THE POST-NEWTONIAN EQUATIONS OF HYDRODYNAMICS IN GENERAL RELATIVITY

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ABSTRACT

The standard Eulerian equations of hydrodynamics are generalized to take into account, consistently with Einstein's field equations, all effects of order $1/c^2$. It is further shown that these post-Newtonian equations allow integrals of motion which are entirely analogous to the Newtonian integrals that express the conservation of mass, linear momentum, angular momentum, and energy. The continued validity of these conservation laws enables a consistent definition of "mass," "momentum," and "energy" in the framework of the post-Newtonian theory.

Besides the equations of motion, an appropriate tensor form of the virial theorem is also derived.

I. INTRODUCTION

Recent investigations (Chandrasekhar 1964*a, b;* see also Misner and Zapolsky 1964; Harrison, Thorne, Wakano, and Wheeler 1965; and Tsuruta, Wright, and Cameron 1965) on the dynamical instability of gaseous masses in the framework of the general theory of relativity have shown that the theory predicts, already in the post-Newtonian approximation, phenomena which are qualitatively different from those that are to be expected on the Newtonian theory: gaseous masses are predicted to become unstable for spherically symmetric radial oscillations much before the Schwarzschild limit is reached.

An exact treatment of the radial oscillations of a gaseous mass in general relativity is possible and has been given (Chandrasekhar 1964*b*). A similar exact treatment of non-radial oscillations is not to be expected: apart from the difficulties associated with the solution of Einstein's field equations with no presupposed symmetry, allowance should also have to be made for the emission of gravitational radiation. However, for the purposes of ascertaining the nature of the effects predicted by general relativity in these contexts, it may suffice to examine the relevant problems in a consistent post-Newtonian approximation, i.e., at a level of approximation in which gravitational radiation plays no role.

For a systematic investigation of the post-Newtonian effects of general relativity on the behavior of hydrodynamic systems, it is clearly necessary to have at our disposal the generalization of the standard Eulerian equations of Newtonian hydrodynamics that will consistently allow for all effects of order $1/c^2$ originating in the exact field equations of Einstein. In this paper, such a set of equations will be derived; these equations provide for hydrodynamics what the theory of Einstein, Infeld, and Hoffmann (for an account of this theory see Infeld and Plebanski 1960 or Landau and Lifshitz 1962) provides for the n-body problem of classical dynamics. In the papers following this one (Chandrasekhar 1965*b, c*), the equations derived in this paper are applied to determine the post-Newtonian effects of general relativity on the equilibrium of uniformly rotating homogeneous masses and on the stability of gaseous masses to radial as well as non-radial oscillations. A preliminary account of these investigations has been published as a Letter (Chandrasekhar 1965*a*).

II. THE METHOD OF APPROXIMATION

In general relativity the nature of a physical system is determined by the assumption one makes about the energy-momentum tensor $T_{ij}(i, j = 0, 1, 2,$ and 3; in this paper the convention will be adopted of letting Latin indices take the values 0, 1, 2, and 3 and the

Greek indices take only the values 1, 2, and 3 referring to the spatial coordinates; also the summation over repeated indices will be restricted to their respective ranges). In selecting the form of T_{ij} one is, however, limited by the requirement that it agree, in a local Minkowskian frame, with the choice of the special theory of relativity for the same physical system.

In this paper we shall be concerned with a system which in the Newtonian limit is governed by the standard Eulerian equations of hydrodynamics for an inviscid fluid, namely,

$$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x_\beta}(\rho v_\beta)=0 \tag{1}$$

and

$$\frac{\partial}{\partial t}(\rho v_\alpha)+\frac{\partial}{\partial x_\beta}(\rho v_\alpha v_\beta)=-\frac{\partial p}{\partial x_\alpha}+\rho\frac{\partial U}{\partial x_\alpha}, \tag{2}$$

where $v_\alpha(\alpha = 1, 2, 3)$ are the usual components of the velocity, ρ is the density, p is the pressure, and U is the gravitational potential determined in terms of ρ by Poisson's equation

$$\nabla^2 U = -4\pi G\rho\,. \tag{3}$$

This requirement on the behavior of the system in the Newtonian limit demands that the energy-momentum tensor have the form

$$T_{ij} = (\epsilon + p)u_i u_j - p g_{ij}\,, \tag{4}$$

where ϵ denotes the energy-density, u_i the covariant four-velocity, and g_{ij} the metric tensor. For our present purposes it is convenient to make the further assumption that there are two parts to the energy-density ϵ: a part ρc^2 associated with a *material density* ρ and a part $\rho\,\Pi$ associated with the *internal energy* of the first and the second laws of thermodynamics. If this separation of ϵ into the two parts is permissible, we may write

$$\epsilon = \rho c^2(1 + \Pi/c^2)\,. \tag{5}$$

When ϵ is written in this form, the density ρ may be taken as referring to that part of the rest mass which does not vary during the motion.

A detailed justification for an assumption of ϵ of the form (5) has been given by Fock (1964). However, Fock's justification applies only to situations when the proportion of antimatter that is present (under conditions, say, of thermal equilibrium) is negligible; if this should not be the case, then the assumption that ϵ is of the form (5) will become untenable and it must be replaced by some other equivalent assumption, such as the *conservation of the baryon number* (cf. Chandrasekhar 1964*b*, p. 422). In this paper we shall continue to make the assumption (5); the modifications in the analysis, if (5) should be replaced by some other assumption, are minor and can be made if desired.

With the choice of the form of T_{ij}, the *entire* behavior of the system is determined, in terms of initial conditions, by Einstein's field equation

$$R_{ij} = -\frac{8\pi G}{c^4}\left(T_{ij}-\tfrac{1}{2}T g_{ij}\right), \tag{6}$$

where R_{ij} is the Ricci tensor and

$$T = T^k{}_k \tag{7}$$

is the trace of $T_i{}^j$.

The theory of Einstein, Infeld, and Hoffmann provides a model for the solution of the field equation by a method of successive approximation based on expansions in powers of $1/c^2$. In this paper the same method will be used to derive the post-Newtonian equations of motion for the hydrodynamic system we are presently considering. In outline, the method is the following.

The principle of equivalence (cf. Schiff 1960) allows one to assert that, quite generally,

$$g_{00} = 1 - \frac{2U}{c^2} + O(c^{-4}), \tag{8}$$

$$g_{\alpha\beta} = -\left(1 + \frac{2U}{c^2}\right)\delta_{\alpha\beta} + O(c^{-4}), \quad \text{and} \quad g_{0\alpha} = O(c^{-3}), \tag{9}$$

where U is the gravitational potential determined in terms of ρ by Poisson's equation (3). Indeed, it is the particular form for g_{00} given by equation (8) that determines the "constant of proportionality" in the field equation as $-8\pi G/c^4$; and, moreover, the identity

$$T^{ij}{}_{;j} = 0 \tag{10}$$

(where $;j$ denotes covariant differentiation with respect to x_j) yields, in this "zeroth" approximation, the Eulerian equations (1) and (2) of Newtonian hydrodynamics.

With the knowledge of g_{00} and $g_{\alpha\beta}$ to $O(c^{-2})$ provided by equations (8) and (9), the components of the energy-momentum tensor can be deduced to an accuracy sufficient for the field equation (6) to determine g_{00} to $O(c^{-4})$ and the dominant term of $O(c^{-3})$ in $g_{0\alpha}$. The resulting improved knowledge of the coefficients of the metric tensor enables the Christoffel symbols to be evaluated, in turn, to an accuracy sufficient for the identity (10) to yield now the desired equations of motion in the post-Newtonian approximation.

III. THE SOLUTION OF THE FIELD EQUATIONS IN THE POST-NEWTONIAN APPROXIMATION

We shall write the coefficients of the metric tensor in the form

$$g_{00} = 1 + h_{00}, \quad g_{0\alpha} = h_{0\alpha}, \quad \text{and} \quad g_{\alpha\beta} = -\delta_{\alpha\beta} + h_{\alpha\beta}, \tag{11}$$

where

$$h_{00} = -\frac{2U}{c^2} + O(c^{-4}), \quad h_{0\alpha} = O(c^{-3}), \quad \text{and} \quad h_{\alpha\beta} = -\frac{2U}{c^2}\delta_{\alpha\beta} + O(c^{-4}). \tag{12}$$

We shall raise and lower the indices of h_{ij} by the Minkowskian metric ($g_{00} = 1$, $g_{0\alpha} = 0$, and $g_{\alpha\beta} = -\delta_{\alpha\beta}$) to obtain

$$h_{00} = h_0{}^0 = h^0{}_0 = h^{00} = -\frac{2U}{c^2} + O(c^{-4}),$$

$$h_{0\alpha} = h^0{}_\alpha = -h_0{}^\alpha = -h^{0\alpha} = O(c^{-3}),$$

$$h_{\alpha\beta} = -h^\alpha{}_\beta = -h_\alpha{}^\beta = h^{\alpha\beta} = -\frac{2U}{c^2}\delta_{\alpha\beta} + O(c^{-4}), \tag{13}$$

and

$$h^\alpha{}_\alpha = h_\alpha{}^\alpha = \frac{6U}{c^2}.$$

The components of the contravariant four-velocity

$$u^i = \frac{dx_i}{ds}, \tag{14}$$

to the accuracy determined by the metric (as we know it at this stage) are

$$u^0 = 1 + \frac{1}{c^2}\left(\tfrac{1}{2}v^2 + U\right) + O(c^{-4})$$

and

$$u^{a}=\left[1+\frac{1}{c^{2}}\left(\tfrac{1}{2}v^{2}+U\right)\right]\frac{v_{a}}{c}+O(c^{-5}). \tag{15}$$

The corresponding covariant components are

$$u_{0}=1+\frac{1}{c^{2}}\left(\tfrac{1}{2}v^{2}-U\right)+O(c^{-4})$$

and

$$u_{a}=-\frac{v_{a}}{c}+O(c^{-3}), \tag{16}$$

where it may be noted here that the timelike coordinate x_0 is related to the coordinate time t by $dx_0 = cdt$.

a) The Components of the Energy-Momentum Tensor

Turning to the components of the energy-momentum tensor, we now find

$$T_{00}=\rho c^{2}\left[1+\frac{1}{c^{2}}\left(v^{2}-2U+\Pi\right)\right]+O(c^{-2}),$$

$$T_{0a} = -\rho c v_{a} + O(c^{-1}), \tag{17}$$

and

$$T_{a\beta} = \rho v_{a} v_{\beta} + \delta_{a\beta} p + O(c^{-2}).$$

Remembering that

$$T = T^{k}{}_{k} = \rho c^{2} + \rho\Pi - 3p, \tag{18}$$

we have in particular

$$T_{00}-\tfrac{1}{2}Tg_{00}=\tfrac{1}{2}\rho c^{2}+\rho\left(v^{2}-U+\tfrac{1}{2}\Pi+\tfrac{3}{2}\frac{p}{\rho}\right)+O(c^{-2}). \tag{19}$$

In the same way, we find that the contravariant components of the energy-momentum tensor are

$$T^{00}=\rho c^{2}\left[1+\frac{1}{c^{2}}\left(v^{2}+2U+\Pi\right)\right]+O(c^{-2}),$$

$$T^{0a}=\rho c\left[1+\frac{1}{c^{2}}\left(v^{2}+2U+\Pi+\frac{p}{\rho}\right)\right]v_{a}+O(c^{-3}), \tag{20}$$

and

$$T^{a\beta}=\rho v_{a}v_{\beta}+p\delta_{a\beta}+\frac{1}{c^{2}}\left[\rho\left(v^{2}+2U+\Pi+\frac{p}{\rho}\right)v_{a}v_{\beta}-2pU\delta_{a\beta}\right]+O(c^{-4}).$$

b) The (0,0)- and the (0,a)-Components of the Ricci Tensor

The general expression for the Ricci tensor is

$$R_{ij}=\tfrac{1}{2}g^{kl}\left(\frac{\partial^{2}g_{ij}}{\partial x_{k}\partial x_{l}}+\frac{\partial^{2}g_{kl}}{\partial x_{i}\partial x_{j}}-\frac{\partial^{2}g_{kj}}{\partial x_{i}\partial x_{l}}-\frac{\partial^{2}g_{il}}{\partial x_{k}\partial x_{l}}\right) \tag{21}$$
$$+g^{kl}g^{mn}\left(\Gamma_{n,ij}\Gamma_{m,kl}-\Gamma_{n,il}\Gamma_{m,kj}\right).$$

In view of the definitions (11), the (0,0)-component of R_{ij} takes the form

$$R_{00} = \tfrac{1}{2} g^{kl} \left(\frac{\partial^2 h_{00}}{\partial x_k \partial x_l} + \frac{\partial^2 h_{kl}}{\partial^2 x_0{}^2} - 2 \frac{\partial^2 h_{k0}}{\partial x_0 \partial x_l} \right) + g^{kl} g^{mn} (\Gamma_{n,00} \Gamma_{m,kl} - \Gamma_{n,0l} \Gamma_{m,0k}) . \quad (22)$$

For our present purposes of determining g_{00} to $O(c^{-4})$, R_{00} must be evaluated correctly to $O(c^{-4})$; and we shall verify that the present information on the metric tensor enables us to do so.

The terms involving the second derivatives of the h's, in equation (22), evaluated consistently with our present knowledge of the g_{ij}'s, give

$$\frac{\partial}{\partial x_0} \left(\tfrac{1}{2} \frac{\partial h_\alpha{}^\alpha}{\partial x_0} - \frac{\partial h_0{}^\alpha}{\partial x_\alpha} \right) - \tfrac{1}{2} (\delta_{\alpha\beta} + h_{\alpha\beta}) \frac{\partial^2 h_{00}}{\partial x_\alpha \partial x_\beta} . \quad (23)$$

We now introduce the *gauge condition*

$$\tfrac{1}{2} \frac{\partial h_\alpha{}^\alpha}{\partial x_0} - \frac{\partial h_0{}^\alpha}{\partial x_\alpha} = 0 \quad (24)$$

—a condition whose satisfaction by the solution to be obtained will have to be verified subsequently.

With the chosen gauge, the terms (23) become

$$- \tfrac{1}{2} \nabla^2 h_{00} - \tfrac{1}{2} h_{\alpha\beta} \frac{\partial^2 h_{00}}{\partial x_\alpha \partial x_\beta} . \quad (25)$$

In the *second* of the two foregoing terms, we can (consistently, in the present approximation) insert for h_{00} and $h_{\alpha\beta}$ their known values. We thus obtain

$$- \tfrac{1}{2} \nabla^2 h_{00} - \frac{2U}{c^4} \delta_{\alpha\beta} \frac{\partial^2 U}{\partial x_\alpha \partial x_\beta} = - \tfrac{1}{2} \nabla^2 h_{00} - \frac{2}{c^4} U \nabla^2 U . \quad (26)$$

Considering next the terms in the Christoffel symbols in equation (22), we find by direct evaluation that, in the chosen gauge,

$$g^{kl} g^{mn} \Gamma_{n,00} \Gamma_{m,kl} = O(c^{-6}) \quad (27)$$

and

$$- g^{kl} g^{mn} \Gamma_{n,0l} \Gamma_{m,0k} = \frac{2}{c^4} \left(\frac{\partial U}{\partial x_\alpha} \right)^2 . \quad (28)$$

Combining the results (26), (27), and (28), we obtain

$$R_{00} = - \tfrac{1}{2} \nabla^2 h_{00} - \frac{2}{c^4} U \nabla^2 U + \frac{2}{c^4} \left(\frac{\partial U}{\partial x_\alpha} \right)^2 . \quad (29)$$

Making use of the identity

$$\nabla^2 U^2 = 2 U \nabla^2 U + 2 \left(\frac{\partial U}{\partial x_\alpha} \right)^2 , \quad (30)$$

we can rewrite the expression for R_{00} in the form

$$R_{00} = \nabla^2 \left(- \tfrac{1}{2} h_{00} + \frac{U^2}{c^4} \right) - \frac{4}{c^4} U \nabla^2 U , \quad (31)$$

or, alternatively, in view of Poisson's equation governing U,

$$R_{00} = \nabla^2 \left(-\tfrac{1}{2} h_{00} + \frac{U^2}{c^4} \right) + \frac{16\pi G}{c^4} \rho U . \tag{32}$$

Turning next to the evaluation of the $(0,a)$-component R_{ij}, we observe that, for our present purposes of determining the dominant term of $O(c^{-3})$ in g_{0a}, it will suffice to consider only the terms in the second derivatives of the h's in equation (21) since the non-linear terms in the Christoffel symbols are at least of $O(c^{-4})$. Thus, we may write

$$R_{0a} = \tfrac{1}{2} g^{kl} \left(\frac{\partial^2 h_{0a}}{\partial x_k \partial x_l} + \frac{\partial^2 h_{kl}}{\partial x_0 \partial x_a} - \frac{\partial^2 h_{ka}}{\partial x_0 \partial x_l} - \frac{\partial^2 h_{0l}}{\partial x_k \partial x_a} \right). \tag{33}$$

On evaluating the terms on the right-hand side of equation (33), consistently in the present approximation, we find

$$R_{0a} = -\tfrac{1}{2} \nabla^2 h_{0a} + \tfrac{1}{2} \frac{\partial}{\partial x_a} \left(\frac{\partial h_\beta{}^\beta}{\partial x_0} - \frac{\partial h_0{}^\beta}{\partial x_\beta} \right) - \tfrac{1}{2} \frac{\partial^2 h_a{}^\beta}{\partial x_0 \partial x_\beta} . \tag{34}$$

The gauge condition (24) simplifies this expression to the form

$$R_{0a} = -\tfrac{1}{2} \nabla^2 h_{0a} + \tfrac{1}{4} \frac{\partial^2 h_\beta{}^\beta}{\partial x_0 \partial x_a} - \tfrac{1}{2} \frac{\partial^2 h_a{}^\beta}{\partial x_0 \partial x_\beta} . \tag{35}$$

We can substitute the known expression for $h_a{}^\beta$ in the two last terms on the right-hand side of equation (35); and we obtain

$$R_{0a} = -\tfrac{1}{2} \nabla^2 h_{0a} + \frac{1}{c} \frac{\partial}{\partial t} \left[\tfrac{1}{4} \frac{\partial}{\partial x_a} \left(\frac{6U}{c^2} \right) - \tfrac{1}{2} \frac{\partial}{\partial x_\beta} \left(\frac{2U}{c^2} \delta_{a\beta} \right) \right], \tag{36}$$

or, after simplification,

$$R_{0a} = -\tfrac{1}{2} \nabla^2 h_{0a} + \frac{1}{2c^3} \frac{\partial^2 U}{\partial t \partial x_a} . \tag{37}$$

c) The Solution of the Field Equations

With the expressions for $T_{00} - \tfrac{1}{2} T g_{00}$ and R_{00} given in equations (19) and (31), the (0,0)-component of the field equation gives

$$\nabla^2 \left(-\tfrac{1}{2} h_{00} + \frac{U^2}{c^4} \right) + \frac{16\pi G}{c^4} \rho U = -\frac{4\pi G}{c^2} \rho - \frac{8\pi G}{c^4} \rho \left(v^2 - U + \tfrac{1}{2}\Pi + \tfrac{3}{2} \frac{p}{\rho} \right). \tag{38}$$

This equation can be rewritten in the form

$$\nabla^2 \left(-\tfrac{1}{2} h_{00} - \frac{U}{c^2} + \frac{U^2}{c^4} \right) = -\frac{8\pi G}{c^4} \rho \phi , \tag{39}$$

where

$$\phi = v^2 + U + \tfrac{1}{2}\Pi + \tfrac{3}{2} \frac{p}{\rho} . \tag{40}$$

We shall find that this quantity ϕ plays an important role in the subsequent developments of the theory.

Defining a "potential" Φ by means of the equation

$$\nabla^2 \Phi = -4\pi G \rho \phi , \tag{41}$$

we can write down, at once, the solution of equation (39); we have

$$h_{00} = -\frac{2U}{c^2} + \frac{1}{c^4}(2U^2 - 4\Phi) + O(c^{-6}). \tag{42}$$

Considering next the $(0,\alpha)$-component of the field equation, we have by equations (17) and (37)

$$-\tfrac{1}{2}\nabla^2 h_{0\alpha} + \frac{1}{2c^3}\frac{\partial^2 U}{\partial t \partial x_\alpha} = -\frac{8\pi G}{c^4} T_{0\alpha} = \frac{8\pi G}{c^3}\rho v_\alpha . \tag{43}$$

Defining the further potentials χ and U_α by means of the equations

$$\nabla^2 \chi = -2U \tag{44}$$

and

$$\nabla^2 U_\alpha = -4\pi G \rho v_\alpha , \tag{45}$$

we can write the solution of equation (43) in the form

$$h_{0\alpha} = \frac{1}{c^3}\left(4U_\alpha - \tfrac{1}{2}\frac{\partial^2 \chi}{\partial t \partial x_\alpha}\right). \tag{46}$$

It may be noted here that χ as defined by equation (44) is the same as the "superpotential" that has been introduced in the theory of Newtonian gravitation in another connection (Chandrasekhar and Lebovitz 1962).

d) The Verification of the Gauge Condition

It remains to verify that the solution for $h_{0\alpha}$ obtained in § III*c* is consistent with the gauge condition (24) introduced at an earlier stage. Inserting, then, the values of $h_0{}^\alpha$ and $h_\alpha{}^\alpha$ given by equations (13) and (46) in equation (24), we obtain

$$\begin{aligned}\tfrac{1}{2}\frac{\partial h_\alpha{}^\alpha}{\partial x_0} - \frac{\partial h_0{}^\alpha}{\partial x_\alpha} &= \frac{3}{c^3}\frac{\partial U}{\partial t} + \frac{1}{c^3}\frac{\partial}{\partial x_\alpha}\left(4U_\alpha - \tfrac{1}{2}\frac{\partial^2\chi}{\partial t \partial x_\alpha}\right) \\ &= \frac{1}{c^3}\left(3\frac{\partial U}{\partial t} + 4\frac{\partial U_\alpha}{\partial x_\alpha} - \tfrac{1}{2}\frac{\partial}{\partial t}\nabla^2\chi\right) \\ &= \frac{4}{c^3}\left(\frac{\partial U}{\partial t} + \frac{\partial U_\alpha}{\partial x_\alpha}\right).\end{aligned} \tag{47}$$

On the other hand, from the equations satisfied by U and U_α,

$$\frac{1}{c^3}\nabla^2\left(\frac{\partial U}{\partial t} + \frac{\partial U_\alpha}{\partial x_\alpha}\right) = -\frac{4\pi G}{c^3}\left[\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_\alpha}(\rho v_\alpha)\right] = 0 , \tag{48}$$

by the Newtonian equation of continuity (which is applicable for evaluating this post-Newtonian term). Since U and U_α are required to be continuously differentiable and further vanish at infinity, it follows from equation (48) that

$$\frac{\partial U}{\partial t} + \frac{\partial U_\alpha}{\partial x_\alpha} = 0 ; \tag{49}$$

and this completes the verification of the gauge condition.

IV. THE METRIC IN THE POST-NEWTONIAN APPROXIMATION AND THE CHRISTOFFEL SYMBOLS

With the solutions for h_{00} and $h_{0\alpha}$ obtained in the preceding section, we can now write

$$g_{00}=1-\frac{2U}{c^2}+\frac{1}{c^4}(2U^2-4\Phi)+O(c^{-6}),$$

$$g_{0\alpha}=\frac{1}{c^3}\left(4U_\alpha-\tfrac{1}{2}\frac{\partial^2\chi}{\partial t\partial x_\alpha}\right)+O(c^{-5}), \tag{50}$$

and

$$g_{\alpha\beta}=-\left(1+\frac{2U}{c^2}\right)\delta_{\alpha\beta}+O(c^{-4}).$$

A comparison of this solution with Schwarzschild's "interior solution" for a spherically symmetric distribution of matter in hydrostatic equilibrium is made in the Appendix.

The spatial part of the metric defined by (cf. Landau and Lifshitz 1962, p. 273)

$$\gamma_{\alpha\beta}=g_{\alpha\beta}-\frac{g_{0\alpha}g_{0\beta}}{g_{00}} \tag{51}$$

agrees with $g_{\alpha\beta}$ to $O(c^{-4})$. From this fact and the known result,

$$g=|g_{ij}|=g_{00}|\gamma_{\alpha\beta}|, \tag{52}$$

it follows that

$$g=-\left(1+\frac{4U}{c^2}\right)+O(c^{-4}). \tag{53}$$

Accordingly,

$$\log\sqrt{-g}=\frac{2U}{c^2}+O(c^{-4}). \tag{54}$$

The contravariant components of the metric tensor can be deduced from the relations

$$g^{0i}g_{i0}=g^{00}g_{00}+g^{0\alpha}g_{0\alpha}=1$$

and

$$g^{0j}g_{j\alpha}=g^{00}g_{0\alpha}+g^{0\beta}g_{\beta\alpha}=0. \tag{55}$$

We find

$$g^{00}=\frac{1}{g_{00}}+O(c^{-6})=1+\frac{2U}{c^2}+\frac{1}{c^4}(2U^2+4\Phi)+O(c^{-6}),$$

$$g^{0\alpha}=g_{\alpha 0}+O(c^{-5})=\frac{1}{c^3}\left(4U_\alpha-\tfrac{1}{2}\frac{\partial^2\chi}{\partial t\partial x_\alpha}\right)+O(c^{-5}), \tag{56}$$

and

$$g^{\alpha\beta}=-\left(1-\frac{2U}{c^2}\right)\delta_{\alpha\beta}+O(c^{-4}).$$

a) *The Christoffel Symbols*

We can now evaluate the Christoffel symbols with the aid of the metric coefficients given in equations (50) and (56). Retaining in each case only the terms to which we are entitled, we find

$$\Gamma^0{}_{00} = -\frac{1}{c^3}\frac{\partial U}{\partial t}, \qquad \Gamma^0{}_{0\alpha} = -\frac{1}{c^2}\frac{\partial U}{\partial x_\alpha},$$

$$\Gamma^0{}_{\alpha\beta} = \frac{1}{2c^3}\left[4\left(\frac{\partial U_\alpha}{\partial x_\beta}+\frac{\partial U_\beta}{\partial x_\alpha}\right)-\frac{\partial^3 \chi}{\partial t \partial x_\alpha \partial x_\beta}+2\delta_{\alpha\beta}\frac{\partial U}{\partial t}\right]$$

$$\Gamma^\alpha{}_{00} = -\frac{1}{c^2}\frac{\partial U}{\partial x_\alpha}+\frac{1}{c^4}\left[\frac{\partial}{\partial x_\alpha}(2U^2-2\Phi)-4\frac{\partial U_\alpha}{\partial t}+\tfrac{1}{2}\frac{\partial^3\chi}{\partial t^2 \partial x_\alpha}\right] \tag{57}$$

$$\Gamma^\alpha{}_{0\beta} = \frac{1}{c^3}\left[\frac{\partial U}{\partial t}\delta_{\alpha\beta}-2\left(\frac{\partial U_\alpha}{\partial x_\beta}-\frac{\partial U_\beta}{\partial x_\alpha}\right)\right],$$

and

$$\Gamma^\alpha{}_{\beta\gamma} = \frac{1}{c^2}\left(\frac{\partial U}{\partial x_\gamma}\delta_{\alpha\beta}+\frac{\partial U}{\partial x_\beta}\delta_{\alpha\gamma}-\frac{\partial U}{\partial x_\alpha}\delta_{\beta\gamma}\right).$$

And finally, we may note for future reference that (cf. eq. [54])

$$y_j = \frac{\partial \log \sqrt{-g}}{\partial x_j} = \frac{2}{c^2}\frac{\partial U}{\partial x_j}; \tag{58}$$

in particular,

$$y_0 = \frac{2}{c^3}\frac{\partial U}{\partial t} \quad \text{and} \quad y_\alpha = \frac{2}{c^2}\frac{\partial U}{\partial x_\alpha}. \tag{59}$$

V. THE EQUATIONS OF HYDRODYNAMICS IN THE POST-NEWTONIAN APPROXIMATION

With the list of the Christoffel symbols given in § IV, the identity

$$T^{ij}{}_{;j} = 0, \tag{60}$$

now gives the desired hydrodynamic equations in the post-Newtonian approximation. Thus the time-component of equation (60), namely,

$$\frac{1}{c}\frac{\partial T^{00}}{\partial t}+\frac{\partial T^{0\alpha}}{\partial x_\alpha}+(\Gamma^0{}_{00}+y_0)T^{00}+(2\Gamma^0{}_{0\alpha}+y_\alpha)T^{0\alpha}+\Gamma^0{}_{\alpha\beta}T^{\alpha\beta}=0, \tag{61}$$

together with the expressions (20) for the contravariant components of the energy-momentum tensor[1] gives

$$\frac{\partial}{\partial t}\left\{\rho\left[1+\frac{1}{c^2}(v^2+2U+\Pi)\right]\right\}+\frac{\partial}{\partial x_\alpha}\left\{\rho v_\alpha\left[1+\frac{1}{c^2}\left(v^2+2U+\Pi+\frac{p}{\rho}\right)\right]\right\} + \frac{1}{c^2}\rho\frac{\partial U}{\partial t}=0. \tag{62}$$

Letting

$$\sigma = \rho\left[1+\frac{1}{c^2}\left(v^2+2U+\Pi+\frac{p}{\rho}\right)\right], \tag{63}$$

[1] Note that T^{00} is $O(c^2)$, $T^{0\alpha}$ is $O(c)$, $T^{\alpha\beta}$ is $O(1)$, $\Gamma^0{}_{00}+y_0$ $(=\partial U/c^3\partial t)$ is $O(c^{-3})$, $2\Gamma^0{}_{0\alpha}+y_\alpha$ is $O(c^{-4})$, and $\Gamma^0{}_{\alpha\beta}$ is $O(c^{-3})$.

we can rewrite equation (62) in the simpler form

$$\frac{\partial \sigma}{\partial t}+\frac{\partial}{\partial x_\alpha}(\sigma v_\alpha)+\frac{1}{c^2}\left(\rho\frac{\partial U}{\partial t}-\frac{\partial p}{\partial t}\right)=0\,. \tag{64}$$

Equation (64) replaces the equation of continuity of Newtonian hydrodynamics; as we shall see later, the equation can indeed be cast in the exact form of an equation of continuity (see eq. [117] below).

Considering next the space component of equation (60), we have

$$\frac{1}{c}\frac{\partial T^{\alpha 0}}{\partial t}+\frac{\partial T^{\alpha\beta}}{\partial x_\beta}+\Gamma^\alpha{}_{00}T^{00}+2\Gamma^\alpha{}_{\beta 0}T^{0\beta}+y_0T^{\alpha 0}+\Gamma^\alpha{}_{\beta\gamma}T^{\beta\gamma}+y_\beta T^{\alpha\beta}=0\,. \tag{65}$$

On inserting the values of the Christoffel symbols and the components of the energy-momentum tensor, we find that equation (65) becomes

$$\begin{aligned}
&\frac{\partial}{\partial t}\left\{\rho\left[1+\frac{1}{c^2}\left(v^2+2U+\Pi+\frac{p}{\rho}\right)\right]v_\alpha\right\}\\
&+\frac{\partial}{\partial x_\beta}\left\{\rho v_\alpha v_\beta+p\,\delta_{\alpha\beta}+\frac{1}{c^2}\left[\rho\left(v^2+2U+\Pi+\frac{p}{\rho}\right)v_\alpha v_\beta-2pU\delta_{\alpha\beta}\right]\right\}\\
&-\rho\left\{\frac{\partial U}{\partial x_\alpha}-\frac{1}{c^2}\left[\frac{\partial}{\partial x_\alpha}(2U^2-2\Phi)-4\frac{\partial U_\alpha}{\partial t}+\tfrac{1}{2}\frac{\partial^3\chi}{\partial t^2\partial x_\alpha}\right]\right\}\left[1+\frac{1}{c^2}(v^2+2U+\Pi)\right]\\
&+\frac{2}{c^2}\rho v_\beta\left[\frac{\partial U}{\partial t}\delta_{\alpha\beta}-2\left(\frac{\partial U_\alpha}{\partial x_\beta}-\frac{\partial U_\beta}{\partial x_\alpha}\right)\right]+\frac{2}{c^2}\rho v_\alpha\frac{\partial U}{\partial t}\\
&+\frac{1}{c^2}\left(\frac{\partial U}{\partial x_\gamma}\delta_{\alpha\beta}+\frac{\partial U}{\partial x_\beta}\delta_{\alpha\gamma}-\frac{\partial U}{\partial x_\alpha}\delta_{\beta\gamma}\right)(\rho v_\beta v_\gamma+p\,\delta_{\beta\gamma})\\
&+\frac{2}{c^2}(\rho v_\alpha v_\beta+p\,\delta_{\alpha\beta})\frac{\partial U}{\partial x_\beta}=0\,.
\end{aligned} \tag{66}$$

On simplifying this equation, we are left with

$$\begin{aligned}
&\frac{\partial}{\partial t}(\sigma v_\alpha)+\frac{\partial}{\partial x_\beta}(\sigma v_\alpha v_\beta)+\left(1-\frac{2U}{c^2}\right)\frac{\partial p}{\partial x_\alpha}-\sigma\left(1+\frac{v^2}{c^2}\right)\frac{\partial U}{\partial x_\alpha}\\
&\quad+\frac{4}{c^2}\rho v_\alpha\left(\frac{\partial U}{\partial t}+v_\beta\frac{\partial U}{\partial x_\beta}\right)-\frac{4}{c^2}\rho\left[\frac{\partial U_\alpha}{\partial t}+v_\beta\left(\frac{\partial U_\alpha}{\partial x_\beta}-\frac{\partial U_\beta}{\partial x_\alpha}\right)\right]\\
&\quad+\frac{1}{c^2}\rho\frac{\partial}{\partial x_\alpha}\left(2U^2-2\Phi+\tfrac{1}{2}\frac{\partial^2\chi}{\partial t^2}\right)=0\,.
\end{aligned} \tag{67}$$

After some rearrangements, equation (67) can be brought to the form

$$\begin{aligned}
&\frac{\partial}{\partial t}(\sigma v_\alpha)+\frac{\partial}{\partial x_\beta}(\sigma v_\alpha v_\beta)+\frac{\partial}{\partial x_\alpha}\left[\left(1+\frac{2U}{c^2}\right)p\right]-\rho\frac{\partial U}{\partial x_\alpha}+\frac{4}{c^2}\rho\frac{d}{dt}(v_\alpha U-U_\alpha)\\
&\quad+\frac{4}{c^2}\rho v_\beta\frac{\partial U_\beta}{\partial x_\alpha}+\frac{1}{2c^2}\rho\frac{\partial^3\chi}{\partial t^2\partial x_\alpha}-\frac{2}{c^2}\rho\left(\phi\frac{\partial U}{\partial x_\alpha}+\frac{\partial\Phi}{\partial x_\alpha}\right)=0\,.
\end{aligned} \tag{68}$$

It will be observed that in equation (68) the superpotential χ occurs differentiated twice with respect to time. To reduce the order of this differentiation we proceed as follows.

We start with the known integral representation of χ, namely (cf. Chandrasekhar and Lebovitz 1962, eq. [19]),

$$\chi(x,t) = -G\int_V \rho(x',t)\,|x-x'|\,dx', \tag{69}$$

where the integration is effected over the entire volume V occupied by the fluid. Differentiating equation (69) with respect to time, we have

$$\frac{\partial \chi}{\partial t} = -G\int_V \frac{\partial \rho(x',t)}{\partial t}\,|x-x'|\,dx'. \tag{70}$$

Since χ occurs only in a post-Newtonian term, we can make use of the Newtonian equations in its reductions. Thus, making use of the equation of continuity (1), we have

$$\frac{\partial \chi}{\partial t} = G\int_V |x-x'|\,\frac{\partial}{\partial x_\mu'}\,[\rho(x',t)\,v_\mu(x')]\,dx'; \tag{71}$$

and an integration by parts now gives

$$\frac{\partial \chi}{\partial t} = G\int_V \rho(x')\,v_\mu(x')\,\frac{x_\mu - x_\mu'}{|x-x'|}\,dx'. \tag{72}$$

Next, differentiating equation (72) with respect to x_α, we obtain

$$\frac{\partial^2 \chi}{\partial t\,\partial x_\alpha} = G\int_V \frac{\rho(x')\,v_\alpha(x')}{|x-x'|}\,dx' - G\int_V \rho(x')\,v_\mu(x')\,\frac{(x_\mu - x_\mu')(x_\alpha - x_\alpha')}{|x-x'|^3}\,dx'. \tag{73}$$

The first term on the right-hand side of this equation is clearly U_α; and defining

$$U_{\beta;\alpha\gamma} = G\int_V \rho(x')\,v_\beta(x')\,\frac{(x_\alpha - x_\alpha')(x_\gamma - x_\gamma')}{|x-x'|^3}\,dx' \tag{74}$$

we can write

$$\frac{\partial^2 \chi}{\partial t\,\partial x_\alpha} = U_\alpha - U_{\beta;\alpha\beta}. \tag{75}$$

From this last equation we obtain

$$\frac{\partial^3 \chi}{\partial t^2\,\partial x_\alpha} = \frac{d}{dt}\,(U_\alpha - U_{\beta;\alpha\beta}) - v_\mu\,\frac{\partial}{\partial x_\mu}\,(U_\alpha - U_{\beta;\alpha\beta}), \tag{76}$$

which effects the required reduction in the order of differentiation with respect to time.

An explicit integral representation of the second term on the right-hand side of equation (76) will be needed in the subsequent analysis. It can be obtained as follows:

From the integral expressions defining U_α and $U_{\beta;\alpha\beta}$, we readily obtain the formulae

$$v_\mu\,\frac{\partial U_\alpha}{\partial x_\mu} = -G\int_V \rho(x')\,v_\alpha(x')\,v_\mu(x)\,\frac{x_\mu - x_\mu'}{|x-x'|^3}\,dx', \tag{77}$$

and

$$\begin{aligned} v_\mu\,\frac{\partial U_{\beta;\alpha\beta}}{\partial x_\mu} = {} & G\int_V \rho(x')\,v_\mu(x)\,v_\mu(x')\,\frac{x_\alpha - x_\alpha'}{|x-x'|^3}\,dx' \\ & + G\int_V \rho(x')\,v_\alpha(x)\,v_\mu(x')\,\frac{x_\mu - x_\mu'}{|x-x'|^3}\,dx' \\ & - 3G\int_V \rho(x')\,v_\mu(x)\,v_\nu(x')(x_\mu - x_\mu')(x_\nu - x_\nu')\,\frac{x_\alpha - x_\alpha'}{|x-x'|^5}\,dx'; \end{aligned} \tag{78}$$

and combining these formulae, we have

$$
\begin{aligned}
v_\mu \frac{\partial}{\partial x_\mu}(U_\alpha - U_{\beta\,\alpha\beta}) = & -G\int_V \rho(x')\, v_\mu(x)\, v_\mu(x') \frac{x_\alpha - x_\alpha'}{|x-x'|^3}\, dx \\
& -G\int_V \rho(x')[v_\alpha(x)\, v_\mu(x') + v_\alpha(x')\, v_\mu(x)] \frac{x_\mu - x_\mu'}{|x-x'|^3}\, dx' \\
& + 3G\int_V \rho(x')[v_\mu(x)\, v_\nu(x')(x_\mu - x_\mu')(x_\nu - x_\nu')] \frac{x_\alpha - x_\alpha'}{|x-x'|^5}\, dx' \\
& = W_\alpha(x) \quad (\text{say}).
\end{aligned}
\tag{79}
$$

Making use of equations (76) and (79), we shall now rewrite equation (68) in the form

$$
\begin{aligned}
& \frac{\partial}{\partial t}(\sigma v_\alpha) + \frac{\partial}{\partial x_\mu}(\sigma v_\alpha v_\mu) + \frac{\partial}{\partial x_\alpha}\left[\left(1+\frac{2U}{c^2}\right)p\right] - \rho\frac{\partial U}{\partial x_\alpha} + \frac{4}{c^2}\rho\frac{d}{dt}(v_\alpha U - U_\alpha) \\
& + \frac{4}{c^2}\rho v_\mu \frac{\partial U_\mu}{\partial x_\alpha} + \frac{1}{2c^2}\rho\frac{d}{dt}(U_\alpha - U_{\mu;\alpha\mu}) - \frac{1}{2c^2}\rho W_\alpha(x) \\
& - \frac{2}{c^2}\rho\left(\phi\frac{\partial U}{\partial x_\alpha} + \frac{\partial \Phi}{\partial x_\alpha}\right) = 0 .
\end{aligned}
\tag{80}
$$

Equation (80) together with the equation

$$
\frac{\partial \sigma}{\partial t} + \frac{\partial}{\partial x_\alpha}(\sigma v_\alpha) + \frac{1}{c^2}\left(\rho\frac{\partial U}{\partial t} - \frac{\partial p}{\partial t}\right) = 0 , \tag{81}
$$

provides the required generalizations of the Eulerian equations (1) and (2) of Newtonian hydrodynamics.

In §§ VII, VIII, IX, and X, we shall show how equations (80) and (81) allow analogues of the classical integrals expressing the conservation of mass, linear momentum, angular momentum, and energy. But to establish these integrals, some auxiliary lemmas are needed; and these are stated and proved in § VI.

VI. SOME AUXILIARY LEMMAS

First we observe that $U(x)$, $U_\alpha(x)$, and $\Phi(x)$ are expressible as integrals in the forms

$$
U(x) = G\int_V \frac{\rho(x')}{|x-x'|}\, dx' , \qquad U_\alpha(x) = G\int_V \frac{\rho(x')\, v_\alpha(x')}{|x-x'|}\, dx' ,
$$

and

$$
\Phi(x) = G\int_V \frac{\rho(x')\,\phi(x')}{|x-x'|}\, dx' . \tag{82}
$$

LEMMA 1:

$$
\int_V \rho(x)\,\phi(x)\,\frac{\partial \Phi(x)}{\partial x_\alpha}\, dx = 0 . \tag{83}
$$

PROOF: We have

$$
\int_V \rho(x)\,\phi(x)\,\frac{\partial \Phi(x)}{\partial x_\alpha}\, dx = -G\int_V\int_V \rho(x)\,\rho(x')\,\phi(x)\,\phi(x')\,\frac{x_\alpha - x_\alpha'}{|x-x'|^3}\, dx\, dx' ; \tag{84}
$$

and the result stated follows from the antisymmetry of the integrand in x and x'.

COROLLARY: In exactly the same way

$$
\int_V \rho(x)\, v_\mu(x)\,\frac{\partial U_\mu(x)}{\partial x_\alpha}\, dx = 0 \tag{85}
$$

(summation or no summation over the repeated index in eq. [85]).

LEMMA 2: *The tensor*

$$\int_V \rho x_\alpha \phi \frac{\partial \Phi}{\partial x_\beta} dx \tag{86}$$

is symmetric in α *and* β.

PROOF: We have

$$\begin{aligned}\int_V \rho(x) x_\alpha \phi(x) \frac{\partial \Phi(x)}{\partial x_\beta} dx &= G \int_V \int_V \rho(x) \rho(x') \phi(x) \phi(x') \frac{x_\alpha (x_\beta' - x_\beta)}{|x - x'|^3} dx dx' \\ &= -\tfrac{1}{2} G \int_V \int_V \rho(x) \rho(x') \phi(x) \phi(x') \frac{(x_\alpha - x_\alpha')(x_\beta - x_\beta')}{|x - x'|^3} dx dx' ;\end{aligned} \tag{87}$$

and the symmetry is manifest.

COROLLARY 1: In case $\phi = 1$, the integral in question defines the potential energy tensor

$$\mathfrak{W}_{\alpha\beta} = -\tfrac{1}{2} \int_V \rho \mathfrak{V}_{\alpha\beta} dx = -\tfrac{1}{2} G \int_V \int_V \rho(x) \rho(x') \frac{(x_\alpha - x_\alpha')(x_\beta - x_\beta')}{|x - x'|^3} dx dx' . \tag{88}$$

COROLLARY 2: In the same way

$$\begin{aligned}\mathfrak{U}_{\mu\mu\cdot\alpha\beta} &= \int_V \rho v_\mu \frac{\partial U_\mu}{\partial x_\alpha} x_\beta dx \\ &= -\tfrac{1}{2} G \int_V \int_V \rho(x) \rho(x') v_\mu(x) v_\mu(x') \frac{(x_\alpha - x_\alpha')(x_\beta - x_\beta')}{|x - x'|^3} dx dx'\end{aligned} \tag{89}$$

is symmetric in α and β.

LEMMA 3:

$$\int_V \rho(x) \Phi(x) dx = \int_V \rho(x) \phi(x) U(x) dx . \tag{90}$$

PROOF: We have

$$\begin{aligned}\int_V \rho(x) \Phi(x) dx &= G \int_V \int_V \rho(x) \rho(x') \frac{\phi(x')}{|x - x'|} dx dx' \\ &= G \int_V dx' \phi(x') \rho(x') \int_V dx \frac{\rho(x)}{|x - x'|} \\ &= \int_V \rho(x') \phi(x') U(x') dx' ;\end{aligned} \tag{91}$$

and this is the result stated.

COROLLARY: In exactly the same way

$$\int_V \rho U_\alpha dx = \int_V \rho v_\alpha U dx . \tag{92}$$

LEMMA 4:

$$\int_V \rho \left(\phi \frac{\partial U}{\partial x_\alpha} + \frac{\partial \Phi}{\partial x_\alpha} \right) dx = 0 . \tag{93}$$

PROOF: We have

$$\int_V \rho \frac{\partial \Phi}{\partial x_\alpha} dx = -G\int_V\int_V \rho(x)\rho(x')\phi(x') \frac{x_\alpha - x_\alpha'}{|x-x'|^3} dx dx'$$
$$= -G\int_V dx' \rho(x')\phi(x') \frac{\partial}{\partial x_\alpha'} \int_V dx \frac{\rho(x)}{|x-x'|} \tag{94}$$
$$= -\int_V \rho(x')\phi(x') \frac{\partial U(x')}{\partial x_\alpha'} dx';$$

and the result stated follows.

LEMMA 5: *The tensor*

$$\langle\Phi\rangle_{\alpha\beta} = \int_V \rho x_\beta \left(\phi \frac{\partial U}{\partial x_\alpha} + \frac{\partial \Phi}{\partial x_\alpha}\right) dx \tag{95}$$

is symmetric in α *and* β.

PROOF: We have

$$\langle\Phi\rangle_{\alpha\beta} = -G\int_V\int_V \rho(x)\rho(x')[\phi(x)+\phi(x')] \frac{x_\beta(x_\alpha - x_\alpha')}{|x-x'|^3} dx dx'$$
$$= -\tfrac{1}{2}G\int_V\int_V \rho(x)\rho(x')[\phi(x)+\phi(x')] \frac{(x_\alpha - x_\alpha')(x_\beta - x_\beta')}{|x-x'|^3} dx dx'; \tag{96}$$

and the symmetry is manifest.

Rewriting the last line of equation (96) in the manner

$$\langle\Phi\rangle_{\alpha\beta} = -G\int_V\int_V \rho(x)\rho(x')\phi(x) \frac{(x_\alpha - x_\alpha')(x_\beta - x_\beta')}{|x-x'|^3} dx dx'$$
$$= -G\int_V dx \rho(x)\phi(x)\int_V dx' \rho(x') \frac{(x_\alpha - x_\alpha')(x_\beta - x_\beta')}{|x-x'|^3}, \tag{97}$$

we can state

COROLLARY:

$$\langle\Phi\rangle_{\alpha\beta} = -\int_V \rho\phi \mathfrak{B}_{\alpha\beta} dx, \tag{98}$$

where $\mathfrak{B}_{\alpha\beta}$ is the Newtonian tensor potential.

LEMMA 6: *The tensor*

$$\mathfrak{U}_{\alpha\beta} = \int_V \rho v_\alpha U_\beta dx \tag{99}$$

is symmetric in α *and* β.

PROOF: We have

$$\mathfrak{U}_{\alpha\beta} = G\int_V\int_V \rho(x)\rho(x') v_\alpha(x) v_\beta(x') \frac{dx dx'}{|x-x'|}$$
$$= \tfrac{1}{2}G\int_V\int_V \rho(x)\rho(x')[v_\alpha(x) v_\beta(x') + v_\alpha(x') v_\beta(x)] \frac{dx dx'}{|x-x'|}; \tag{100}$$

and the symmetry is manifest.

COROLLARY:

$$\mathfrak{U}_{\alpha\alpha} = -2\mathfrak{U}_{\mu\mu;\alpha\alpha}. \tag{101}$$

This result is an immediate consequence of equations (89) and (100).

For purposes of convenient reference we shall state as Lemmas 7 and 8 two well-known theorems.

LEMMA 7: *If $F(x,t)$ is any continuous bounded function of the arguments, then for integrations effected over any arbitrarily specified volume in V*

$$\frac{d}{dt}\int F(x,t)\,dx=\int\left(\frac{dF}{dt}+F\operatorname{div}\boldsymbol{v}\right)dx=\int\left[\frac{\partial F}{\partial t}+\operatorname{div}(F\boldsymbol{v})\right]dx\,. \tag{102}$$

If F should satisfy the "*equation of continuity,*"

$$\frac{\partial F}{\partial t}+\operatorname{div}(F\boldsymbol{v})=0\,, \tag{103}$$

then the integral in question naturally vanishes.

LEMMA 8: *If F satisfies the equation of continuity and f is some other function, then, for integrations effected over any arbitrarily specified volume in V,*

$$\frac{d}{dt}\int Ff\,dx=\int F\frac{df}{dt}\,dx\,. \tag{104}$$

COROLLARY: In evaluating a post-Newtonian term, we may always suppose that

$$\frac{d}{dt}\int \rho f\,dx=\int \rho\frac{df}{dt}\,dx\,, \tag{105}$$

since ρ satisfies the equation of continuity in the Newtonian approximation.

LEMMA 9: *In the Newtonian framework*

$$\begin{aligned}\int_V\rho v_\alpha\left(\phi\frac{\partial U}{\partial x_\alpha}+\frac{\partial\Phi}{\partial x_\alpha}\right)dx&=\int_V\rho\phi\frac{dU}{dt}\,dx\\&=\frac{d}{dt}\int_V\rho\phi U\,dx-\int_V\rho U\frac{d\phi}{dt}\,dx\,.\end{aligned} \tag{106}$$

PROOF: We have

$$\begin{aligned}&\int_V\rho v_\alpha\left(\phi\frac{\partial U}{\partial x_\alpha}+\frac{\partial\Phi}{\partial x_\alpha}\right)dx\\&=-G\int_V\int_V\rho(x)\rho(x')[\phi(x)+\phi(x')]v_\alpha(x)\frac{x_\alpha-x_\alpha'}{|x-x'|^3}\,dx\,dx'\\&=-\tfrac{1}{2}G\int_V\int_V\rho(x)\rho(x')[\phi(x)+\phi(x')]\frac{[v_\alpha(x)-v_\alpha(x')](x_\alpha-x_\alpha')}{|x-x'|^3}\,dx\,dx'\\&=+\tfrac{1}{2}G\int_V\int_V\rho(x)\rho(x')[\phi(x)+\phi(x')]\frac{d}{dt}\frac{1}{|x-x'|}\,dx\,dx'\\&=G\int_V dx\,\rho(x)\phi(x)\int_V dx'\rho(x')\frac{d}{dt}\frac{1}{|x-x'|}\\&=G\int_V dx\,\rho(x)\phi(x)\frac{d}{dt}\int dx'\frac{\rho(x')}{|x-x'|}\\&=\int_V\rho\phi\frac{dU}{dt}\,dx=\int_V\rho\frac{d}{dt}(\phi U)\,dx-\int_V\rho U\frac{d\phi}{dt}\,dx\\&=\frac{d}{dt}\int_V\rho\phi U\,dx-\int_V\rho U\frac{d\phi}{dt}\,dx\,.\end{aligned} \tag{107}$$ [2]

[2] Even though only the argument x (or x') of the functions is explicitly noted in these and similar equations, all functions which occur are in fact functions of t as well.

Note that by setting $\phi = 1$ in equation (106), we obtain the known result,

$$2\int_V \rho\, v_\alpha \frac{\partial U}{\partial x_\alpha}\, d\boldsymbol{x} = \frac{d}{dt}\int_V \rho\, U\, d\boldsymbol{x}\,. \tag{108}$$

VII. THE EQUATION OF CONTINUITY AND THE CONSERVATION OF MASS

We now return to the equation

$$\frac{\partial \sigma}{\partial t}+\frac{\partial}{\partial x_\alpha}(\sigma v_\alpha)+\frac{1}{c^2}\left(\rho\frac{\partial U}{\partial t}-\frac{\partial p}{\partial t}\right)=0 \tag{109}$$

that replaces, in the post-Newtonian approximation, the equation of continuity of Newtonian hydrodynamics.

We may transform the terms in equation (109) that occur explicitly with the factor $1/c^2$ with the aid of equations valid in the Newtonian limit. Thus,

$$\begin{aligned}\frac{1}{c^2}\left(\rho\frac{\partial U}{\partial t}-\frac{\partial p}{\partial t}\right)&=\frac{1}{c^2}\left[\rho\frac{dU}{dt}-\frac{dp}{dt}-v_\alpha\left(\rho\frac{\partial U}{\partial x_\alpha}-\frac{\partial p}{\partial x_\alpha}\right)\right]\\ &=\frac{1}{c^2}\left[\rho\left(\frac{dU}{dt}-v_\alpha\frac{dv_\alpha}{dt}\right)-\frac{dp}{dt}\right]\\ &=\frac{1}{c^2}\left[\rho\frac{d}{dt}(U-\tfrac{1}{2}v^2)-\frac{dp}{dt}\right];\end{aligned} \tag{110}$$

and we may rewrite equation (109) in the form

$$\begin{aligned}&\left[\frac{d}{dt}+(\operatorname{div}\boldsymbol{v})\right]\left\{\rho\left[1+\frac{1}{c^2}(v^2+2U+\Pi)\right]+\frac{1}{c^2}p\right\}\\ &\quad+\frac{1}{c^2}\left[\rho\frac{d}{dt}(U-\tfrac{1}{2}v^2)-\frac{dp}{dt}\right]=0\,,\end{aligned} \tag{111}$$

where we have inserted for σ its value (63). The terms in dp/dt in equation (111) cancel, and we are left with

$$\begin{aligned}&\left[\frac{d}{dt}+(\operatorname{div}\boldsymbol{v})\right]\left\{\rho\left[1+\frac{1}{c^2}(v^2+2U+\Pi)\right]\right\}\\ &\quad+\frac{1}{c^2}\left[\rho\frac{d}{dt}(U-\tfrac{1}{2}v^2)+p\operatorname{div}\boldsymbol{v}\right]=0\,.\end{aligned} \tag{112}$$

Now it is an exact relation[3] in this theory that

$$\rho\frac{d\Pi}{dt}=\frac{p}{\rho}\frac{d\rho}{dt}. \tag{113}$$

When evaluating a post-Newtonian term (as we are at present), we can replace equation (113) by its Newtonian equivalent

$$\rho\frac{d\Pi}{dt}=-p\operatorname{div}\boldsymbol{v}\,. \tag{114}$$

[3] The relation is no more than a statement of the first law of thermodynamics under the non-dissipative conditions assumed (cf. Fock 1964, pp. 104 and 420).

With the aid of this last relation, equation (112) becomes

$$\left[\frac{d}{dt}+(\operatorname{div} \boldsymbol{v})\right]\left\{\rho\left[1+\frac{1}{c^2}(v^2+2U+\Pi)\right]\right\}-\frac{1}{c^2}\rho\frac{d}{dt}(\tfrac{1}{2}v^2-U+\Pi)=0\,. \tag{115}$$

By some further simplifications, equation (115) can be brought to the form

$$\begin{gathered}\left[\frac{d}{dt}+(\operatorname{div} \boldsymbol{v})\right]\left\{\rho\left[1+\frac{1}{c^2}(\tfrac{1}{2}v^2+3U)\right]\right\}\\+\frac{1}{c^2}(\tfrac{1}{2}v^2-U+\Pi)\left(\frac{d\rho}{dt}+\rho\operatorname{div} \boldsymbol{v}\right)=0\,.\end{gathered} \tag{116}$$

The second term in equation (116) can be set equal to zero in view of the Newtonian equation of continuity and the fact that the term occurs with a factor $1/c^2$. Thus, an equivalent form of equation (109) is

$$\frac{\partial\rho^*}{\partial t}+\frac{\partial}{\partial x_\alpha}(\rho^* v_\alpha)=0\,, \tag{117}$$

where

$$\rho^*=\rho\left[1+\frac{1}{c^2}(\tfrac{1}{2}v^2+3U)\right]. \tag{118}$$

Equation (117) has exactly the form of a classical equation of continuity. We may, therefore, conclude: *in the post-Newtonian approximation, the mass defined in terms of the density ρ^* is conserved.*[4]

Since ρ^* satisfies the equation of continuity, we have the following important special case of Lemma 8.

LEMMA 10: *In the post-Newtonian approximation (i.e., correctly to order $1/c^2$)*

$$\frac{d}{dt}\int_V \rho^* f\,dx=\int_V \rho^*\frac{df}{dt}\,dx\,, \tag{119}$$

where f is any continuous function defined in V and the integrals in question converge.

VIII. THE CONSERVATION OF THE TOTAL LINEAR MOMENTUM

An integral analogous to the Newtonian integral, which expresses the conservation of the total linear momentum of the system, follows from integrating the equation of motion (80) over the volume V.

By Lemmas 7 and 8

$$\int_V\left[\frac{\partial}{\partial t}(\sigma v_\alpha)+\frac{\partial}{\partial x_\mu}(\sigma v_\alpha v_\mu)\right]dx=\frac{d}{dt}\int_V \sigma v_\alpha dx\,. \tag{120}$$

$$\frac{1}{2c^2}\int_V\rho\frac{d}{dt}(U_\alpha-U_{\mu;\alpha\mu})dx=\frac{1}{2c^2}\frac{d}{dt}\int_V\rho(U_\alpha-U_{\mu;\alpha\mu})dx\,, \tag{121}$$

and

$$\frac{4}{c^2}\int_V\rho\frac{d}{dt}(v_\alpha U-U_\alpha)dx=\frac{4}{c^2}\frac{d}{dt}\int_V\rho(v_\alpha U-U_\alpha)dx\,; \tag{122}$$

and the last of these integrals vanishes by Lemma 3 (Cor.). The integral

$$\int_V\frac{\partial}{\partial x_\alpha}\left[\left(1+\frac{2U}{c^2}\right)p\right]dx \tag{123}$$

[4] This result is implicit in some work of Fock (1964, p 249).

vanishes by virtue of the condition that $p = 0$ on the boundary of the configuration. And the integrals over V of

$$\rho \frac{\partial U}{\partial x_\alpha}, \qquad \rho v_\mu \frac{\partial U_\mu}{\partial x_\alpha}, \qquad \text{and} \qquad \rho \left(\phi \frac{\partial U}{\partial x_\alpha} + \frac{\partial \Phi}{\partial x_\alpha}\right) \tag{124}$$

all vanish by Lemmas 1 and 4. The last remaining integral of $\rho W_\alpha(x)$ over V also vanishes; this fact is apparent from equation (79) defining the function $W_\alpha(x)$: for

$$\int_V \rho(x) W_\alpha(x) dx \tag{125}$$

when expressed as a double integral over x and x' has an integrand which is manifestly antisymmetric in x and x'. Therefore, the result of integrating the equation of motion (80) over the volume is

$$\frac{d}{dt}\int_V \left[\sigma v_\alpha + \frac{1}{2c^2} \rho (U_\alpha - U_{\mu;\alpha\mu})\right] dx = 0 . \tag{126}$$

In other words

$$\int_V \left[\sigma v_\alpha + \frac{1}{2c^2} \rho (U_\alpha - U_{\mu\,\alpha\mu})\right] dx = \text{constant} . \tag{127}$$

This integral of the equations of motion may be considered as expressing *the conservation of the total linear momentum of the system*. And it suggests that we may define the integrand of the quantity which is conserved as the linear momentum of the fluid element per unit volume. However, it will appear (see § IX below) that a more satisfactory definition of the *linear momentum* is

$$\pi_\alpha = \sigma v_\alpha + \frac{1}{2c^2} \rho (U_\alpha - U_{\mu\,\alpha\mu}) + \frac{4}{c^2} \rho (v_\alpha U - U_\alpha) , \tag{128}$$

even though the volume integral of the last term vanishes: it contributes, as we shall see, to the total angular momentum of the system.

IX. THE CONSERVATION OF ANGULAR MOMENTUM AND THE TENSOR VIRIAL THEOREM

We shall now show that equation (80) allows an integral which may be interpreted as expressing the conservation of the total angular momentum of the system. To this end, we multiply equation (80) by x_β and integrate over the volume V occupied by the fluid. We shall find that the resulting equation consists of terms which are either the derivatives with respect to time of certain integrals or integrals which are symmetric in α and β. Thus, considering in turn the various terms in equation (80), we obtain:

$$\begin{aligned} \int_V x_\beta \left[\frac{\partial}{\partial t}(\sigma v_\alpha) + \operatorname{div}(\sigma v_\alpha \boldsymbol{v})\right] dx & \\ = \frac{d}{dt}\int_V \sigma v_\alpha x_\beta dx - \int_V \sigma v_\alpha v_\beta dx & \\ = \frac{d}{dt}\int_V \sigma v_\alpha x_\beta dx - 2\mathfrak{T}_{\alpha\beta}\,(\text{say}) & \qquad \text{(by Lemmas 7 and 8)} ; \end{aligned} \tag{129}$$

$$\begin{aligned} \int_V x_\beta \frac{\partial}{\partial x_\alpha}\left[\left(1 + \frac{2U}{c^2}\right) p\right] dx &= -\delta_{\alpha\beta}\int_V \left(1 + \frac{2U}{c^2}\right) p\, dx \\ &= -\delta_{\alpha\beta} P\,(\text{say}) ; \end{aligned} \tag{130}$$

$$-\int_V \rho x_\beta \frac{\partial U}{\partial x_\alpha} d x = -\mathfrak{W}_{\alpha\beta} \qquad \text{(by Lemma 2, Cor. 1)} ; \tag{131}$$

$$\frac{4}{c^2} \int_V \rho x_\beta v_\mu \frac{\partial U_\mu}{\partial x_\alpha} d x = \frac{4}{c^2} \mathfrak{U}_{\mu\mu\cdot\alpha\beta} \qquad \text{(by Lemma 2, Cor. 2)} ; \tag{132}$$

$$-\frac{2}{c^2} \int_V \rho x_\beta \left(\phi \frac{\partial U}{\partial x_\alpha} + \frac{\partial \Phi}{\partial x_\alpha}\right) d x = -\frac{2}{c^2} \langle \Phi \rangle_{\alpha\beta} \qquad \text{(by Lemma 5)} ; \tag{133}$$

$$\begin{aligned} \frac{4}{c^2} \int_V \rho x_\beta \frac{d}{dt} (v_\alpha U - U_\alpha) d x &= \frac{4}{c^2} \frac{d}{dt} \int_V \rho x_\beta (v_\alpha U - U_\alpha) d x \\ &+ \frac{4}{c^2} \mathfrak{U}_{\alpha\beta} - \frac{4}{c^2} \int_V \rho v_\alpha v_\beta U d x \qquad \text{(by Lemmas 6 and 8)} ; \end{aligned} \tag{134}$$

$$\begin{aligned} \frac{1}{2c^2} \int_V \rho x_\beta \frac{d}{dt} (U_\alpha - U_{\mu;\alpha\mu}) d x &= \frac{1}{2c^2} \frac{d}{dt} \int_V \rho x_\beta (U_\alpha - U_{\mu;\alpha\mu}) d x - \frac{1}{2c^2} \mathfrak{U}_{\alpha\beta} \\ &+ \frac{G}{2c^2} \int_V \int_V \rho(x) \rho(x') v_\mu(x') (x_\mu - x_\mu') \frac{v_\beta(x)(x_\alpha - x_\alpha')}{|x - x'|^3} d x d x' \\ &\qquad \text{(by Lemmas 6, 8 and eq. [74])} ; \end{aligned} \tag{135}$$

$$\begin{aligned} -\frac{1}{2c^2} \int_V \rho x_\beta W_\alpha(x) d x &= -\frac{1}{2c^2} \mathfrak{U}_{\mu\mu\cdot\alpha\beta} \\ &+ \frac{G}{2c^2} \int_V \int_V \rho(x) \rho(x') v_\mu(x') (x_\mu - x_\mu') \frac{v_\alpha(x)(x_\beta - x_\beta')}{|x - x'|^3} d x d x' \\ &- \frac{3G}{4c^2} \int_V \int_V \rho(x) \rho(x') v_\mu(x) (x_\mu - x_\mu') v_\nu(x') (x_\nu - x_\nu') \frac{(x_\alpha - x_\alpha')(x_\beta - x_\beta')}{|x - x'|^5} d x d x' \\ &\qquad \text{(by Lemma 2 and eq. [79])} . \end{aligned} \tag{136}$$

By combining the foregoing results, we obtain

$$\begin{aligned} \frac{d}{dt} \int_V x_\beta \left[\sigma v_\alpha + \frac{1}{2c^2} \rho (U_\alpha - U_{\mu;\alpha\mu}) + \frac{4}{c^2} \rho (v_\alpha U - U_\alpha)\right] d x &= 2\mathfrak{T}_{\alpha\beta} + \mathfrak{W}_{\alpha\beta} \\ + \delta_{\alpha\beta} P + \frac{1}{c^2} \Big[4W_{\alpha\beta} + 2\langle \Phi \rangle_{\alpha\beta} - \tfrac{7}{2}(\mathfrak{U}_{\alpha\beta} + \mathfrak{U}_{\mu\mu;\alpha\beta}) - \tfrac{1}{2} Q_{\alpha\beta} + \tfrac{3}{4} Z_{\alpha\beta}\Big] , \end{aligned} \tag{137}$$

where the quantities newly introduced have the meanings

$$Q_{\alpha\beta} = G \int_V \int_V \rho(x) \rho(x') v_\mu(x') (x_\mu - x_\mu') \frac{v_\alpha(x)(x_\beta - x_\beta') + v_\beta(x)(x_\alpha - x_\alpha')}{|x - x'|^3} d x d x' \tag{138}$$

$$Z_{\alpha\beta} = G \int_V \int_V \rho(x) \rho(x') v_\mu(x) (x_\mu - x_\mu') v_\nu(x') (x_\nu - x_\nu') \frac{(x_\alpha - x_\alpha')(x_\beta - x_\beta')}{|x - x'|^5} d x d x' \tag{139}$$

and

$$W_{\alpha\beta} = \int_V \rho v_\alpha v_\beta U d x . \tag{140}$$

These new tensors, like those already defined, are symmetric in α and β.

We recognize that the terms in square brackets (under the integral sign) on the left-hand side of equation (137) constitute the linear momentum per unit volume defined in equation (128). And since all the tensors on the right-hand side of equation (137) are symmetric in α and β, it follows from taking the antisymmetric part of the equation that

$$\frac{d}{dt}\int_V (x_\beta \pi_\alpha - x_\alpha \pi_\beta)\, d\boldsymbol{x} = 0 . \tag{141}$$

This integral of the equations of motion expresses *the conservation of the total angular momentum of the system* with the linear momentum as defined in equation (128).

a) The Virial Theorem

Equation (137) represents the post-Newtonian generalization of the classical tensor virial theorem (cf. Chandrasekhar 1964*c*). Under stationary conditions, the theorem gives

$$2\mathfrak{T}_{\alpha\beta}+\mathfrak{W}_{\alpha\beta}+\delta_{\alpha\beta}P+\frac{1}{c^2}\left[4W_{\alpha\beta}+2\langle\Phi\rangle_{\alpha\beta}-\tfrac{7}{2}(\mathfrak{U}_{\alpha\beta}+\mathfrak{U}_{\mu\mu;\alpha\beta})-\tfrac{1}{2}Q_{\alpha\beta}+\tfrac{3}{4}Z_{\alpha\beta}\right]=0 . \tag{142}$$

The contracted version of this equation is

$$2\mathfrak{T}+\mathfrak{W}+3P+\frac{1}{c^2}\left[4W_{\alpha\alpha}+2\langle\Phi\rangle_{\alpha\alpha}-\tfrac{7}{2}(\mathfrak{U}_{\alpha\alpha}+\mathfrak{U}_{\mu\mu;\alpha\alpha})-\tfrac{1}{2}Q_{\alpha\alpha}+\tfrac{3}{4}Z_{\alpha\alpha}\right]=0 , \tag{143}$$

where

$$\mathfrak{T}=\tfrac{1}{2}\int_V \sigma v^2 d\boldsymbol{x} , \qquad \mathfrak{W}=-\tfrac{1}{2}\int_V \rho U d\boldsymbol{x} , \qquad W_{\alpha\alpha}=\int_V \rho v^2 U d\boldsymbol{x} ,$$

$$\mathfrak{U}_{\alpha\alpha}=-2\mathfrak{U}_{\mu\mu;\alpha\alpha}=\int_V \rho v_\alpha U_\alpha d\boldsymbol{x} , \qquad \langle\Phi\rangle_{\alpha\alpha}=-\int_V \rho\phi U d\boldsymbol{x} , \tag{144}$$

and

$$Q_{\alpha\alpha}=2Z_{\alpha\alpha}=2G\int_V\int_V \rho(\boldsymbol{x})\rho(\boldsymbol{x}')\frac{v_\mu(\boldsymbol{x})(x_\mu-x_\mu')\,v_\nu(\boldsymbol{x}')(x_\nu-x_\nu')}{|\boldsymbol{x}-\boldsymbol{x}'|^3}\,d\boldsymbol{x}\,d\boldsymbol{x}' .$$

In view of the foregoing relations equation (143) takes the form

$$\begin{aligned}\int_V \sigma v^2 d\boldsymbol{x} - \tfrac{1}{2}\int_V \rho U d\boldsymbol{x} + 3\int_V p\left(1+\frac{2U}{c^2}\right)d\boldsymbol{x}+\frac{1}{c^2}\Big[4\int_V \rho v^2 U d\boldsymbol{x}\\ -2\int_V \rho U\left(v^2+U+\tfrac{1}{2}\Pi+\tfrac{3}{2}\frac{p}{\rho}\right)d\boldsymbol{x}-\tfrac{7}{4}\int_V \rho v_\alpha U_\alpha d\boldsymbol{x}-\tfrac{1}{4}Z_{\alpha\alpha}\Big]=0 ,\end{aligned} \tag{145}$$

where we have further substituted for ϕ in accordance with its definition (eq. [40]). On further simplification, equation (145) becomes

$$\begin{aligned}\int_V \rho v^2 d\boldsymbol{x} - \tfrac{1}{2}\int \rho U d\boldsymbol{x} + 3\int_V p\, d\boldsymbol{x}+\frac{1}{c^2}\Big\{\int_V \rho v^2\left(v^2+\Pi+\frac{p}{\rho}\right)d\boldsymbol{x}\\ -\int_V \rho U\left(2U+\Pi-3\frac{p}{\rho}\right)d\boldsymbol{x}+4\int_V \rho v^2 U d\boldsymbol{x}-\tfrac{7}{4}\int_V \rho v_\alpha U_\alpha d\boldsymbol{x}\\ -\tfrac{1}{4}G\int_V\int_V \rho(\boldsymbol{x})\rho(\boldsymbol{x}')\frac{[\boldsymbol{v}(\boldsymbol{x})\cdot(\boldsymbol{x}-\boldsymbol{x}')][\boldsymbol{v}(\boldsymbol{x}')\cdot(\boldsymbol{x}-\boldsymbol{x}')]}{|\boldsymbol{x}-\boldsymbol{x}'|^3}\,d\boldsymbol{x}\,d\boldsymbol{x}'\Big\}=0 .\end{aligned} \tag{146}$$

We shall find that the tensor and the scalar forms of the virial theorem are useful in exhibiting the post-Newtonian effects of rotation on the equilibrium of bodies (see Chandrasekhar 1965*b*).

X. THE CONSERVATION OF ENERGY

An integral analogous to the Newtonian integral which expresses the conservation of energy follows from equation (80) by integration over the volume V after multiplication by v_α and contraction. The reductions are somewhat involved; accordingly, we shall preface the enumeration of the results of the integrations of the various terms of equations (80) by some lemmas (elementary in themselves) which must be used in their reductions.

LEMMA 11:

$$-\rho \operatorname{div} \boldsymbol{v} = \frac{d\rho}{dt} + \frac{1}{c^2} \rho \frac{d}{dt} \left(\tfrac{1}{2} v^2 + 3U \right). \tag{147}$$

Equation (147) is simply another form of the equation of continuity satisfied by

$$\rho^* = \rho \left[1 + \frac{1}{c^2} \left(\tfrac{1}{2} v^2 + 3U \right) \right]. \tag{148}$$

LEMMA 12:

$$-p \operatorname{div} \boldsymbol{v} = \rho^* \frac{d\Pi}{dt} - \frac{1}{c^2} \rho \left(\tfrac{1}{2} v^2 + 3U \right) \frac{d\Pi}{dt} + \frac{1}{c^2} p \frac{d}{dt} \left(\tfrac{1}{2} v^2 + 3U \right). \tag{149}$$

Equation (149) follows from eliminating $d\rho/dt$ between equations (113) and (147).

DEFINITION: Let U^* be the gravitational potential arising from the distribution of the density ρ^* so that

$$\nabla^2 U^* = -4\pi G \rho^* = -4\pi G \rho \left[1 + \frac{1}{c^2} \left(\tfrac{1}{2} v^2 + 3U \right) \right]. \tag{150}$$

Writing

$$U^* = U + \frac{1}{c^2} Q, \tag{151}$$

we infer from the equation governing U and U^* that

$$\nabla^2 Q = -4\pi G \rho q, \qquad \text{where} \qquad q = \tfrac{1}{2} v^2 + 3U. \tag{152}$$

We now state some further lemmas.

LEMMA 13:

$$\rho \frac{\partial U}{\partial x_\alpha} = \rho^* \frac{\partial U^*}{\partial x_\alpha} - \frac{1}{c^2} \rho \left(q \frac{\partial U}{\partial x_\alpha} + \frac{\partial Q}{\partial x_\alpha} \right). \tag{153}$$

Equation (153) is a direct consequence of the definitions of the various quantities.

LEMMA 14:

$$\int_V \rho^* v_\alpha \frac{\partial U^*}{\partial x_\alpha} d\boldsymbol{x} = \tfrac{1}{2} \frac{d}{dt} \int_V \rho^* U^* d\boldsymbol{x}. \tag{154}$$

PROOF: By Lemma 10,

$$\begin{aligned} \int_V \rho^* v_\alpha \frac{\partial U^*}{\partial x_\alpha} d\boldsymbol{x} &= -\tfrac{1}{2} G \int_V \int_V \rho^*(\boldsymbol{x}) \rho^*(\boldsymbol{x}') \frac{[v_\alpha(\boldsymbol{x}) - v_\alpha(\boldsymbol{x}')](x_\alpha - x_\alpha')}{|\boldsymbol{x} - \boldsymbol{x}'|^3} d\boldsymbol{x} d\boldsymbol{x}' \\ &= \tfrac{1}{2} G \int_V \int_V \rho^*(\boldsymbol{x}) \rho^*(\boldsymbol{x}') \frac{d}{dt} \frac{1}{|\boldsymbol{x} - \boldsymbol{x}'|} d\boldsymbol{x} d\boldsymbol{x}' \\ &= \tfrac{1}{2} G \frac{d}{dt} \int_V \int_V \frac{\rho^*(\boldsymbol{x}) \rho^*(\boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|} d\boldsymbol{x} d\boldsymbol{x}'; \end{aligned} \tag{155}$$

and the result stated follows. It is to be noted particularly that Lemma 14 is correct to terms inclusive of $O(1/c^2)$: it is a genuine post-Newtonian result.

LEMMA 15: *In the Newtonian approximation*

$$\int_V \rho v_\alpha \left(q \frac{\partial U}{\partial x_\alpha} + \frac{\partial Q}{\partial x_\alpha} \right) dx = \int_V \rho q \frac{dU}{dt} dx = \int_V \rho \left(\tfrac{1}{2} v^2 + 3U \right) \frac{dU}{dt} dx . \tag{156}$$

The proof of this lemma follows exactly on the lines of that of Lemma 9: the roles of ϕ and Φ in that lemma are now played by q and Q, respectively.

We now enumerate the results of the integrations of the various terms of equation (80) after multiplication by v_α and contraction.

$$\begin{aligned} \int_V v_\alpha \left[\frac{\partial}{\partial t} (\sigma v_\alpha) + \operatorname{div}(\sigma v_\alpha \boldsymbol{v}) \right] dx &= \int_V \left[\frac{\partial}{\partial t} (\sigma v^2) + \operatorname{div}(\sigma v^2 \boldsymbol{v}) \right] dx \\ &\quad - \tfrac{1}{2} \int_V \rho^* \frac{dv^2}{dt} dx - \frac{1}{2c^2} \int_V \rho \left(\tfrac{1}{2} v^2 + \Pi - U + \frac{p}{\rho} \right) \frac{dv^2}{dt} dx \\ &= \frac{d}{dt} \int_V \left(\sigma v^2 - \tfrac{1}{2} \rho^* v^2 - \frac{1}{8c^2} \rho v^4 \right) dx - \frac{1}{2c^2} \int_V \rho \left(\Pi - U + \frac{p}{\rho} \right) \frac{dv^2}{dt} dx \end{aligned} \tag{157}$$

(by Lemmas 1, 8, and 10) ;

$$-\int_V \rho v_\alpha \frac{\partial U}{\partial x_\alpha} dx = \frac{d}{dt} \int_V \left(-\tfrac{1}{2} \rho^* U^* + \frac{3}{2c^2} \rho U^2 \right) dx + \frac{1}{2c^2} \int_V \rho v^2 \frac{dU}{dt} dx \tag{158}$$

(by Lemmas 13, 14, and 15) ;

$$\begin{aligned} -\frac{2}{c^2} \int_V \rho v_\alpha \left(\phi \frac{\partial U}{\partial x_\alpha} + \frac{\partial \Phi}{\partial x_\alpha} \right) dx &= -\frac{2}{c^2} \int_V \rho \phi \frac{dU}{dt} dx = -\frac{1}{c^2} \frac{d}{dt} \int_V \rho U^2 dx \\ &\quad - \frac{1}{c^2} \int_V [\rho (2v^2 + \Pi) + 3p] \frac{dU}{dt} dx \quad \text{(by Lemmas 9 and 10 and eq. [40]) ;} \end{aligned} \tag{159}$$

$$\begin{aligned} \int_V v_\alpha \frac{\partial}{\partial x_\alpha} \left[\left(1 + \frac{2U}{c^2} \right) p \right] dx &= \frac{d}{dt} \int_V \rho^* \Pi dx - \frac{1}{c^2} \int_V \rho U \frac{d\Pi}{dt} dx \\ &\quad - \frac{1}{2c^2} \int_V \rho v^2 \frac{d\Pi}{dt} dx + \frac{1}{c^2} \int_V p \frac{d}{dt} \left(\tfrac{1}{2} v^2 + 3U \right) dx \quad \text{(by Lemmas 10 and 12) ;} \end{aligned} \tag{160}$$

$$\frac{4}{c^2} \int_V \rho v_\alpha v_\beta \frac{\partial U_\beta}{\partial x_\alpha} dx = \frac{2G}{c^2} \int_V \int_V \rho(x) \rho(x') v_\beta(x) v_\beta(x') \frac{d}{dt} \frac{1}{|x - x'|} dx dx' ; \tag{161}$$

$$\begin{aligned} \frac{4}{c^2} \int_V \rho v_\alpha \frac{d}{dt} (v_\alpha U - U_\alpha) dx &= \frac{4}{c^2} \frac{d}{dt} \int_V \rho (v^2 U - v_\alpha U_\alpha) dx + \frac{2}{c^2} \frac{d}{dt} \int_V \rho v_\alpha U_\alpha dx \\ &\quad - \frac{2}{c^2} \int_V \rho U \frac{dv^2}{dt} dx - \frac{2G}{c^2} \int_V \int_V \rho(x) \rho(x') v_\alpha(x) v_\alpha(x') \frac{d}{dt} \frac{1}{|x - x'|} dx dx' ; \end{aligned} \tag{162}$$

$$\begin{aligned} -\frac{1}{2c^2} \int_V \rho v_\alpha W_\alpha dx &= -\frac{G}{4c^2} \int_V \int_V \rho(x) \rho(x') v_\mu(x) v_\mu(x') \frac{d}{dt} \frac{1}{|x - x'|} dx dx' \\ &\quad - \frac{G}{2c^2} \int_V \int_V \rho(x) \rho(x') v_\alpha(x') (x_\alpha - x_\alpha') \frac{x_\mu - x_\mu'}{|x - x'|^3} \frac{dv_\mu(x)}{dt} dx dx' \\ &\quad + \frac{G}{4c^2} \frac{d}{dt} \int_V \int_V \rho(x) \rho(x') v_\mu(x) v_\nu(x') (x_\mu - x_\mu')(x_\nu - x_\nu') \frac{dx dx'}{|x - x'|^3} ; \end{aligned} \tag{163}$$

and

$$\frac{1}{2c^2}\int_V \rho\, v_\alpha \frac{d}{dt}\,(U_\alpha - U_{\mu;\alpha\mu})\,dx = \frac{1}{2c^2}\frac{d}{dt}\int_V \rho\, v_\alpha (U_\alpha - U_{\mu;\alpha\mu})\,dx$$

$$-\frac{1}{4c^2}\frac{d}{dt}\int_V \rho\, v_\alpha U_\alpha dx + \frac{G}{4c^2}\int_V\int_V \rho(x)\rho(x')\,v_\alpha(x)\,v_\alpha(x')\,\frac{d}{dt}\,\frac{1}{|x-x'|}\,dx\,dx' \quad (164)$$

$$+\frac{G}{2c^2}\int_V\int_V \rho(x)\rho(x')\,v_\mu(x')(x_\mu - x_\mu')\,\frac{x_\alpha - x_\alpha'}{|x-x'|^3}\,\frac{d\,v_\alpha(x)}{dt}\,dx\,dx'\,.$$

Now adding the contributions (157)–(164), we find that we are left with

$$\frac{d}{dt}\int_V\Big[(\sigma - \tfrac{1}{2}\rho^*)\,v^2 + \rho^*\Pi - \tfrac{1}{2}\rho^* U^*$$
$$+\frac{1}{c^2}\,\rho\,(-\tfrac{1}{8}v^4 + \tfrac{1}{2}U^2 - \Pi U - \tfrac{1}{2}v^2\Pi + \tfrac{5}{2}v^2 U - \tfrac{7}{4}v_\alpha U_\alpha - \tfrac{1}{4}v_\alpha U_{\mu;\alpha\mu})\Big]dx = 0\,. \quad (165)$$

Equation (165) may be interpreted as expressing *the conservation of the total energy of the system;* and it suggests that we may define

$$\mathfrak{E} = (\sigma - \tfrac{1}{2}\rho^*)\,v^2 + \rho^*\Pi - \tfrac{1}{2}\rho^* U^*$$
$$+\frac{1}{c^2}\,\rho\,(-\tfrac{1}{8}v^4 + \tfrac{1}{2}U^2 - \Pi U - \tfrac{1}{2}v^2\Pi + \tfrac{5}{2}v^2 U - \tfrac{7}{4}v_\alpha U_\alpha - \tfrac{1}{4}v_\alpha U_{\mu\,\alpha\mu})\,, \quad (166)$$

as the energy per unit volume of the fluid. That various "cross-terms" should appear in such an expression is to be expected; but one could hardly have guessed the appearance of the terms in U_α and $U_{\mu;\alpha\mu}$ or the numerical coefficients of the other terms. Fortunately, the "guessing" is not needed: Einstein's field equations treated in the most direct manner lead to the correct expression.

XI. CONCLUDING REMARKS

The manner in which the post-Newtonian equations of hydrodynamics have been derived brings out with particular clarity the essential "boot-strap" character of Einstein's general theory, a character which is somewhat obscured in the corresponding derivations of Einstein, Infeld, and Hoffmann by the necessity in that treatment of formulating the equations governing the motions of the singularities in the field. The severe directness of the present treatment exemplifies, once again, the marvelous simplicity and the inherent self-consistency of Einstein's theory.

It is clear that the equations derived in this paper can be applied to investigate the post-Newtonian effects of general relativity on the hydrodynamic behavior of large-scale systems. Examples of such applications will be found in the two papers following.

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APPENDIX

COMPARISON OF THE POST-NEWTONIAN METRIC WITH SCHWARZSCHILD'S EXACT SOLUTION

Karl Schwarzschild's exact solution for the metric in the interior of a spherically symmetric stationary distribution of matter (described by the same energy-momentum tensor as that used in this paper) is given by

$$d s^2 = e^{\nu}(d x_0)^2 - \xi^2(d\theta^2 + \sin^2\theta d\phi^2) - e^{\lambda} d\xi^2 \tag{A.1}$$

where ν and λ are functions of the radial coordinate ξ determined by the equations

$$e^{-\lambda} = 1 - \frac{2G\mathfrak{M}(\xi)}{c^2\xi}, \tag{A.2}$$

$$\mathfrak{M}(\xi) = \frac{4\pi}{c^2}\int_0^{\xi} \epsilon\xi^2 d\xi = 4\pi\int_0^{\xi} \rho\left(1 + \frac{\Pi}{c^2}\right)\xi^2 d\xi, \tag{A.3}$$

and

$$\tfrac{1}{2}c^2\left[1 - \frac{2G\mathfrak{M}(\xi)}{c^2\xi}\right]\frac{d\nu}{d\xi} = \frac{G\mathfrak{M}(\xi)}{\xi^2} + \frac{4\pi G}{c^2}p\xi. \tag{A.4}$$

For a comparison of this solution with that obtained in this paper (appropriately specialized to the case of spherical symmetry and hydrostatic equilibrium), we must first transform the metric (A.1) to isotropic coordinates. The required transformation is

$$\log Ar = \int e^{\lambda/2}\frac{d\xi}{\xi} = \int\frac{d\xi}{\xi\sqrt{[1 - 2G\mathfrak{M}(\xi)/c^2\xi]}}, \tag{A.5}$$

where the constant A is to be determined by the condition that in the Newtonian limit r and ξ must coincide. In the post-Newtonian approximation the transformation (as deduced from eq. [A.5]) is

$$\xi = r\left(1 + \frac{U}{c^2}\right) \quad \text{where} \quad U(r) = \int_r^{\infty}\frac{GM(r)}{r^2}dr. \tag{A.6}$$

The transformation to the coordinate r reduces the spatial part of the metric (A.1) to

$$-\left(1 + \frac{2U}{c^2}\right)(dx_1^2 + dx_2^2 + dx_3^2); \tag{A.7}$$

and this is in agreement with the solution used in this paper.

According to the solution obtained in § IV, $g_{0\alpha} = 0$, when there are no internal motions. Therefore, it remains to verify that to $O(c^{-4})$ Schwarzschild's solution agrees with the solution given in the text, namely,

$$g_{00} = 1 - \frac{2U}{c^2} + \frac{1}{c^4}(2U^2 - 4\Phi). \tag{A.8}$$

First we observe that the equation

$$\frac{d\mathfrak{M}}{d\xi} = 4\pi\rho\left(1 + \frac{\Pi}{c^2}\right)\xi^2 \tag{A.9}$$

determining $\mathfrak{M}(\xi)$, when expressed in terms of the variable r, becomes

$$\frac{d\mathfrak{M}}{dr} = 4\pi\rho\left[1 + \frac{1}{c^2}\left(\Pi + 3U + r\frac{dU}{dr}\right)\right]r^2. \tag{A.10}$$

The integral of this equation, after some reductions involving several integrations by parts, can be brought to the form

$$\mathfrak{M}(r)=M(r)+\frac{1}{c^2}\left[\delta\,\mathfrak{M}(r)+3UM(r)-\tfrac{1}{2}\frac{GM^2(r)}{r}+\tfrac{5}{2}\int_0^r\frac{GM^2(r)}{r^2}dr\right], \tag{A.11}$$

where

$$M(r)=4\pi\int_0^r\rho r^2dr \quad\text{and}\quad d\,\mathfrak{M}(r)=4\pi\int_0^r\rho\Pi r^2dr\,. \tag{A.12}$$

Next, considering equation (A.4) and transforming it to the variable r, we have

$$\tfrac{1}{2}c^2\frac{d\nu}{dr}=\left(1-\frac{U}{c^2}+\frac{1}{c^2}\,r\,\frac{dU}{dr}\right)\frac{G\mathfrak{M}(r)}{r^2}+\frac{4\pi G}{c^2}\,pr+\frac{2G^2M^2(r)}{c^2r^3}\,. \tag{A.13}$$

After some considerable reductions, the integral of this equation can be brought to the form

$$-\nu=\frac{2U}{c^2}+\frac{2}{c^4}\left[U^2+\int_r^\infty\frac{G\delta\,\mathfrak{M}(r)}{r^2}dr+3\int_r^\infty\frac{G^2M^2(r)}{r^3}dr\right.$$
$$\left.+\frac{5}{2r}\int_0^r\frac{G^2M^2(r)}{r^2}dr+4\pi G\int_r^\infty prdr\right]. \tag{A.14}$$

Inclusive of terms of $O(c^{-4})$ Schwarzschild's metric, therefore, gives

$$g_{00}=1+\nu+\tfrac{1}{2}\nu^2+\ldots=1-\frac{2U}{c^2}+\frac{2U^2}{c^4}-\frac{2}{c^4}[\ldots.]+O(c^{-6}), \tag{A.15}$$

where [. . . .] denotes the quantity in square brackets in equation (A.14). Agreement with the solution (A.8) requires, then, that

$$2\Phi=U^2+\int_r^\infty\frac{G\delta\,\mathfrak{M}(r)}{r^2}dr+3\int_r^\infty\frac{G^2M^2(r)}{r^3}dr+\frac{5}{2r}\int_0^r\frac{G^2M^2(r)}{r^2}dr$$
$$+4\pi G\int_r^\infty prdr\,. \tag{A.16}$$

But as defined in the text, Φ is determined, in the present case of spherical symmetry and hydrostatic equilibrium, by the equation

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Phi}{dr}\right)=-4\pi G\rho\left(U+\tfrac{1}{2}\Pi+\tfrac{3}{2}\frac{p}{\rho}\right); \tag{A.17}$$

and it can be verified that equation (A.16) does, indeed, represent the solution of equation (A.17).

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