpha \frac{\partial p}{\partial x_\alpha} dx} \\ & + \underset{\text{(V)}}{\frac{1}{c^2}\int_V (3\gamma-2) p \operatorname{div} \xi \; \xi_\alpha \frac{\partial U}{\partial x_\alpha} dx} - \underset{\text{(VI)}}{\int_V \rho \xi_\alpha \frac{\partial \Delta U}{\partial x_\alpha} dx} \\ & - \underset{\text{(VII)}}{\frac{2}{c^2}\int_V \rho \phi \xi_\alpha \frac{\partial \Delta U}{\partial x_\alpha} dx} - \underset{\text{(VIII)}}{\frac{2}{c^2}\int_V \rho \xi_\alpha \frac{\partial \Delta \Phi}{\partial x_\alpha} dx}. \end{aligned} \tag{33}$$

As we have stated, equation (33) can be used for a variational determination of ω^2.

IV. THE FORM OF THE VARIATIONAL PRINCIPLE FOR THE NORMAL MODES BELONGING TO THE DIFFERENT SPHERICAL HARMONICS

We shall now analyze the Lagrangian displacement ξ into normal modes belonging to the different vector spherical harmonics and write (Chandrasekhar 1961, 1964*c;* see also Lebovitz 1965*b* where the substitution is rigorously justified)

$$\xi_r = \frac{\psi(r)}{r^2} Y_l^m(\vartheta,\varphi),$$

$$\xi_\vartheta = \frac{1}{l(l+1)r}\frac{d\chi(r)}{dr}\frac{\partial Y_l^m(\vartheta,\varphi)}{\partial\vartheta}, \tag{34}$$

and

$$\xi_\varphi = \frac{1}{l(l+1)r\sin\vartheta}\frac{d\chi(r)}{dr}\frac{\partial Y_l^m(\vartheta,\varphi)}{\partial\varphi},$$

where $\psi(r)$ and $\chi(r)$ are two radial functions and $Y_l^m(\vartheta, \varphi)$ is a spherical harmonic. And we may note here for future reference that for ξ given by equations (34)

$$x\cdot\xi = r\xi_r = \frac{\psi}{r} Y_l^m(\vartheta,\varphi) \quad \text{and} \quad \operatorname{div}\xi = \frac{1}{r^2}\frac{d}{dr}(\psi-\chi) Y_l^m(\vartheta,\varphi). \tag{35}$$

It is clearly to be expected that, when the displacement ξ belongs to a particular vector spherical harmonic, the Lagrangian and the Eulerian changes in a scalar quantity (such as ρ or U) will be expressible as products of a radial function and a spherical harmonic thus:

$$\Delta F(r,\vartheta,\varphi) = \Delta F(r) Y_l^m(\vartheta,\varphi) \quad \text{and} \quad \delta F(r,\vartheta,\varphi) = \delta F(r) Y_l^m(\vartheta,\varphi), \tag{36}$$

where F stands for a typical scalar. We shall call $\Delta F(r)$ and $\delta F(r)$ *the radial functions defining* the respective changes. Moreover, since any function $F(r)$ describing the unperturbed state is, by assumption, spherically symmetric it follows that

$$\Delta F(r,\vartheta,\varphi) = \delta F(r,\vartheta,\varphi) + \xi_r \frac{dF}{dr}. \tag{37}$$

By the first of the equations (35), the corresponding relation between the radial functions $\Delta F(r)$ and $\delta F(r)$ is

$$\Delta F(r) = \delta F(r) + \frac{\psi}{r^2}\frac{dF}{dr}. \tag{38}$$

This relation holds for any quantity defined in the equilibrium state.

In view of equations (19) and (35), we may now write

$$\Delta\rho(r) = -\rho\left[\frac{1}{r^2}\frac{d}{dr}(\psi-\chi) + \frac{3}{c^2}\Delta U(r)\right]; \tag{39}$$

and we have similar expressions for $\Delta p(r)$ and $\rho\Delta\Pi(r)$. And for future reference, we may note here that equation (25) written in terms of the defining radial functions is

$$\Delta\phi(r) = \Delta U(r) - \tfrac{1}{2}\frac{p}{\rho}(3\gamma-2)\frac{1}{r^2}\frac{d}{dr}(\psi-\chi) + O(c^{-2}). \tag{40}$$

a) The Radial Function Defining $\delta U(r,\vartheta,\varphi)$

The radial function defining $\delta U(r,\vartheta,\varphi)$ is known in the Newtonian approximation (cf. Chandrasekhar and Lebovitz 1964, eqs. [A.14] and [A.15]); we have

$$\delta U(r) = \frac{4\pi G}{2l+1}\left[\frac{J_l(r)}{r^{l+1}} - r^l K_l(r)\right], \tag{41}$$

where

$$J_l(r) = \int_0^r \rho(s)\, s^l\left[l\frac{\psi(s)}{s} + \frac{d\chi(s)}{ds}\right]ds$$

and

$$K_l(r) = \int_r^R \frac{\rho(s)}{s^{l+1}}\left[(l+1)\frac{\psi(s)}{s} - \frac{d\chi(s)}{ds}\right]ds. \tag{42}$$

For the evaluation of the terms III, IV, and VII in equation (33), the knowledge of $\delta U(r)$ provided by equation (41) is sufficient since these terms have already a factor $1/c^2$. But the term VI requires to be evaluated correctly to $O(1/c^2)$; and for this purpose the additional term

$$-\frac{3}{c^2}G\int_V \frac{\rho(x')\Delta U(x')}{|x-x'|}dx', \tag{43}$$

in the expression for δU given in equation (27) should be included. In evaluating this last expression, the Newtonian value for ΔU derived from equation (41) is clearly sufficient. By expanding $|x-x'|^{-1}$ in spherical harmonics, we find that

$$G\int_V \frac{\rho(x')\Delta U(x')}{|x-x'|}dx' = \frac{4\pi G}{2l+1}Y_l^m(\vartheta,\varphi)\left[\frac{Q_l(r)}{r^{l+1}} + r^l P_l(r)\right], \tag{44}$$

where

$$Q_l(r) = \int_0^r \rho(s)\Delta U(s)\, s^{l+2}ds \quad \text{and} \quad P_l(r) = \int_r^R \rho(s)\Delta U(s)\frac{ds}{s^{l-1}}. \tag{45}$$

Therefore, to $O(1/c^2)$

$$\delta U(r)=\frac{4\pi G}{2l+1}\left\{\frac{J_l(r)}{r^{l+1}}-r^l K_l(r)-\frac{3}{c^2}\left[\frac{Q_l(r)}{r^{l+1}}+r^l P_l(r)\right]\right\}. \tag{46}$$

The following relations which readily follow from the definitions of the functions J_l, K_l, Q_l, and P_l are essential in the subsequent reductions:

$$\frac{d}{dr}\left(\frac{J_l}{r^{l+1}}-r^l K_l\right)=-\frac{l+1}{r^{l+2}}J_l-l r^{l-1}K_l+(2l+1)\frac{\rho\psi}{r^2}$$

and

$$\frac{d}{dr}\left(\frac{Q_l}{r^{l+1}}+r^l P_l\right)=-\frac{l+1}{r^{l+2}}Q_l+l r^{l-1}P_l. \tag{47}$$

b) The Radial Function Defining $\delta\Phi(r,\vartheta,\varphi)$

The evaluation of $\delta\Phi$ in accordance with equation (28) proceeds along very similar lines. We find (cf. eqs. [41], [42], [44], and [45])

$$\delta\Phi(r)=\frac{4\pi G}{2l+1}\left[\frac{\mathfrak{J}_l(r)}{r^{l+1}}-r^l\mathfrak{K}_l(r)+\frac{\mathfrak{F}_l(r)}{r^{l+1}}+r^l\mathfrak{G}_l(r)\right], \tag{48}$$

where

$$\mathfrak{J}_l(r)=\int_0^r \rho(s)\phi(s)s^l\left[l\frac{\psi(s)}{s}+\frac{d\chi(s)}{ds}\right]ds,$$

$$\mathfrak{K}_l(r)=\int_r^R \frac{\rho(s)\phi(s)}{s^{l+1}}\left[(l+1)\frac{\psi(s)}{s}-\frac{d\chi(s)}{ds}\right]ds, \tag{49}$$

$$\mathfrak{F}_l(r)=\int_0^r \rho(s)\Delta\phi(s)s^{l+2}ds \quad\text{and}\quad \mathfrak{G}_l(r)\int_r^R \rho(s)\Delta\phi(s)\frac{ds}{s^{l-1}}. \tag{50}$$

Among these newly defined functions (and J_l and K_l already defined) there exist the relations

$$\frac{d\mathfrak{J}_l}{dr}=\phi\frac{dJ_l}{dr},\qquad \frac{d\mathfrak{K}_l}{dr}=\phi\frac{dK_l}{dr},$$

$$\frac{d}{dr}\left(\frac{\mathfrak{J}_l}{r^{l+1}}-r^l\mathfrak{K}_l\right)=-\frac{l+1}{r^{l+2}}\mathfrak{J}_l-l r^{l-1}\mathfrak{K}_l(r)+(2l+1)\frac{\rho\phi\psi}{r^2}, \tag{51}$$

and

$$\frac{d}{dr}\left(\frac{\mathfrak{F}_l}{r^{l+1}}+r^l\mathfrak{G}_l\right)=-\frac{l+1}{r^{l+2}}\mathfrak{F}_l+l r^{l-1}\mathfrak{G}_l.$$

Since $\delta\Phi$ occurs in a post-Newtonian term, it will clearly suffice to use for $\Delta\phi$ in equations (50) its Newtonian value (40).

c) The Explicit Form of the Variational Principle

We shall now return to equation (33) and consider the form it takes when the displacement ξ belongs to a particular vector spherical harmonic. When the expressions for the various quantities (such as div ξ, ξ_r, ΔU, etc.) appropriate to the chosen displacement are inserted into the different integrands, the integrations over the angles are immediate: in all cases, the result will simply be the normalization integral of the spherical harmonic $Y_l^m(\vartheta,\varphi)$, namely,

$$N_{lm}=\frac{4\pi}{2l+1}\frac{(l+|m|)!}{(l-|m|)!}; \tag{52}$$

and we shall be left with integrals over r in which the relevant changes are replaced by their defining radial functions.

Suppressing the common factor N_{lm} which will occur with every term, we shall enumerate the radial integrals which remain after the angle integrations have been performed. The results for the first five terms on the right-hand side of equation (33) can be written down simply by inspection. Thus

$$\text{I:} \quad \int_0^R \gamma p\left(1+\frac{2U}{c^2}\right)\left[\frac{d}{dr}(\psi-\chi)\right]^2 \frac{dr}{r^2}, \tag{53}$$

$$\text{II:} \quad \int_0^R \frac{d}{dr}\left[\left(1+\frac{2U}{c^2}\right)p\right]\psi\frac{d}{dr}(\psi-\chi)\frac{dr}{r^2}, \tag{54}$$

$$\text{III:} \quad \frac{1}{c^2}\int_0^R (3\gamma-2)p\Delta U(r)\frac{d}{dr}(\psi-\chi)dr, \tag{55}$$

$$\text{IV:} \quad \frac{1}{c^2}\int_0^R \Delta U(r)\frac{dp}{dr}\psi dr, \tag{56}$$

$$\text{V:} \quad \frac{1}{c^2}\int_0^R (3\gamma-2)p\frac{dU}{dr}\psi\frac{d}{dr}(\psi-\chi)\frac{dr}{r^2}. \tag{57}$$

The evaluation of the remaining terms VI, VII, and VIII is less straightforward. The procedure, however, is the same as that set out in detail in an earlier paper (Chandrasekhar and Lebovitz 1964, pp. 1526–1527) with respect to the sixth term in the Newtonian approximation. The essential "point" to note about the reductions is the use of the relations (47) and (51) at the "right" places.

The sixth term in equation (33) must be evaluated now to $O(1/c^2)$ for which purpose the full expression for $\delta U(r)$ given in equation (46) must be used. However, it is sufficient to evaluate the terms VII and VIII in the Newtonian approximation. We find

$$\text{VI:} \quad \int_0^R \rho\frac{dU}{dr}\left[4\frac{\psi^2}{r^3}-\frac{\psi}{r^2}\frac{d}{dr}(\psi+\chi)\right]dr-\frac{4\pi G}{2l+1}\int_0^R\left(J_l\frac{dK_l}{dr}-K_l\frac{dJ_l}{dr}\right)dr + \frac{3}{c^2}\int_0^R \rho\Delta U(r)\delta U(r)r^2 dr; \tag{58}$$

$$\text{VII:} \quad \frac{2}{c^2}\int_0^R \rho\phi\frac{dU}{dr}\left[4\frac{\psi^2}{r^3}-\frac{\psi}{r^2}\frac{d}{dr}(\psi+\chi)\right]dr - \frac{8\pi G}{(2l+1)c^2}\int_0^R \phi\left(J_l\frac{dK_l}{dr}-K_l\frac{dJ_l}{dr}\right)dr; \tag{59}$$

$$\text{VIII:} \quad \frac{2}{c^2}\int_0^R \rho\frac{d\Phi}{dr}\left[4\frac{\psi^2}{r^3}-\frac{\psi}{r^2}\frac{d}{dr}(\psi+\chi)\right]dr - \frac{8\pi G}{(2l+1)c^2}\int_0^R \phi\left(J_l\frac{dK_l}{dr}-K_l\frac{dJ_l}{dr}\right)dr - \frac{2}{c^2}\int_0^R \rho\Delta\phi(r)\delta U(r)r^2 dr. \tag{60}$$

Combining the foregoing results of the integrations over the angles we find, after some substantial reductions and rearrangements, the surprisingly simple result:

$$
\begin{aligned}
Q\omega^2 = &\int_0^R \gamma p \left(1+\frac{2U}{c^2}\right)\left[\frac{d}{dr}(\psi-\chi)\right]^2 \frac{dr}{r^2} \\
&+2\int_0^R \frac{d}{dr}\left[\left(1+\frac{2U}{c^2}\right)p\right]\left(2\frac{\psi^2}{r}-\psi\frac{d\chi}{dr}\right)\frac{dr}{r^2} \\
&-\frac{4\pi G}{2l+1}\int_0^R \left(1+\frac{4}{c^2}\phi\right)\left(J_l\frac{dK_l}{dr}-K_l\frac{dJ_l}{dr}\right)dr \\
&+\frac{1}{c^2}\left\{\int_0^R \rho[\Delta U(r)]^2 r^2 dr + 2\int_0^R (3\gamma-2)p\Delta U(r)\frac{d}{dr}(\psi-\chi)dr\right\}.
\end{aligned}
\tag{61}
$$

Since $\Delta U(r)$ occurs in equation (61) only in the post-Newtonian terms, it will clearly suffice to use its Newtonian value

$$
\Delta U(r) = \frac{4\pi G}{2l+1}\left[\frac{J_l(r)}{r^{l+1}} - r^l K_l(r)\right] + \frac{\psi}{r^2}\frac{dU}{dr}, \tag{62}
$$

where $J_l(r)$ and $K_l(r)$ are defined in equations (42).

The quantity Q, that occurs as a factor of ω^2 in equation (61), is defined in equation (31) (apart from a factor N_{lm} which is presumed to have been suppressed). We shall not consider its reduction in this paper since its precise value is not needed for our present purposes of determining the conditions for marginal stability.

And finally, it may be noted that if the terms in $1/c^2$ are suppressed in equation (61), we recover the equation which obtains in the corresponding Newtonian theory (cf. Chandrasekhar and Lebovitz 1964, eq. [A.18]).

V. THE POST-NEWTONIAN CONDITION FOR THE ONSET OF DYNAMICAL INSTABILITY

We shall first consider the case of radial oscillations and show how equation (61) enables us to obtain a very simple condition for the onset of dynamical instability in the post-Newtonian approximation. As we shall see presently, the evaluation of this condition, for the particular case $\gamma=$ constant, does not require any knowledge of the equilibrium configuration that cannot be derived entirely in the Newtonian framework.

First, we observe that when we are dealing with radial oscillations

$$
l=0 \quad \text{and} \quad \chi=0\,. \tag{63}
$$

From equations (42) it now follows that

$$
J_0=0 \quad \text{and} \quad K_0=\int_r^R \rho(s)\frac{\psi(s)}{s^2}ds\,. \tag{64}
$$

Accordingly, equation (61) reduces in this case to

$$
\begin{aligned}
Q\omega^2 = &\int_0^R \gamma p\left(1+\frac{2U}{c^2}\right)\left(\frac{d\psi}{dr}\right)^2\frac{dr}{r^2} + 4\int_0^R \frac{\psi^2}{r^3}\frac{d}{dr}\left[\left(1+\frac{2U}{c^2}\right)p\right]dr \\
&+\frac{1}{c^2}\left\{\int_0^R \rho[\Delta U(r)]^2 r^2 dr + 2\int_0^R (3\gamma-2)p\Delta U(r)\frac{d\psi}{dr}dr\right\},
\end{aligned}
\tag{65}
$$

where (cf. eq. [62])

$$
\Delta U(r) = -4\pi G\int_r^R \rho(s)\psi(s)\frac{ds}{s^2} + \frac{\psi}{r^2}\frac{dU}{dr}. \tag{66}
$$

With the further substitutions

$$\psi = r^3\eta \quad \text{and} \quad \xi_r = r\eta\,, \tag{67}$$

equation (65) can be reduced to the form

$$\begin{aligned} Q\omega^2 = \int_0^R p\left(1+\frac{2U}{c^2}\right)\left[\gamma r^4\left(\frac{d\eta}{dr}\right)^2 + (3\gamma-4)\frac{d}{dr}(r^3\eta^2)\right]dr \\ +\frac{1}{c^2}\left\{\int_0^R \rho[\Delta U(r)]^2 r^2 dr + 2\int_0^R (3\gamma-2)p\Delta U(r)\frac{d}{dr}(r^3\eta)dr\right\}, \end{aligned} \tag{68}$$

where it might be recalled that p and ρ are the distributions of pressure and density in the equilibrium configuration in *the post-Newtonian approximation* (and, therefore, include terms of order $1/c^2$).

The condition for marginal stability follows from equation (68) by setting $\omega^2 = 0$. In the particular case γ = constant[1] (and it will appear that *only in this case*), the criterion for marginal stability can be deduced from information about the equilibrium configuration obtained solely in the Newtonian approximation. The reason is that when γ = constant, the corresponding criterion in the Newtonian approximation is $\gamma = \frac{4}{3}$ and, moreover, the proper solution for η is a constant. Accordingly, in the post-Newtonian approximation, we should have

$$\gamma - \tfrac{4}{3} = O(c^{-2}) \quad \text{and} \quad \eta = \text{constant} + O(c^{-2})\,. \tag{69}$$

Under these circumstances, equation (68) gives for marginal stability the condition

$$\begin{aligned} (3\gamma-4)\int_0^R p\left(1+\frac{2U}{c^2}\right)\frac{d}{dr}(r^3\eta^2)dr \\ = -\frac{1}{c^2}\left\{\int_0^R \rho[\Delta U(r)]^2 r^2 dr + 12\int_0^R p\Delta U(r) r^2 dr\right\}, \end{aligned} \tag{70}$$

where, now,

$$\Delta U(r) = -4\pi G\int_r^R \rho r\, dr + r\frac{dU}{dr}\,. \tag{71}$$

[Note that the term $(d\eta/dr)^2$ in equation (68) makes no contribution in the present approximation since it is of order c^{-4}.] It is now apparent that in the integral on the right-hand side of equation (70), we may ignore the post-Newtonian corrections to p, neglect the term $2U/c^2$, and set η = constant. We thus obtain the condition

$$9(\gamma-\tfrac{4}{3})\int_0^R p r^2 dr = -\frac{1}{c^2}\left\{\int_0^R \rho[\Delta U(r)]^2 r^2 dr + 12\int_0^R p\Delta U(r) r^2 dr\right\}. \tag{72}$$

Recalling the formula

$$\mathfrak{W} = -12\pi\int_0^R p r^2 dr\,, \tag{73}$$

for the gravitational potential energy of a configuration in hydrostatic equilibrium, we can rewrite the condition (74) in the form

$$\gamma - \tfrac{4}{3} = \frac{1}{3c^2\mathfrak{W}}\left\{\int_0^R [\Delta U(r)]^2 dM(r) + 12\int_0^R \frac{p}{\rho}\Delta U(r)\, dM(r)\right\}; \tag{74}$$

[1] It should perhaps be stated here, explicitly, that no assumption concerning γ (its constancy or otherwise) has been made during the entire development of this theory.

and this criterion for the onset of dynamical instability involves no knowledge of the equilibrium configuration beyond the Newtonian framework. However, the reduction of the general criterion to information obtained in the Newtonian approximation has been possible in the case γ = constant for very special reasons; it is not to be expected that when γ is variable a similar reduction will be possible (cf. Fowler 1964 where one may obtain the impression that the reduction is quite generally possible).

a) The Evaluation of the Criterion (74) for Polytropes

The criterion (74) for the onset of dynamical instability can be obtained in an explicit form for the polytropes. Thus expressing all the quantities (r, ρ, p, and U) in units appropriate for reduction to the standard Emden variables, ξ and θ, we find that equation (74) becomes

$$\gamma - \tfrac{4}{3} = -\frac{2GM}{Rc^2}\frac{(5-n)}{18(n+1)\xi_1^4|\theta_1'|^3}\left\{(n+1)\int_0^{\xi_1}\theta^n[\Delta U(\xi)]^2\xi^2 d\xi + 12\int_0^{\xi_1}\theta^{n+1}\Delta U(\xi)\xi^2 d\xi\right\}, \tag{75}$$

TABLE 1*

VALUES OF THE CONSTANT K

n	K	n	K
0 .	0 452381 (=19/42)	2 5	0 900302
1 0	565382 (5654)	3 0.	1 12447 (1 1245)
1 5	645063	3 25	1 28503
2 0	0 751296 (7513)	3 5	1 49954

* The values in parentheses are those derived in Chandrasekhar (1964*b*).

where n is the polytropic index, ξ_1 is the first zero of the Lane-Emden function θ_n, and θ_1' is the value of the derivative of θ_n at ξ_1; also, M is the mass and R is the radius of the configuration. For the case under consideration (in the non-dimensional units used)

$$\Delta U(\xi) = -\int_\xi^{\xi_1}\theta^n \xi d\xi + \xi\frac{d\theta}{d\xi} = -(\theta + \xi_1|\theta_1'|). \tag{76}$$

Inserting this expression for $\Delta U(\xi)$ in equation (74), we find after some reductions that the condition for marginal stability can be written in the form (cf. Chandrasekhar 1964*b*, eq. [87])

$$R = \frac{K}{\gamma - \frac{4}{3}}\frac{2GM}{c^2},$$

where

$$K = \frac{5-n}{18}\left[\frac{2(11-n)}{(n+1)\xi_1^4|\theta_1'|^3}\int_0^{\xi_1}\theta\left(\frac{d\theta}{d\xi}\right)^2\xi^2 d\xi + 1\right]. \tag{77}$$

The values of the constant K evaluated with the aid of the foregoing formula, for different values of n, are listed in Table 1.

In the earlier paper (Chandrasekhar 1964*b*, § VIII*a*) the constant K was evaluated on a specific model of the relativistic polytropes due to Tooper (1964); and it appeared that the value of K depended explicitly on the function (ϕ in the notation of that paper) that describes the post-Newtonian correction to the non-relativistic Lane-Emden function. The present theory clearly shows that there can be no such dependence; and the close agreement (see Table 1) of the values of K derived in that paper (for $n = 1$, 2, and 3) with those now obtained with the aid of equation (77) confirms that this is indeed the case!

VI. THE CONDITION FOR THE OCCURRENCE OF A NEUTRAL MODE OF OSCILLATION FOR $l \geq 1$: THE POST-NEWTONIAN FORM OF SCHWARZSCHILD'S CRITERION

We shall now obtain the condition for the occurrence of a neutral mode of non-radial oscillation belonging to $l \geq 1$. The method we shall use will closely parallel the one recently devised by Lebovitz (1965*b*) in his examination of the same problem in the Newtonian framework.

Setting $\omega^2 = 0$ in equation (16), we obtain

$$\text{grad}\, \Delta P - \frac{\Delta\rho}{\rho}\, \text{grad}\, P - \rho\, \text{grad}\, \Delta U - \frac{2}{c^2}\, \rho\, (\Delta\phi\, \text{grad}\, U + \phi\, \text{grad}\, \Delta U + \text{grad}\, \Delta\Phi) = 0\,, \tag{78}$$

where for brevity we have written

$$P = \left(1 + \frac{2U}{c^2}\right) p\,. \tag{79}$$

We rewrite equation (78) in the form

$$\text{grad}\left[\Delta P - \rho\Delta U - \frac{2}{c^2}\, \rho\, (\phi\Delta U + \Delta\Phi)\right] = \frac{\Delta\rho}{\rho}\, \text{grad}\, P - \Delta U\, \text{grad}\, \rho + \frac{2}{c^2}[\, \rho\Delta\phi\, \text{grad}\, U - \Delta U\, \text{grad}(\rho\phi) - \Delta\Phi\, \text{grad}\, \rho\,]\,. \tag{80}$$

Since P, ρ, U, and $\rho\phi$ are functions of r only, the vector on the right-hand side of equation (80) has a component only in the radial direction and none in the transverse ϑ- or φ-directions. It follows from this fact that

$$\Delta P - \rho\Delta U - \frac{2}{c^2}\, \rho\, (\phi\Delta U + \Delta\Phi) = F(r) \tag{81}$$

and

$$\frac{\Delta\rho}{\rho}\frac{dP}{dr} - \Delta U\, \frac{d\rho}{dr} + \frac{2}{c^2}\left[\rho\Delta\phi\, \frac{dU}{dr} - \Delta U\, \frac{d(\rho\phi)}{dr} - \Delta\Phi\, \frac{d\rho}{dr}\right] = \frac{dF(r)}{dr}\,, \tag{82}$$

where $F(r)$ is some function of r. On the other hand, we have seen in § IV that, when the Lagrangian displacement is analyzed in vector spherical harmonics, the resulting Lagrangian change in a scalar quantity (such as P, U, or Φ) is expressed as a product of a radial function and a spherical harmonic. Thus, the quantity on the left-hand side of equation (81) has a spherical harmonic as a factor; it cannot, therefore, be a function of r only except in the case $l = 0$. Excluding, then the case $l = 0$, we must have

$$F(r) = 0 \qquad (l \geq 1)\,, \tag{83}$$

and equations (81) and (82) become

$$\Delta P = \rho\left[\Delta U + \frac{2}{c^2}(\phi\Delta U + \Delta\Phi)\right] \tag{84}$$

and

$$\frac{\Delta\rho}{\rho}\frac{dP}{dr}=\frac{d\rho}{dr}\left[\Delta U+\frac{2}{c^2}(\phi\Delta U+\Delta\Phi)\right]+\frac{2}{c^2}\rho\left(\Delta U\frac{d\phi}{dr}-\Delta\phi\frac{dU}{dr}\right). \tag{85}$$

We observe that ρ in equation (84) and $d\rho/dr$ in equation (85) occur with the same factor. Eliminating this factor, we obtain

$$\frac{\Delta\rho}{\rho}\frac{dP}{dr}=\frac{\Delta P}{\rho}\frac{d\rho}{dr}+\frac{2}{c^2}\rho\left(\Delta U\frac{d\phi}{dr}-\Delta\phi\frac{dU}{dr}\right), \tag{86}$$

and restoring to P its meaning, we have

$$\frac{\Delta\rho}{\rho}\left[\left(1+\frac{2U}{c^2}\right)\frac{dp}{dr}+\frac{2}{c^2}p\frac{dU}{dr}\right]=\frac{1}{\rho}\frac{d\rho}{dr}\left[\left(1+\frac{2U}{c^2}\right)\Delta p+\frac{2}{c^2}p\Delta U\right]$$
$$+\frac{2}{c^2}\rho\left(\Delta U\frac{d\phi}{dr}-\Delta\phi\frac{dU}{dr}\right). \tag{87}$$

Dividing equation (87) by $(1+2U/c^2)$, we obtain (correctly to order $1/c^2$)

$$\frac{\Delta\rho}{\rho}\left(\frac{dp}{dr}+\frac{2}{c^2}p\frac{dU}{dr}\right)=\frac{1}{\rho}\frac{d\rho}{dr}\left(\Delta p+\frac{2}{c^2}p\Delta U\right)+\frac{2}{c^2}\rho\left(\Delta U\frac{d\phi}{dr}-\Delta\phi\frac{dU}{dr}\right). \tag{88}$$

Making use of the relation

$$\Delta p=\gamma\frac{p}{\rho}\Delta\rho\,, \tag{89}$$

which underlies the present development, and letting

$$S(r)=\frac{dp}{dr}-\gamma\frac{p}{\rho}\frac{d\rho}{dr} \tag{90}$$

denote the *Schwarzschild discriminant*, we can transform equation (88) to the form

$$\frac{\Delta\rho}{\rho}S(r)=\frac{2}{c^2}\left[\left(\frac{p}{\rho}\frac{d\rho}{dr}+\rho\frac{d\phi}{dr}\right)\Delta U-\left(\frac{p}{\rho}\Delta\rho+\rho\Delta\phi\right)\frac{dU}{dr}\right]. \tag{91}$$

It is clearly sufficient to use the Newtonian values for $\Delta\rho$, $\Delta\phi$, and ΔU on the right-hand side of equation (91). Inserting then the values

$$\Delta\rho=-\rho\operatorname{div}\xi\quad\text{and}\quad\rho\Delta\phi=\rho\Delta U-\tfrac{1}{2}(3\gamma-2)\,p\operatorname{div}\xi\,, \tag{92}$$

we obtain

$$\frac{\Delta\rho}{\rho}S(r)=\frac{2}{c^2}\left[\tfrac{3}{2}\gamma p\frac{dU}{dr}\operatorname{div}\xi+\left(\frac{p}{\rho}\frac{d\rho}{dr}-\rho\frac{dU}{dr}+\rho\frac{d\phi}{dr}\right)\Delta U\right]. \tag{93}$$

Before we can proceed further, it is necessary to write out the term $d\phi/dr$ (in eq. [93]) explicitly in terms of the other variables. For this purpose, it is convenient to define a new ratio Γ by the relation

$$\Pi=\frac{1}{\Gamma-1}\frac{p}{\rho}. \tag{94}$$

The ratio Γ as defined by equation (94) is not generally the same as the "ratio of the specific heats" γ defined by equation (89) and appropriate to conditions when changes take place adiabatically: the thermodynamic definition of the latter ratio is

$$\gamma=\left[\frac{\partial(\log p)}{\partial(\log\rho)}\right]_s, \tag{95}$$

where the subscript s denotes that the derivative in question is with respect to constant entropy (s). For later use, we shall define here the further ratio

$$\Gamma_3=1+\left[\frac{\partial(\log T)}{\partial(\log\rho)}\right]_s. \tag{96}$$[2]

For a mixture of gas and radiation, the expressions for the three ratios γ, Γ_3, and Γ are (cf. Chandrasekhar 1939, eqs. [131] and [141], pp. 57 and 58)

$$\gamma=\beta+\frac{(4-3\beta)^2(\gamma_g-1)}{\beta+12(\gamma_g-1)(1-\beta)},$$

$$\Gamma_3=1+\frac{(4-3\beta)(\gamma_g-1)}{\beta+3(\gamma_g-1)(1-\beta)}, \tag{97}$$

and

$$\Gamma=1+\frac{\gamma_g-1}{\beta+3(\gamma_g-1)(1-\beta)},$$

where γ_g is the ratio of the specific heats of the gas as conventionally defined and β is the ratio of the gas pressure to the total pressure.

After this digression on the representation of Π, we return to equation (93) and first note that

$$\phi=U+\tfrac{1}{2}\Pi+\tfrac{3}{2}\frac{p}{\rho}=U+\frac{3\Gamma-2}{2(\Gamma-1)}\frac{p}{\rho}. \tag{98}$$

A differentiation of this equation now gives

$$\frac{d\phi}{dr}=\frac{5\Gamma-4}{2(\Gamma-1)}\frac{dU}{dr}-\frac{3\Gamma-2}{2(\Gamma-1)}\frac{p}{\rho^2}\frac{d\rho}{dr}-\frac{p}{2\rho(\Gamma-1)^2}\frac{d\Gamma}{dr}. \tag{99}$$

Substituting this expression for $d\phi/dr$ in equation (93), we find

$$\frac{\Delta\rho}{\rho}S(r)=\frac{1}{c^2}\left\{3\gamma p\frac{dU}{dr}\operatorname{div}\xi+\frac{\Delta U}{\Gamma-1}\left[(3\Gamma-2)\rho\frac{dU}{dr}-\Gamma\frac{p}{\rho}\frac{d\rho}{dr}-\frac{p}{\Gamma-1}\frac{d\Gamma}{dr}\right]\right\}. \tag{100}$$

Now consider the neutral modes belonging to a particular vector spherical harmonic. Then equation (100), in terms of the defining radial functions, becomes

$$-\left[\frac{1}{r^2}\frac{d}{dr}(\psi-\chi)+\frac{3}{c^2}\Delta U(r)\right]S(r)=\frac{1}{c^2}\left\{\frac{3\gamma p}{r^2}\frac{dU}{dr}\frac{d}{dr}(\psi-\chi)+\frac{\Delta U(r)}{\Gamma-1}\left[(3\Gamma-2)\rho\frac{dU}{dr}-\Gamma\frac{p}{\rho}\frac{d\rho}{dr}-\frac{p}{\Gamma-1}\frac{d\Gamma}{dr}\right]\right\}. \tag{101}$$

[2] The ratios γ and Γ_3 defined here have the same meanings as the adiabatic coefficients Γ_1 and Γ_3 defined in Chandrasekhar (1939, eqs. [123] and [125], p. 56) for the particular case of a mixture of gas and radiation pressure.

It is known from Lebovitz's discussion of this same problem in the Newtonian framework that a necessary and sufficient condition for the occurrence of a neutral mode of oscillation is that $S(r)$ vanishes over some finite interval of r; and, further, that when this happens, a proper solution for the displacement is obtained by choosing $\psi(r)$ arbitarily in the interval in which $S(r)$ vanishes (and zero outside this interval) and determining χ by the relation

$$\frac{\psi}{\rho}\frac{d\rho}{dr}=-\frac{d}{dr}(\psi-\chi). \tag{102}$$

In ascertaining the criterion for the occurrence of a neutral mode that is implied by equation (101), we shall continue to make the same assumptions concerning ψ and χ, namely, that ψ is chosen arbitrarily (in the first instance) and that χ is determined, in terms of ψ, by means of equation (102), or its equivalent

$$\rho\frac{d\chi}{dr}=\frac{d}{dr}(\rho\psi). \tag{103}$$

The particular feature which distinguishes this manner of relating ψ and χ is that $\delta\rho$, in the Newtonian limit, vanishes: for the radial function $\delta\rho(r)$ defining the Eulerian change in ρ is given by (cf. eq. [39])

$$\delta\rho(r)=-\rho\left[\frac{1}{r^2}\frac{d}{dr}(\psi-\chi)+\frac{\psi}{r^2\rho}\frac{d\rho}{dr}+\frac{3}{c^2}\Delta U(r)\right] \tag{104}$$

and when the relation (102) obtains,

$$\delta\rho(r)=-\frac{3}{c^2}\rho\Delta U(r). \tag{105}$$

Since $\delta\rho$ vanishes in the Newtonian limit, δU must also vanish in the same limit (see eq. [115] below). Hence when ψ and χ are related in the chosen manner,

$$\Delta U(r)=\frac{\psi}{r^2}\frac{dU}{dr}+O(c^{-2}), \tag{106}$$

and

$$\Delta\rho(r)=\frac{\psi}{r^2}\left(\frac{d\rho}{dr}-\frac{3}{c^2}\rho\frac{dU}{dr}\right). \tag{107}$$

Inserting the foregoing relations in equation (101), we find after some rearranging

$$\frac{\psi}{r^2}\left(\frac{1}{\rho}\frac{d\rho}{dr}-\frac{3}{c^2}\frac{dU}{dr}\right)S(r)=\frac{1}{c^2}\frac{\psi}{r^2}\frac{dU}{dr}\left[\frac{3\Gamma-2}{\Gamma-1}S(r)\right.$$
$$\left.+\frac{\gamma-\Gamma}{\Gamma-1}\frac{p}{\rho}\frac{d\rho}{dr}-\frac{p}{(\Gamma-1)^2}\frac{d\Gamma}{dr}\right]. \tag{108}$$

We now rewrite equation (108) in the form

$$\frac{\psi}{r^2}\left\{\left(\frac{1}{\rho}\frac{d\rho}{dr}-\frac{1}{c^2}\frac{6\Gamma-5}{\Gamma-1}\frac{dU}{dr}\right)S(r)\right.$$
$$\left.+\frac{1}{c^2}\frac{p}{\rho}\frac{d\rho}{dr}\frac{dU}{dr}\left[\frac{\Gamma-\gamma}{\Gamma-1}+\frac{1}{(\Gamma-1)^2}\frac{d\Gamma/dr}{d(\log\rho)/dr}\right]\right\}=0. \tag{109}$$

From equation (109) it now follows that *to $O(c^{-2})$, a sufficient condition for the occurrence of a neutral mode of non-radial oscillation is that*

$$S(r) = -\frac{1}{c^2}\frac{p}{\Gamma-1}\frac{dU}{dr}\left(\Gamma-\gamma+\frac{1}{\Gamma-1}\frac{d\Gamma/dr}{d\rho/\rho dr}\right) \tag{110}$$

over some finite interval of r: for $\psi(r)$ can be chosen arbitrarily over the interval in which equation (110) holds (and zero outside this interval) and $\chi(r)$ can be determined, in terms of ψ, with the aid of equation (103). By using equation (94), we can rewrite the condition (110) in the form

$$S(r) = -\frac{\Pi}{c^2}\frac{dp}{dr}\left(\Gamma-\gamma+\frac{1}{\Gamma-1}\frac{d\Gamma/dr}{d\rho/\rho dr}\right). \tag{111}$$

The classical criterion of Schwarzschild, which requires $S(r)$ to vanish over some finite interval of r to insure a marginal state with respect to the onset of convective instability, is replaced by the requirement (111) in the post-Newtonian approximation of Einstein's field equations. However, it will appear (see § VIII below) that the physical content of the two criteria is the same.

VII. THE DERIVATION OF THE POST-NEWTONIAN CRITERION FOR CONVECTIVE INSTABILITY FROM THE VARIATIONAL PRINCIPLE

It is instructive to derive the condition (111) from the variational principle: it will enable us to establish at the same time that *if*

$$S(r) < -\frac{\Pi}{c^2}\frac{dp}{dr}\left(\Gamma-\gamma+\frac{1}{\Gamma-1}\frac{d\Gamma/dr}{d\rho/\rho dr}\right)+O(c^{-4}) \tag{112}$$

over some finite interval of r, then there exist unstable modes of non-radial oscillation for every $l \geq 1$.

In evaluating the variational expression for ω^2, we shall continue to suppose that ψ and χ are related to the manner required by equations (102) and (103). But first, we may note some special features of the chosen form of the displacement.

i) Making use of equation (103), we find that

$$\rho r^l\left(l\frac{\psi}{r}+\frac{d\chi}{dr}\right)=\frac{d}{dr}(r^l\rho\psi)$$

and

$$\frac{\rho}{r^{l+1}}\left[(l+1)\frac{\psi}{r}-\frac{d\chi}{dr}\right]=-\frac{d}{dr}\left(\frac{\rho\psi}{r^{l+1}}\right). \tag{113}$$

From equation (42) it now follows that

$$J_l = r^l\rho\psi \quad \text{and} \quad K_l=\frac{\rho\psi}{r^{l+1}}. \tag{114}$$

ii) In view of equations (114),

$$\delta U(r) = \frac{4\pi G}{2l+1}\left(\frac{J_l}{r^{l+1}}-r^l K_l\right)=0 \tag{115}$$

—a fact to which we have already drawn attention in writing equation (106).

iii) We also have the relation

$$J_l\frac{dK_l}{dr}-K_l\frac{dJ_l}{dr}=-(2l+1)\frac{\rho^2\psi^2}{r^2}. \tag{116}$$

Turning now to the evaluation of the different terms on the right-hand side of equation (61), we find that the first term gives (as may be directly verified)

$$\int_0^R \gamma p \left(1+\frac{2U}{c^2}\right)\frac{\psi^2}{\rho^2}\left(\frac{d\rho}{dr}\right)^2\frac{dr}{r^2} = -\int_0^R S(r)\frac{\psi^2}{r^2\rho}\frac{d\rho}{dr}dr + \frac{2}{c^2}\int_0^R \gamma p U\frac{\psi^2}{r^2\rho^2}\left(\frac{d\rho}{dr}\right)^2 dr - \frac{2}{c^2}\int_0^R \frac{d(pU)}{dr}\frac{\psi^2}{r^2\rho}\frac{d\rho}{dr}dr + \int_0^R \frac{d}{dr}\left[\left(1+\frac{2U}{c^2}\right)p\right]\frac{\psi^2}{r^2\rho}\frac{d\rho}{dr}dr. \tag{117}$$

The last term on the right-hand side of equation (117) combines with the second term on the right-hand side of equation (61) to give

$$-\int_0^R \frac{r^2}{\rho}\frac{d}{dr}\left[\left(1+\frac{2U}{c^2}\right)p\right]\frac{d}{dr}\left(\frac{\rho\psi^2}{r^4}\right)dr. \tag{118}$$

By using equation (1) governing equilibrium, we can write, instead,

$$-\int_0^R r^2\left[\frac{dU}{dr}+\frac{2}{c^2}\left(\phi\frac{dU}{dr}+\frac{d\Phi}{dr}\right)\right]\frac{d}{dr}\left(\frac{\rho\psi^2}{r^4}\right)dr. \tag{119}$$

An integration by parts, followed by reductions in which use is made of equations (3), gives

$$-4\pi G\int_0^R\left(1+\frac{4}{c^2}\phi\right)\frac{\rho^2\psi^2}{r^2}dr+\frac{2}{c^2}\int_0^R \rho\frac{\psi^2}{r^2}\frac{d\phi}{dr}\frac{dU}{dr}dr. \tag{120}$$

By equation (116), the first integral in (120) cancels the third term on the right-hand side of equation (61). Thus, altogether we have

$$Q\omega^2 = -\int_0^R S(r)\frac{\psi^2}{r^2\rho}\frac{d\rho}{dr}dr+\frac{2}{c^2}\int_0^R \gamma pU\frac{\psi^2}{r^2\rho^2}\left(\frac{d\rho}{dr}\right)^2 dr - \frac{2}{c^2}\int_0^R\frac{d(pU)}{dr}\frac{\psi^2}{r^2\rho}\frac{d\rho}{dr}dr+\frac{2}{c^2}\int_0^R\rho\frac{\psi^2}{r^2}\frac{d\phi}{dr}\frac{dU}{dr}dr + \frac{1}{c^2}\int_0^R\rho\frac{\psi^2}{r^2}\left(\frac{dU}{dr}\right)^2 dr-\frac{2}{c^2}\int_0^R(3\gamma-2)p\frac{\psi^2}{r^2\rho}\frac{d\rho}{dr}\frac{dU}{dr}dr. \tag{121}$$

On some further simplifications equation (121) becomes

$$Q\omega^2 = -\int_0^R\frac{\psi^2}{r^2}\left\{\frac{1}{\rho}\frac{d\rho}{dr}S(r)+\frac{1}{c^2}\left[2(3\gamma-1)\frac{p}{\rho}\frac{d\rho}{dr}\frac{dU}{dr}+2\frac{U}{\rho}\frac{d\rho}{dr}\frac{dp}{dr} - \rho\left(\frac{dU}{dr}\right)^2-2\gamma\frac{pU}{\rho^2}\left(\frac{d\rho}{dr}\right)^2-2\rho\frac{d\phi}{dr}\frac{dU}{dr}\right]\right\}dr. \tag{122}$$

Now substituting for $d\phi/dr$ in accordance with equation (99), we obtain after some further reductions and rearrangements (cf. eq. [109])

$$Q\omega^2 = -\int_0^R\frac{\psi^2}{r^2}\left\{\left[\frac{1}{\rho}\frac{d\rho}{dr}-\frac{1}{c^2}\left(\frac{6\Gamma-5}{\Gamma-1}\frac{dU}{dr}-2\frac{U}{\rho}\frac{d\rho}{dr}\right)\right]S(r) + \frac{1}{c^2}\frac{p}{\rho}\frac{d\rho}{dr}\frac{dU}{dr}\left[\frac{\Gamma-\gamma}{\Gamma-1}+\frac{1}{(\Gamma-1)^2}\frac{d\Gamma/dr}{d\rho/\rho dr}\right]\right\}dr. \tag{123}$$

From this last equation it follows that *if to* $O(c^{-4})$

$$\mathfrak{S}(r)=S(r)+\frac{\Pi}{c^2}\frac{dp}{dr}\left(\Gamma-\gamma+\frac{1}{\Gamma-1}\frac{d\Gamma/dr}{d\rho/\rho dr}\right) \tag{124}$$

vanishes over some finite interval of r, then $\omega^2 = 0$ *is possible for a non-trivial* ψ*: it need not vanish over the interval in which* $\mathfrak{S}(r)$ *vanishes.* A neutral mode of the chosen form, therefore, exists under the conditions stated. Moreover, by neglecting $S(r)/c^2$ as $O(c^{-4})$ and rewriting equation (123) in the form

$$Q\omega^2=-\int_0^R \frac{\psi^2}{r^2\rho}\frac{d\rho}{dr}\mathfrak{S}(r)\,dr\,, \tag{125}$$

we can infer that *if* $\mathfrak{S}(r) < 0$ *over a finite interval of r, then there exist non-trivial modes of oscillation that belong to* $\omega^2 < 0$ *and lead to instability.*

VIII. THE EQUIVALENCE OF THE NEWTONIAN AND THE POST-NEWTONIAN CONDITIONS FOR CONVECTIVE INSTABILITY

Schwarzschild's criterion for the onset of convective instability by modes of non-radial oscillations is based on the discriminant

$$S(r)=\frac{dp}{dr}-\gamma\frac{p}{\rho}\frac{d\rho}{dr}; \tag{126}$$

in the post-Newtonian theory, the criterion is based, instead, on the discriminant

$$\mathfrak{S}(r)=S(r)+\frac{\Pi}{c^2}\frac{dp}{dr}\left(\Gamma-\gamma+\frac{1}{\Gamma-1}\frac{d\Gamma/dr}{d\rho/\rho dr}\right), \tag{127}$$

where, it may be recalled, Π is the internal energy, γ is the "ratio of the specific heats" as defined by equation (95), and Γ is defined by the relation $\rho\Pi = p/(\Gamma - 1)$.

We shall now show that the conditions derived from the two discriminants (126) and (127) are completely equivalent. To show this equivalence, we first observe that from the definition of Π as the thermodynamic internal energy, it follows that

$$\left[\frac{\partial\Pi}{\partial(1/\rho)}\right]_s=-p\,, \tag{128}$$

where the subscript s indicates that the partial derivative is with respect to constant entropy (s). Inserting for Π in accordance with equation (94), we find that the foregoing relation gives (cf. eq. [95])

$$\Gamma-\gamma+\frac{1}{\Gamma-1}\left[\frac{\partial\Gamma}{\partial(\log\rho)}\right]_s=0\,. \tag{129}$$

On the other hand, by choosing ρ and s as the independent thermodynamic variables, we obtain

$$\begin{aligned}\frac{1}{\Gamma-1}\frac{d\Gamma/dr}{d(\log\rho)/dr}&=\frac{1}{\Gamma-1}\left[\frac{\partial\Gamma}{\partial(\log\rho)}\right]_s+\left[\frac{\partial}{\partial s}\log(\Gamma-1)\right]_\rho\frac{ds/dr}{d(\log\rho)/dr}\\&=\frac{1}{\Gamma-1}\left[\frac{\partial\Gamma}{\partial(\log\rho)}\right]_s+\left[\frac{1}{p}\left(\frac{\partial p}{\partial s}\right)_\rho-\frac{1}{\Pi}\left(\frac{\partial\Pi}{\partial s}\right)_\rho\right]\frac{ds/dr}{d(\log\rho)/dr}\,.\end{aligned} \tag{130}$$

By using one of Maxwell's thermodynamic relations, we have (cf. eq. [96])

$$\left(\frac{\partial p}{\partial s}\right)_\rho=\rho^2\left(\frac{\partial T}{\partial\rho}\right)_s=\rho T\left[\frac{\partial(\log T)}{\partial(\log\rho)}\right]_s=\rho T(\Gamma_3-1)\,. \tag{131}$$

Equation (130) now gives

$$\frac{1}{\Gamma-1}\frac{d\Gamma/dr}{d(\log\rho)/dr}=\frac{1}{\Gamma-1}\left[\frac{\partial\Gamma}{\partial(\log\rho)}\right]_s+\frac{\rho T}{p}(\Gamma_3-\Gamma)\frac{ds/dr}{d(\log\rho)/dr}. \tag{132}$$

By combining equations (127), (129), and (132), we now obtain

$$\mathfrak{S}(r)=S(r)+\frac{\Pi}{c^2}\frac{\rho T}{p}(\Gamma_3-\Gamma)\frac{dp}{dr}\frac{ds/dr}{d(\log\rho)/dr}. \tag{133}$$

But the Schwarzschild discriminant $S(r)$ is directly related to ds/dr: for, with the choice of ρ and s as independent variables (cf. eq. [95])

$$\frac{dp}{dr}=\left(\frac{\partial p}{\partial\rho}\right)_s\frac{d\rho}{dr}+\left(\frac{\partial p}{\partial s}\right)_\rho\frac{ds}{dr}=\gamma\frac{p}{\rho}\frac{d\rho}{dr}+\rho^2\left(\frac{\partial T}{\partial\rho}\right)_s\frac{ds}{dr}, \tag{134}$$

or (cf. eq. [131])

$$S(r)=\rho T(\Gamma_3-1)\frac{ds}{dr}. \tag{135}$$

Eliminating ds/dr from equation (133) with the aid of this last relation, we obtain

$$\mathfrak{S}(r)=S(r)\left[1+\frac{\Pi}{c^2}\frac{\Gamma_3-\Gamma}{\Gamma_3-1}\frac{d(\log p)/dr}{d(\log\rho)/dr}\right]. \tag{136}$$

From this proportionality of the Newtonian and the post-Newtonian discriminants, it follows that *the physical condition for the occurrence of convective instability is unaltered by general relativity in the approximation considered.*

In conclusion, I wish to record my indebtedness to Dr. N. R. Lebovitz for many helpful discussions and to Dr. M. J. Clement for checking the analysis of this paper.

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Note added August 10, 1965. At the International Conference on General Relativity and Gravitation in London, July, 1965, the author, while presenting the post-Newtonian discriminant $\mathfrak{S}(r)$, interpreted it, incorrectly, as implying physical conditions, for the onset of convective instability, different from Schwarzschild's. The correct interpretation described in § VIII was found very soon afterward; and it was also found independently by Dr. J. Bardeen (California Institute of Technology) and by Dr. R. Tooper (Illinois Institute of Technology).

APPENDIX

THE SELF-ADJOINT NATURE OF THE CHARACTERISTIC VALUE PROBLEM FOR ω^2

We shall now show that the characteristic value problem to which the determination of ω^2 was reduced in § II is a self-adjoint one. For this purpose the Eulerian form (12) of the equation governing ξ is slightly more convenient than the Lagrangian form. And in using the Eulerian form of the equation, we shall further assume that the density ρ vanishes on the boundary of the configuration; this assumption has the consequence that δp also vanishes on the original unperturbed boundary.

Let $\omega^{(1)}$ denote a particular characteristic value; and let the proper solutions belonging to it be distinguished by the same superscript. Consider equation (12) belonging to $\omega^{(1)}$ and after multiplication by $\xi_a^{(2)}$, belonging to a different characteristic value $\omega^{(2)}$, and contraction, integrate over the volume V occupied by the fluid. The terms on the left-hand side give

$$
\begin{aligned}
&[\omega^{(1)}]^2 \left\{ \int_V \sigma \xi_\alpha^{(1)} \xi_\alpha^{(2)} dx + \frac{G}{2c^2} \int_V \int_V \rho(x) \rho(x') \frac{\xi_\alpha^{(1)}(x) \xi_\alpha^{(2)}(x')}{|x - x'|} dx dx' \right. \\
&- \frac{G}{2c^2} \int_V \int_V \rho(x) \rho(x') \xi_\alpha^{(1)}(x)(x_\alpha - x_\alpha') \xi_\beta^{(2)}(x')(x_\beta - x_\beta') \frac{dx dx'}{|x - x'|^3} \\
&\left. + \frac{G}{2c^2} \int_V \int_V \rho(x) \rho(x') [\xi_\alpha^{(1)}(x) - \xi_\alpha^{(1)}(x')][\xi_\alpha^{(2)}(x) - \xi_\alpha^{(2)}(x')] \frac{dx dx'}{|x - x'|} \right\} \\
&= [\omega^{(1)}]^2 Q^{(1,2)} ,
\end{aligned} \tag{A.1}
$$

where $Q^{(1,2)}$ is manifestly symmetric in the superscripts 1 and 2. The reductions of the terms on the right-hand side of equation (12) are less straightforward: integrations by parts followed by substitutions for the different Eulerian changes, simplifications, and rearrangements are involved. We find

$$
\begin{aligned}
&\int_V \xi_\alpha^{(2)} \frac{\partial}{\partial x_\alpha} \left[\left(1 + \frac{2U}{c^2}\right) \delta p^{(1)} + \frac{2}{c^2} p \delta U^{(1)} \right] dx \\
&= \int_V \gamma \left(1 + \frac{2U}{c^2}\right) p \operatorname{div} \boldsymbol{\xi}^{(1)} \operatorname{div} \boldsymbol{\xi}^{(2)} dx + \frac{3}{c^2} \int \gamma p \Delta U^{(1)} \operatorname{div} \boldsymbol{\xi}^{(2)} dx \\
&+ \int_V \left(1 + \frac{2U}{c^2}\right) \frac{dp}{dr} \frac{[x \cdot \boldsymbol{\xi}^{(1)}]}{r} \operatorname{div} \boldsymbol{\xi}^{(2)} dx - \frac{2}{c^2} \int_V p \delta U^{(1)} \operatorname{div} \boldsymbol{\xi}^{(2)} dx ;
\end{aligned} \tag{A.2}
$$

$$
\begin{aligned}
&- \int_V \frac{\delta \rho^{(1)}}{\rho} \xi_\alpha^{(2)} \frac{\partial}{\partial x_\alpha} \left[\left(1 + \frac{2U}{c^2}\right) p \right] dx \\
&= \int_V \frac{d\rho}{dr} \frac{d}{dr} \left[\left(1 + \frac{2U}{c^2}\right) p \right] \frac{[x \cdot \boldsymbol{\xi}^{(1)}][x \cdot \boldsymbol{\xi}^{(2)}]}{r^2 \rho} dx \\
&+ \int_V \left(1 + \frac{2U}{c^2}\right) \frac{dp}{dr} \frac{x \cdot \boldsymbol{\xi}^{(2)}}{r} \operatorname{div} \boldsymbol{\xi}^{(1)} dx \\
&+ \frac{2}{c^2} \int_V p \frac{dU}{dr} \frac{x \cdot \boldsymbol{\xi}^{(2)}}{r} \operatorname{div} \boldsymbol{\xi}^{(1)} dx + \frac{3}{c^2} \int_V \rho \Delta U^{(1)} [\boldsymbol{\xi}^{(2)} \cdot \operatorname{grad} U] dx ;
\end{aligned} \tag{A.3}
$$

$$
\begin{aligned}
&- \int_V \rho \xi_\alpha^{(2)} \frac{\partial}{\partial x_\alpha} \delta U^{(1)} dx = -G \int_V \int_V \frac{\delta \rho^{(1)}(x) \delta \rho^{(2)}(x')}{|x - x'|} dx dx' \\
&- \frac{3}{c^2} \int_V \rho \Delta U^{(1)} \Delta U^{(2)} dx + \frac{3}{c^2} \int_V \rho \Delta U^{(2)} [\boldsymbol{\xi}^{(1)} \cdot \operatorname{grad} U] dx ;
\end{aligned} \tag{A.4}
$$

and

$$
\begin{aligned}
&- \frac{2}{c^2} \int_V \rho \xi_\alpha^{(2)} \left[\delta \phi^{(1)} \frac{\partial U}{\partial x_\alpha} + \phi \frac{\partial}{\partial x_\alpha} \delta U^{(1)} + \frac{\partial}{\partial x_\alpha} \delta \Phi^{(1)} \right] dx \\
&= - \frac{2G}{c^2} \int_V \int_V \delta \rho^{(1)}(x) \delta \rho^{(2)}(x') \frac{\phi(x) + \phi(x')}{|x - x'|} dx dx' \\
&- \frac{2}{c^2} \int_V \rho \delta \phi^{(1)} \Delta U^{(2)} dx + \frac{2}{c^2} \int_V \rho [\boldsymbol{\xi}^{(2)} \cdot \operatorname{grad} \phi] d U^{(1)} dx .
\end{aligned} \tag{A.5}
$$

On combining the foregoing results, we find after some further rearrangements and simplifications

$$
\begin{aligned}
[\omega^{(1)}]^2 Q^{(1,2)} = & \int_V \gamma \left(1+\frac{2U}{c^2}\right) p \operatorname{div} \xi^{(1)} \operatorname{div} \xi^{(2)} dx \\
& + \int_V \frac{1}{\rho}\frac{d\rho}{dr}\frac{d}{dr}\left[\left(1+\frac{2U}{c^2}\right)p\right]\frac{[x\cdot\xi^{(1)}][x\cdot\xi^{(2)}]}{r^2} dx \\
& + \int_V \left(1+\frac{2U}{c^2}\right)\frac{dp}{dr}\left[\frac{x\cdot\xi^{(1)}}{r}\operatorname{div}\xi^{(2)}+\frac{x\cdot\xi^{(2)}}{r}\operatorname{div}\xi^{(1)}\right]dx \\
& - G\int_V\int_V \left\{1+\frac{2}{c^2}[\phi(x)+\phi(x')]\right\}\frac{\delta\rho^{(1)}(x)\,\delta\rho^{(2)}(x')}{|x-x'|}dx\,dx' \\
& + \frac{1}{c^2}\int_V \rho\frac{d}{dr}(3U+2\phi)\left[\frac{x\cdot\xi^{(1)}}{r}\Delta U^{(2)}+\frac{x\cdot\xi^{(2)}}{r}\Delta U^{(1)}\right]dx \\
& - \frac{2}{c^2}\int_V \rho\frac{dU}{dr}\left[\frac{x\cdot\xi^{(1)}}{r}\Delta\Pi^{(2)}+\frac{x\cdot\xi^{(2)}}{r}\Delta\Pi^{(1)}\right]dx \\
& - \frac{1}{c^2}\int_V (3\gamma-2)\rho[\Delta U^{(1)}\Delta\Pi^{(2)}+\Delta U^{(2)}\Delta\Pi^{(1)}]dx \\
& - \frac{2}{c^2}\int_V \rho\frac{d\phi}{dr}\frac{dU}{dr}\frac{[x\cdot\xi^{(1)}][x\cdot\xi^{(2)}]}{r^2}dx - \frac{5}{c^2}\int_V \rho\Delta U^{(1)}\Delta U^{(2)}dx\,.
\end{aligned}
\tag{A.6}
$$

The right-hand side of equation (A.6) is manifestly symmetric in the superscripts 1 and 2; accordingly,

$$\{[\omega^{(1)}]^2-[\omega^{(2)}]^2\}Q^{(1,2)}=0\,; \tag{A.7}$$

and we conclude

$$Q^{(1,2)}=0 \qquad [\omega^{(1)}\neq\omega^{(2)}]. \tag{A.8}$$

Equation (A.8) expresses an orthogonality property of the proper solutions belonging to different characteristic values and establishes the self-adjoint nature of the underlying characteristic value problem. And this self-adjoint nature of the problem is sufficient to insure that a variational base for determining ω^2 is obtained by multiplying the original equation (12) by ξ_α, contracting, and integrating over the volume (i.e., by simply suppressing the superscripts 1 and 2 in eq. [A.6]). In the text this procedure was carried out but with the Lagrangian form of the equation.

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