THE POST-NEWTONIAN EFFECTS OF GENERAL RELATIVITY ON THE EQUILIBRIUM OF UNIFORMLY ROTATING BODIES

I. THE MACLAURIN SPHEROIDS AND THE VIRIAL THEOREM

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ABSTRACT

The post-Newtonian effects of general relativity on the equilibrium of uniformly rotating bodies are considered with the aid of a suitably generalized version of the classical tensor virial theorem. An exact relation exhibiting the relativistic effects is obtained; and it shows that, if the figure of equilibrium is approximated by a spheroid, the effect of general relativity is to attribute to the spheroid a larger angular velocity than the Newtonian value.

I. INTRODUCTION

In the preceding paper (Chandrasekhar 1965; this paper will be referred to hereinafter as "Paper I") the equations of hydrodynamics in the post-Newtonian approximation of Einstein's field equations have been derived. Under stationary conditions the equations are (cf. Paper I, eqs. [68] and [117])

$$\frac{\partial}{\partial x_\mu}(\sigma v_\alpha v_\mu)+\frac{\partial}{\partial x_\alpha}\left[\left(1+\frac{2U}{c^2}\right)p\right]-\rho\frac{\partial U}{\partial x_\alpha}+\frac{4}{c^2}\rho v_\mu\frac{\partial U_\mu}{\partial x_\alpha}$$
$$+\frac{4}{c^2}\rho v_\mu\frac{\partial}{\partial x_\mu}(v_\alpha U-U_\alpha)-\frac{2}{c^2}\rho\left(\phi\frac{\partial U}{\partial x_\alpha}+\frac{\partial\Phi}{\partial x_\alpha}\right)=0\,, \tag{1}$$

and

$$\frac{\partial}{\partial x_\mu}(\rho^* v_\mu)=0\,, \tag{2}$$

where

$$\sigma=\rho\left[1+\frac{1}{c^2}\left(v^2+2U+\Pi+\frac{p}{\rho}\right)\right],$$
$$\rho^*=\rho\left[1+\frac{1}{c^2}\left(\tfrac{1}{2}v^2+3U\right)\right], \tag{3}$$
$$\phi=v^2+U+\tfrac{1}{2}\Pi+\tfrac{3}{2}\frac{p}{\rho}\,,$$

and U_α and Φ are defined as solutions of the equations

$$\nabla^2 U_\alpha=-4\pi G\rho v_\alpha \qquad \text{and} \qquad \nabla^2\Phi=-4\pi G\rho\phi\,; \tag{4}$$

also ρ denotes the density, p the pressure, $\rho\Pi$ the internal energy, v_α the components of the velocity, and U the gravitational potential determined in terms of ρ.

In this paper the foregoing equations will be used to ascertain the post-Newtonian effects of general relativity on the equilibrium of uniformly rotating bodies in the special case the energy density $\epsilon(=\rho c^2+\rho\Pi)$ is a constant throughout the configuration. (It might be noted, parenthetically, in this connection that the assumption $\epsilon=$ constant is formally equivalent to the assumption $\rho=$ constant and $\Pi=0$; on the latter assumption ρ must be assigned the meaning of ϵ/c^2.)

It is well known that on the Newtonian theory oblate spheroidal forms are permissible figures of equilibrium of uniformly rotating bodies of constant density: these are the *spheroids of Maclaurin.* It is also known that, in this instance, the exact figure of equilibrium (for an assigned angular velocity of rotation Ω) is determined by the virial equation

$$\Omega^2 = \frac{\mathfrak{W}_{33} - \mathfrak{W}_{11}}{I_{11}}, \tag{5}$$

where $\mathfrak{W}_{ij}$ and I_{ij} denote the potential energy and the moment of inertia tensors. The modifications in the relation (5), introduced by general relativity in the post-Newtonian approximation, will be the principal concern of this paper. In deriving these modifications, the methods and notations of Chandrasekhar and Lebovitz, in their investigations on the various classical sequences of ellipsoidal figures of equilibrium of homogeneous bodies, will be used; and familiarity with those methods and notations[1] will be assumed.

II. THE POST-NEWTONIAN FORM OF THE VIRIAL THEOREM FOR A UNIFORMLY ROTATING BODY

The general form of the virial theorem in the post-Newtonian approximation was derived in Paper I (eq. [137]). However, for our present purposes, it is convenient to derive the equation, *ab initio,* from equation (1). Multiplying, then, equation (1) by x_β and integrating over the volume V occupied by the fluid, we obtain by the same transformations as were used in Paper I (see particularly § VI)

$$\begin{aligned}\int_V \sigma v_\alpha v_\beta dx + \mathfrak{W}_{\alpha\beta} + \frac{4}{c^2}\int_V \rho v_\alpha v_\beta U dx - \frac{4}{c^2}\int_V \rho v_\beta U_\alpha dx \\ - \frac{4}{c^2}\int_V \rho x_\beta \frac{\partial U_\mu}{\partial x_\alpha} v_\mu dx - \frac{2}{c^2}\int_V \rho\phi \mathfrak{B}_{\alpha\beta} dx = -\delta_{\alpha\beta}\int_V \left(1 + \frac{2U}{c^2}\right) p dx ,\end{aligned} \tag{6}$$

where

$$\mathfrak{B}_{\alpha\beta}(x) = G\int_V \rho(x') \frac{(x_\alpha - x_\alpha')(x_\beta - x_\beta')}{|x - x'|^3} dx' \tag{7}$$

is the Newtonian tensor potential.

We shall suppose that the motions in the body are those that correspond to a uniform rotation with an angular velocity Ω about the x_3-direction. Then

$$v_1 = -\Omega x_2, \; v_2 = +\Omega x_1, \; v_3 = 0 , \quad \text{and} \quad v^2 = \Omega^2(x_1^2 + x_2^2) . \tag{8}$$

We shall further suppose that the configuration is axisymmetric about the axis of rotation; in this case the equation of continuity (2) is identically satisfied.

For the motions specified in equation (8),

$$\sigma = \rho + \frac{1}{c^2}\rho\left[\Omega^2(x_1^2 + x_2^2) + 2U + \Pi + \frac{p}{\rho}\right] \tag{9}$$

and

$$\phi = \Omega^2(x_1^2 + x_2^2) + U + \tfrac{1}{2}\Pi + \tfrac{3}{2}\frac{p}{\rho}. \tag{10}$$

Also in the notation of Chandrasekhar and Lebovitz (1962*a*, eq. [1])

$$U_1 = -\Omega\mathfrak{D}_2 \quad \text{and} \quad U_2 = +\Omega\mathfrak{D}_1 . \tag{11}$$

[1] For an abbreviated account see Chandrasekhar (1964).

Under the conditions considered, the non-diagonal components of equation (6) vanish identically. Moreover, the assumed axisymmetry of the configuration about the x_3-axis implies that the (1, 1)- and the (2, 2)-components of equation (6) are not independent. Therefore, it will suffice to consider only the (1, 1)- and the (3, 3)-components; these equations

$$\Omega^2 \int_V \sigma x_2^2 dx + \mathfrak{W}_{11} + \frac{4\Omega^2}{c^2} \int_V \rho \left(x_2^2 U - x_2 \mathfrak{D}_2 - x_1^2 \frac{\partial \mathfrak{D}_1}{\partial x_1} - x_1 x_2 \frac{\partial \mathfrak{D}_2}{\partial x_1} \right) dx - \frac{2}{c^2} \int_V \rho \phi \mathfrak{B}_{11} dx = - \int_V p \left(1 + \frac{2U}{c^2} \right) dx \tag{12}$$

and

$$\mathfrak{W}_{33} - \frac{4\Omega^2}{c^2} \int_V \rho x_3 \left(x_1 \frac{\partial \mathfrak{D}_1}{\partial x_3} + x_2 \frac{\partial \mathfrak{D}_2}{\partial x_3} \right) dx - \frac{2}{c^2} \int_V \rho \phi \mathfrak{B}_{33} dx = - \int_V p \left(1 + \frac{2U}{c^2} \right) dx . \tag{13}$$

From equations (12) and (13), we obtain the relation,

$$\Omega^2 \int_V \sigma x_2^2 dx + \frac{4\Omega^2}{c^2} \int \rho \left(x_2^2 U - x_2 \mathfrak{D}_2 - x_1^2 \frac{\partial \mathfrak{D}_1}{\partial x_1} - x_1 x_2 \frac{\partial \mathfrak{D}_2}{\partial x_1} \right) dx + \frac{4\Omega^2}{c^2} \int_V \rho x_3 \left(x_1 \frac{\partial \mathfrak{D}_1}{\partial x_3} + x_2 \frac{\partial \mathfrak{D}_2}{\partial x_3} \right) dx - \frac{2}{c^2} \int_V \rho \phi (\mathfrak{B}_{11} - \mathfrak{B}_{33}) dx = \mathfrak{W}_{33} - \mathfrak{W}_{11} , \tag{14}$$

that, in the post-Newtonian approximation, replaces equation (5).

III. THE POST-NEWTONIAN TERMS IN THE VIRIAL EQUATION IN THE CASE OF UNIFORM DENSITY

We shall now evaluate the post-Newtonian terms in equation (14) in the case the density ρ is a constant throughout the configuration. In evaluating these terms, we may legitimately use relations which obtain in the Newtonian limit when the equilibrium figure is a spheroid. The relations (valid for an ellipsoid with semi-axes a_1, a_2, and a_3) that we shall need are (see Chandrasekhar and Lebovitz 1962*b*, eqs. [47], [49], and [53])

$$U = I - \sum_{\mu=1}^{3} A_\mu x_\mu^2 \qquad \left(\text{where } I = \sum_{\mu=1}^{3} a_\mu^2 A_\mu \right), \tag{15}$$

$$\mathfrak{B}_{\alpha\alpha} = 2 B_{\alpha\alpha} x_\alpha^2 + \left(A_\alpha - \sum_{\mu=1}^{3} A_{\alpha\mu} x_\mu^2 \right) a_\alpha^2 , \tag{16}$$

and

$$\mathfrak{D}_\alpha = x_\alpha \left(A_\alpha - \sum_{\mu=1}^{3} A_{\alpha\mu} x_\mu^2 \right) a_\alpha^2 \tag{17}$$

(*summation over repeated indices* only *when indicated, here and in the sequel*) where a common factor $\pi G \rho$, in the expressions for the potentials, has been suppressed, and the index symbols A_α, $A_{\alpha\beta}$, etc. (defined in the paper quoted[2]), are so normalized that $\Sigma A_\alpha = 2$ (instead of $2/a_1 a_2 a_3$).

[2] The symbol $B_{\alpha\beta}$ is not defined in that paper; it is, however, simply related to the A-symbols by $B_{\alpha\beta} = A_\alpha - a_\beta^2 A_{\alpha\beta}$ and is, like $A_{\alpha\beta}$, symmetric in α and β (see Chandrasekhar and Lebovitz 1963, eqs. [111]–[114]).

In addition to the relations (15)–(17) we shall need the distribution of the pressure in the equilibrium configuration; this is given by (with the same factor $\pi G\rho$ suppressed)

$$p=\rho\left[A_3a_3^2-\sum_{\mu=1}^{3}A_\mu x_\mu^2+\tfrac{1}{2}\Omega^2(x_1^2+x_2^2)\right], \tag{18}$$

where, moreover,

$$a_1^2(\tfrac{1}{2}\Omega^2-A_1)=a_2^2(\tfrac{1}{2}\Omega^2-A_2)=-a_3^2A_3\,. \tag{19}$$

The suppression of the factor $\pi G\rho$ in equations (15)–(18) implies now that Ω^2 is measured in the unit $\pi G\rho$.

Consistent with our assumption of axisymmetry, we should have set $a_1=a_2$ and $A_1=A_2$ in equations (18) and (19). However, for the purpose of retaining a certain formal symmetry in the equations (profuse with terms) we shall continue to distinguish the subscripts 1 and 2; their equivalence will be assumed at the end of the calculations.

Making use of the solution (17) for $\mathfrak{D}_\alpha$ we find

$$x_2^2U-x_2\mathfrak{D}_2-x_1\left(x_1\frac{\partial\mathfrak{D}_1}{\partial x_1}+x_2\frac{\partial\mathfrak{D}_2}{\partial x_1}\right)=(I-a_2^2A_2)x_2^2$$
$$-a_1^2A_1x_1^2-\sum_{\mu=1}^{3}B_{2\mu}x_2^2x_\mu^2+3a_1^2A_{11}x_1^4+(a_1^2+2a_2^2)A_{12}x_1^2x_2^2+a_1^2A_{13}x_1^2x_3^2 \tag{20}$$

and

$$x_3\left(x_1\frac{\partial\mathfrak{D}_1}{\partial x_3}+x_2\frac{\partial\mathfrak{D}_2}{\partial x_3}\right)=-2a_1^2A_{13}x_1^2x_3^2-2a_2^2A_{23}x_2^2x_3^2\,. \tag{21}$$

Evaluating the integrals of these expressions over the volume of the ellipsoid, we find

$$\int_V\rho\left(x_2^2U-x_2\mathfrak{D}_2-x_1^2\frac{\partial\mathfrak{D}_1}{\partial x_1}-x_1x_2\frac{\partial\mathfrak{D}_2}{\partial x_1}\right)dx$$
$$=\frac{4\pi a_1a_2a_3}{105}\rho\Big[7(I-a_2^2A_2)a_2^2+2a_1^4B_{13}-a_1^2a_2^2B_{12}-3a_2^4B_{22}$$
$$-a_2^2a_3^2B_{23}+2a_1^2a_2^2(a_2^2-a_1^2)A_{12}\Big] \tag{22}$$

and

$$-\int_V\rho x_3\left(x_1\frac{\partial\mathfrak{D}_1}{\partial x_3}+x_2\frac{\partial\mathfrak{D}_2}{\partial x_3}\right)dx=\frac{8\pi a_1a_2a_3}{105}\rho a_3^2(a_1^4A_{13}+a_2^4A_{23})\,. \tag{23}$$

The remaining terms in equations (14) involve ϕ and σ. In accordance with the parenthetical remarks in § I, we shall set $\Pi=0$ (and assign to ρ the meaning of ϵ/c^2); and we obtain with the present solutions for U and p/ρ,

$$\sigma=\rho+\frac{1}{c^2}\rho\left[\tfrac{3}{2}\Omega^2(x_1^2+x_2^2)-3\sum_{\mu=1}^{3}A_\mu x_\mu^2+2I+a_3^2A_3\right] \tag{24}$$

and

$$\phi=\tfrac{7}{4}\Omega^2(x_1^2+x_2^2)-\tfrac{5}{2}\sum_{\mu=1}^{3}A_\mu x_\mu^2+I+\tfrac{3}{2}a_3^2A_3\,. \tag{25}$$

Making use of equation (24), we find

$$\int_V\sigma x_2^2dx=I_{22}+\frac{8\pi a_1a_2a_3}{105\,c^2}\rho a_2^2(7I-4a_3^2A_3), \tag{26}$$

where I_{22} is the (2, 2)-component of the moment of inertia tensor evaluated in terms of ρ. Similarly, we find (after some lengthy reductions) that

$$\int_V \rho\phi\mathfrak{W}_{11}dx = \frac{4\pi a_1a_2a_3}{105}\rho[a_1^2(2B_{11}-a_1^2A_{11})(10a_1^2A_1+8a_2^2A_2+a_3^2A_3) \\ -a_1^2a_2^2A_{12}(8a_1^2A_1+10a_2^2A_2+a_3^2A_3) \\ -a_1^2a_3^2A_{13}(8a_1^2A_1+8a_2^2A_2+3a_3^2A_3) \\ +21a_1^2A_1(2a_1^2A_1+2a_2^2A_2+a_3^2A_3)] \tag{27}$$

and

$$\int_V \rho\phi\mathfrak{W}_{33}dx = \frac{4\pi a_1a_2a_3}{105}\rho[-a_1^2a_3^2A_{31}(10a_1^2A_1+8a_2^2A_2+a_3^2A_3) \\ -a_2^2a_3^2A_{32}(8a_1^2A_1+10a_2^2A_2+a_3^2A_3) \\ +a_3^2(2B_{33}-a_3^2A_{33})(8a_1^2A^1+8a_2^2A_2+3a_3^2A_3) \\ +21a_3^2A_3(2a_1^2A_1+2a_2^2A_2+a_3^2A_3)]\,. \tag{28}$$

Inserting the foregoing results of the various integrations in equation (14), we find

$$\Omega^2 = \frac{\mathfrak{W}_{33}-\mathfrak{W}_{11}}{I_{22}} + \frac{2M}{35I_{22}c^2}(\pi G\rho)^2[a_1^2(3B_{11}-B_{13})(10a_1^2A_1+8a_2^2A_2+a_3^2A_3) \\ +a_2^2(B_{12}-B_{32})(8a_1^2A_1+10a_2^2A_2+a_3^2A_3) \\ -a_3^2(3B_{33}-B_{13})(8a_1^2A_1+8a_2^2A_2+3a_3^2A_3) \\ +21(a_1^2-a_3^2)B_{13}(2a_1^2A_1+2a_2^2A_2+a_3^2A_3)] \\ -\frac{4\Omega^2M}{35I_{22}c^2}(\pi G\rho)[10.5a_2^2I-7a_2^4A_2-2a_2^2a_3^2A_3-2a_3^2(a_1^4A_{13}+a_2^4A_{23}) \\ +2a_1^4B_{13}-a_1^2a_2^2B_{12}-3a_2^4B_{22}-a_2^2a_3^2B_{23}+2a_1^2a_2^2(a_2^2-a_1^2)A_{12}]\,, \tag{29}$$

where the factor $\pi G\rho$ (which had been suppressed) has been restored and M $(=\frac{4}{3}\pi a_1a_2a_3\rho)$ is the mass of the ellipsoid. In accordance with our assumption of axisymmetry, we now set $a_1 = a_2$; and remembering that, then, the value of any index symbol is unaltered if the subscript 2 is replaced by 1 wherever it occurs, we find that equation (29) becomes

$$\Omega^2 = \frac{\mathfrak{W}_{33}-\mathfrak{W}_{11}}{I_{11}} + \frac{2M}{35I_{11}c^2}(\pi G\rho)^2[2a_1^2(2B_{11}-B_{13})(18a_1^2A_1+a_3^2A_3) \\ -a_3^2(3B_{33}-B_{13})(16a_1^2A_1+3a_3^2A_3)+21(a_1^2-a_3^2)B_{13}(4a_1^2A_1+a_3^2A_3)] \\ -\frac{4\Omega^2M}{35I_{11}c^2}(\pi G\rho)a_1^2[14a_1^2A_1+8.5a_3^2A_3+(2a_1^2-a_3^2)B_{13} \\ -4a_1^2B_{11}-4a_1^2a_3^2A_{13}]\,. \tag{30}$$

Equation (30) can be rewritten in the form

$$\frac{\Omega^2}{\pi G\rho} = \frac{\mathfrak{W}_{33}-\mathfrak{W}_{11}}{\pi G\rho I_{11}} + \frac{R_s}{a_1}E(e)\,, \tag{31}$$

where

$$R_s = \frac{2GM}{c^2} \tag{32}$$

is the Schwarzschild limit for the mass M and

$$\begin{aligned} E(e) = \frac{3}{28a_1^4}\frac{a_1}{a_3}\{ & 2a_1^2(2B_{11}-B_{13})(18a_1^2A_1+a_3^2A_3) \\ & -a_3^2(3B_{33}-B_{13})(16a_1^2A_1+3a_3^2A_3)+21(a_1^2-a_3^2)B_{13}(4a_1^2A_1+a_3^2A_3) \\ & -\frac{2\Omega^2}{\pi G\rho}a_1^2[14a_1^2A_1+8.5a_3^2A_3+(2a_1^2-a_3^2)B_{13}-4a_1^2B_{11}-4a_1^2a_3^2A_{13}]\} \end{aligned} \tag{33}$$

is a function only of the eccentricity of the spheroid.

TABLE 1*

THE FUNCTION $E(e)$

e	$(\Omega^2/\pi G\rho)_{Mc}$	$E(e)$	e	$(\Omega^2/\pi G\rho)_{Mc}$	$E(e)$
0 ..	0	0	0 75	0 31947	0 17187
0 20	0 02146	0 00933	80	36316	20258
.25	03363	01475	82	38059	21561
30	04862	02153	84.	39761	22889
35	06647	02979	86	41378	24215
.40	08727	03967	88	42845	25498
45 .	11108	05133	90	44053	.26663
50 .	13799	06497	92	44816	.27572
.55	.16807	08081	94	44785	27959
60	.20135	09913	96	43193	27232
65 .	23783	12022	0 98	0 37802	0 23698
0 70	0 27734	0 14439			

* The entries in the column headed $(\Omega^2/\pi G\rho)_{Mc}$ refer to the Maclaurin spheroid in the Newtonian limit

In Table 1 the function $E(e)$ is tabulated. From this table it follows that, if the figure of the rotating body (in the post-Newtonian theory) is approximated by a spheroid, then the effect of general relativity is to attribute to a spheroid of given eccentricity a value of $\Omega^2/\pi G\rho$ that is larger than the Newtonian value by precisely the amount $R_s E(e)/a_1$.

The derivation of equation (31) does not, of course, solve the problem of the equilibrium fully. But equation (31)—exact in the framework of the present theory—does provide some information on the nature and the magnitude of the effect that is to be expected. A complete solution of the problem will inevitably depend on an explicit solution of the original equation (1); and to this matter we shall address ourselves in a future paper.

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