

THE EQUILIBRIUM AND THE STABILITY OF THE DEDEKIND ELLIPSOIDS

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ABSTRACT

Figures of equilibrium of liquid masses with internal motions of uniform vorticity define the Dedekind sequence. It has been known for a long time that the Dedekind sequence of stationary ellipsoids is congruent to the Jacobian sequence of uniformly rotating ellipsoids. It is shown in this paper that the characteristic frequencies of oscillation of a Dedekind ellipsoid, belonging to the second harmonics, are identical with those of a Jacobi ellipsoid having the same figure; but that the points at which instability sets in by a mode of oscillation belonging to the third harmonics are different along the two sequences.

I. INTRODUCTION

In editing for publication a posthumous paper of Dirichlet (1860) on "Untersuchungen über ein Problem der Hydrodynamik," Dedekind (1860) proved a remarkable theorem to the effect that different states of fluid motion can preserve the same ellipsoidal figure of a self-gravitating liquid mass. And Love (1888) showed that the simplest example of Dedekind's theorem is provided by the geometrical congruence of the *Jacobian sequence* of uniformly rotating ellipsoids and the *Dedekind sequence* of stationary¹ ellipsoids with internal motions of uniform vorticity.

Since the Jacobian and the Dedekind sequences, in spite of their geometrical congruence, represent physically distinct systems, we should expect to distinguish them by differences in their normal modes of oscillation. It is, therefore, a noteworthy fact (which will be established in this paper) that the physical difference in the two sequences is *not* manifested among the modes of oscillation belonging to the second harmonics: the characteristic frequencies belonging to these modes are identical for a Jacobi and a Dedekind ellipsoid having the same figure. However, the difference between the two sequences is manifested among the modes of oscillation belonging to the third harmonics; and this fact will be established by showing that the points where instability by a mode of oscillation belonging to the third harmonics sets in are different along the two sequences.

This paper is devoted, then, to an examination of the equilibrium and the stability of the Dedekind ellipsoids. And the method of treatment, as in the recent examinations of the other classical sequences, will be based on the tensor virial equations of the second and the third orders (for a general summary of these methods, see Chandrasekhar 1964).

II. THE DEDEKIND ELLIPSOIDS AND THE SECOND-ORDER VIRIAL THEOREM

In his *A Treatise on Hydrodynamics* (Vol. 2, chap. xv), Basset (1888) gives an account of the related investigations of Dirichlet, Dedekind, and Riemann (see also Lamb 1932). It is, however, convenient to have the principal properties of the Dedekind ellipsoids derived in the context of this paper.

The state of fluid motions, characterized by a uniform vorticity ζ about the x_3 -axis, which we shall consider is

$$u_1 = -q\zeta x_2, \quad u_2 = (1 - q)\zeta x_1, \quad \text{and} \quad u_3 = 0, \quad (1)$$

¹ Stationary, that is, in an inertial frame of reference.

where q is some constant. The condition that this motion preserves the ellipsoidal boundary,

$$F(\mathbf{x}) = \frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \frac{x_3^2}{a_3^2} - 1 = 0, \quad (2)$$

is

$$u_j \frac{\partial F}{\partial x_j} = 2x_1x_2 \left(-\frac{q}{a_1^2} + \frac{1-q}{a_2^2} \right) \zeta = 0, \quad (3)$$

or

$$q = \frac{a_1^2}{a_1^2 + a_2^2}. \quad (4)$$

We shall write

$$u_1 = Q_1x_2 \quad \text{and} \quad u_2 = Q_2x_1, \quad (5)$$

where

$$Q_1 = -\frac{a_1^2}{a_1^2 + a_2^2} \zeta \quad \text{and} \quad Q_2 = +\frac{a_2^2}{a_1^2 + a_2^2} \zeta. \quad (6)$$

To obtain the condition that the ellipsoid is also in gravitational equilibrium, we shall make use of the second-order virial theorem. According to this theorem

$$\frac{d}{dt} \int_V \rho u_i x_j d\mathbf{x} = 2\mathfrak{T}_{ij} + \mathfrak{W}_{ij} + \Pi\delta_{ij}, \quad (7)$$

where

$$\mathfrak{T}_{ij} = \frac{1}{2} \int_V \rho u_i u_j d\mathbf{x}, \quad (8)$$

and

$$\mathfrak{W}_{ij} = -\frac{1}{2} \int_V \rho \mathfrak{W}_{ij} d\mathbf{x} = -\frac{1}{2} G \int_V \int_V \rho(\mathbf{x}) \rho(\mathbf{x}') \frac{(x_i - x_i')(x_j - x_j')}{|\mathbf{x} - \mathbf{x}'|^3} d\mathbf{x} d\mathbf{x}' \quad (9)$$

are the kinetic- and the potential-energy tensors, and

$$\Pi = \int_V p d\mathbf{x}. \quad (10)$$

(In the foregoing equations, the integrations are effected over the entire volume V occupied by the fluid.)

Under conditions of equilibrium, the virial theorem gives

$$2\mathfrak{T}_{ij} + \mathfrak{W}_{ij} = -\Pi\delta_{ij}. \quad (11)$$

For the case we are presently considering, the tensors \mathfrak{T}_{ij} and \mathfrak{W}_{ij} are diagonal; and moreover,

$$\mathfrak{T}_{11} = \frac{1}{2} Q_1^2 I_{22}, \quad \mathfrak{T}_{22} = \frac{1}{2} Q_2^2 I_{11}, \quad \text{and} \quad \mathfrak{T}_{33} = 0, \quad (12)$$

where I_{ij} denotes, as usual, the moment of inertia tensor. Equation (11), under these circumstances, gives

$$Q_1^2 I_{22} + \mathfrak{W}_{11} = Q_2^2 I_{11} + \mathfrak{W}_{22} = \mathfrak{W}_{33} = -\Pi. \quad (13)$$

The geometry of the equilibrium ellipsoids is determined by the equation

$$\frac{Q_1^2 I_{22}}{Q_2^2 I_{11}} = \frac{\mathfrak{W}_{33} - \mathfrak{W}_{11}}{\mathfrak{W}_{33} - \mathfrak{W}_{22}}; \quad (14)$$

and the associated vorticity is determined by the equation

$$Q_1^2 I_{22} - Q_2^2 I_{11} = \mathfrak{W}_{22} - \mathfrak{W}_{11}. \quad (15)$$

The expressions for the components of I_{ij} and \mathfrak{B}_{ij} for homogeneous ellipsoids have been given in an earlier paper (Chandrasekhar and Lebovitz 1962, eqs. [57] and [58]). With the aid of these expressions and of the definitions of Q_1 and Q_2 , equations (14) and (15) give

$$\frac{a_1^2}{a_2^2} = \frac{A_1 a_1^2 - A_3 a_3^2}{A_2 a_2^2 - A_3 a_3^2} \quad (16)$$

and

$$\frac{a_1^2 a_2^2}{(a_1^2 + a_2^2)^2} \frac{\zeta^2}{\pi G \rho a_1 a_2 a_3} = 2 \frac{A_1 a_1^2 - A_2 a_2^2}{(a_1^2 - a_2^2)} = 2B_{12}, \quad (17)$$

where the various index symbols (A_i , A_{ij} , B_{ij} , etc.) have their standard meanings.

We now observe that equation (16) is the same as the equation which determines the geometry of the Jacobi ellipsoid (cf. Chandrasekhar 1962, eq. [AI,3]); the two sequences of ellipsoids are, therefore, congruent. Moreover, by comparing equation (17) with the corresponding equation which determines the angular velocity of rotation Ω of the Jacobi ellipsoid (cf. Chandrasekhar 1962, eq. [AI, 7]), we obtain the relation

$$\zeta^2 = \frac{(a_1^2 + a_2^2)^2}{a_1^2 a_2^2} \Omega^2. \quad (18)$$

Equation (18) relates the Ω of a Jacobi ellipsoid with the ζ of a congruent Dedekind ellipsoid.

And finally, we may note the following alternative form of equation (17) which we shall find useful:

$$\frac{Q_1 Q_2}{\pi G \rho a_1 a_2 a_3} = -2B_{12}. \quad (19)$$

III. THE SECOND-ORDER VIRIAL EQUATIONS GOVERNING SMALL OSCILLATIONS ABOUT EQUILIBRIUM

Suppose that an equilibrium ellipsoid determined consistently with respect to equations (16) and (17) is slightly perturbed. Let the ensuing changes be described in terms of a Lagrangian displacement ξ . Then, a fluid element, originally at \mathbf{x} with a velocity \mathbf{u} , will find itself at $\mathbf{x} + \xi(\mathbf{x}, t)$ with a velocity $\mathbf{u} + \Delta\mathbf{u}$ where (to the first order)

$$\Delta u_i = \frac{\partial \xi_i}{\partial t} + u_j \frac{\partial \xi_i}{\partial x_j} = \frac{D \xi_i}{Dt}. \quad (20)$$

The virial equation (7), linearized to the first order in the displacement ξ , gives

$$\frac{d}{dt} \int_V \rho (\Delta u_i x_j + u_i \xi_j) d\mathbf{x} = 2\delta \mathfrak{T}_{ij} + \delta \mathfrak{B}_{ij} + \delta \Pi \delta_{ij}, \quad (21)$$

where $\delta \mathfrak{T}_{ij}$, $\delta \mathfrak{B}_{ij}$, and $\delta \Pi$ are the first variations in the respective quantities caused by the displacement ξ and the attendant change $\Delta\mathbf{u}$ in the velocity.

It is known that $\delta \mathfrak{B}_{ij}$ can be expressed in terms of the symmetrized virials

$$V_{ij} = V_{i;j} + V_{j;i} = \int_V \rho \xi_i x_j d\mathbf{x} + \int_V \rho \xi_j x_i d\mathbf{x}. \quad (22)$$

Thus (cf. Chandrasekhar and Lebovitz 1963*b*, eqs. [47] and [48])

$$\delta \mathfrak{B}_{ij} = -2B_{ij} V_{ij} \quad (i \neq j), \quad (23)$$

and

$$\delta \mathfrak{B}_{ii} = - (2B_{ii} - a_i^2 A_{ii}) V_{ii} + a_i^2 \sum_{l \neq i} A_{il} V_{ll} \quad (24)$$

(no summation over repeated indices in eqs. [23] and [24]).

(Note that in writing eqs. [23] and [24] a common factor $\pi G \rho a_1 a_2 a_3$ has been suppressed.) We shall now show how

$$\int_V \rho \Delta u_i x_j d\mathbf{x} \quad \text{and} \quad \delta \mathfrak{T}_{ij} \quad (25)$$

can be similarly expressed in terms of the virials $V_{i;j}$ in case u_i (in the unperturbed configuration) is a linear function of the coordinates.

Considering the time rate of change of $V_{i;j}$, we have

$$\frac{dV_{i;j}}{dt} = \frac{d}{dt} \int_V \rho \xi_i x_j d\mathbf{x} = \int_V \rho \frac{D\xi_i}{Dt} x_j d\mathbf{x} + \int_V \rho \xi_i u_j d\mathbf{x}, \quad (26)$$

or, in view of the relation (20),

$$\frac{dV_{i;j}}{dt} = \int_V \rho \Delta u_i x_j d\mathbf{x} + \int_V \rho \xi_i u_j d\mathbf{x}. \quad (27)$$

On the assumption that in the equilibrium configuration we have a linear relation of the form

$$u_j = Q_{jl} x_l, \quad (28)$$

where the Q_{jl} 's are certain constants, equation (27) gives

$$\int_V \rho \Delta u_i x_j d\mathbf{x} = \frac{dV_{i;j}}{dt} - Q_{jl} V_{i;l}. \quad (29)$$

By making use of equation (29), we can now express $\delta \mathfrak{T}_{ij}$ in terms of the virials. Thus,

$$\begin{aligned} 2\delta \mathfrak{T}_{ij} &= \int_V \rho \Delta u_i u_j d\mathbf{x} + \int_V \rho \Delta u_j u_i d\mathbf{x} \\ &= Q_{jl} \int_V \rho \Delta u_i x_l d\mathbf{x} + Q_{il} \int_V \rho \Delta u_j x_l d\mathbf{x} \\ &= Q_{jl} \left(\frac{dV_{i;l}}{dt} - Q_{lk} V_{i;k} \right) + Q_{il} \left(\frac{dV_{j;l}}{dt} - Q_{lk} V_{j;k} \right), \end{aligned} \quad (30)$$

or if \mathbf{Q}^2 represents the square of the matrix \mathbf{Q} , we can write

$$2\delta \mathfrak{T}_{ij} = Q_{jl} \frac{dV_{i;l}}{dt} + Q_{il} \frac{dV_{j;l}}{dt} - (\mathbf{Q}^2_{jk} V_{i;k} + \mathbf{Q}^2_{ik} V_{j;k}). \quad (31)$$

Now, making use of equations (29) and (31), we can rewrite equation (21) in the form

$$\begin{aligned} \frac{d^2 V_{i;j}}{dt^2} - Q_{jk} \frac{dV_{i;k}}{dt} + Q_{ik} \frac{dV_{j;k}}{dt} &= Q_{jk} \frac{dV_{i;k}}{dt} + Q_{ik} \frac{dV_{j;k}}{dt} - (\mathbf{Q}^2_{jk} V_{i;k} + \mathbf{Q}^2_{ik} V_{j;k}) \\ &\quad + \delta \mathfrak{B}_{ij} + \delta_{ij} \delta \Pi, \end{aligned} \quad (32)$$

or

$$\frac{d^2 V_{i;j}}{dt^2} - 2Q_{jk} \frac{dV_{i;k}}{dt} + \mathbf{Q}^2_{jk} V_{i;k} + \mathbf{Q}^2_{ik} V_{j;k} = \delta \mathfrak{B}_{ij} + \delta_{ij} \delta \Pi. \quad (33)$$

When the time dependence of the perturbation is of the form $e^{\lambda t}$, where λ is a parameter whose characteristic values are to be determined, equation (33) becomes

$$\lambda^2 V_{i;j} - 2\lambda Q_{jk} V_{i;k} + Q^2_{jk} V_{i;k} + Q^2_{ik} V_{j;k} = \delta \mathfrak{B}_{ij} + \delta_{ij} \delta \Pi. \quad (34)$$

And this equation must be supplemented by the condition

$$\frac{V_{11}}{a_1^2} + \frac{V_{22}}{a_2^2} + \frac{V_{33}}{a_3^2} = 0 \quad (35)$$

required by the solenoidal character of ξ .

For the particular cases of the Dedekind ellipsoids, the matrices Q and Q^2 have the forms

$$Q = \begin{pmatrix} 0 & Q_1 & 0 \\ Q_2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad Q^2 = \begin{pmatrix} Q_1 Q_2 & 0 & 0 \\ 0 & Q_1 Q_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (36)$$

and the explicit forms which equation (34) takes for the different components are relatively simple. Thus, the equations which are even in the index 3 are

$$\lambda^2 V_{3;3} = \delta \mathfrak{B}_{33} + \delta \Pi, \quad (37)$$

$$\lambda^2 V_{1;1} - 2\lambda Q_1 V_{1;2} + Q_1 Q_2 V_{11} = \delta \mathfrak{B}_{11} + \delta \Pi, \quad (38)$$

$$\lambda^2 V_{2;2} - 2\lambda Q_2 V_{2;1} + Q_1 Q_2 V_{22} = \delta \mathfrak{B}_{22} + \delta \Pi, \quad (39)$$

$$\lambda^2 V_{1;2} - 2\lambda Q_2 V_{1;1} + Q_1 Q_2 V_{12} = \delta \mathfrak{B}_{12} = -2B_{12} V_{12}, \quad (40)$$

$$\lambda^2 V_{2;1} - 2\lambda Q_1 V_{2;2} + Q_1 Q_2 V_{12} = \delta \mathfrak{B}_{12} = -2B_{12} V_{12}, \quad (41)$$

where we have substituted for $\delta \mathfrak{B}_{12}$ in accordance with equation (23). Similarly, the equations which are odd in the index 3 are

$$\lambda^2 V_{1;3} + Q_1 Q_2 V_{3;1} = \delta \mathfrak{B}_{13} = -2B_{13} V_{13}, \quad (42)$$

$$\lambda^2 V_{2;3} + Q_1 Q_2 V_{3;2} = \delta \mathfrak{B}_{23} = -2B_{23} V_{23}, \quad (43)$$

$$\lambda^2 V_{3;1} - 2\lambda Q_1 V_{3;2} + Q_1 Q_2 V_{3;1} = \delta \mathfrak{B}_{13} = -2B_{13} V_{13}, \quad (44)$$

$$\lambda^2 V_{3;2} - 2\lambda Q_2 V_{3;1} + Q_1 Q_2 V_{3;2} = \delta \mathfrak{B}_{23} = -2B_{23} V_{23}. \quad (45)$$

Since in writing equations (23) and (24) a common factor $\pi G \rho a_1 a_2 a_3$ has been suppressed, it is clear that in considering equations (37)–(45), we may alternatively suppose that λ^2 and ζ^2 are measured in the unit $\pi G \rho$ and that the index symbols are so “normalized” that $A_1 + A_2 + A_3 = 2$ (instead of $2/a_1 a_2 a_3$). This convention will be consistently adopted in the rest of this paper. On this convention, equation (19), for example, becomes

$$Q_1 Q_2 = -2B_{12}. \quad (46)$$

IV. THE CHARACTERISTIC FREQUENCIES OF OSCILLATION OF THE DEDEKIND ELLIPSOID BELONGING TO THE SECOND HARMONICS AND THEIR IDENTITY WITH THOSE OF THE JACOBI ELLIPSOID

Considering first the equations governing the even modes, we observe that in view of the relation (46), equations (40) and (41) become

$$\lambda^2 V_{1;2} = \lambda Q_2 V_{11} \quad \text{and} \quad \lambda^2 V_{2;1} = \lambda Q_1 V_{22}. \quad (47)$$

Excluding the possibility that λ may be zero—a possibility to which we shall return presently—we may conclude from equations (47) that

$$\lambda V_{1;2} = Q_2 V_{11} \quad \text{and} \quad \lambda V_{2;1} = Q_1 V_{22}. \quad (48)$$

Eliminating $V_{1;2}$ and $V_{2;1}$ from equations (38) and (39) with the aid of these last relations, we are left with

$$\left(\frac{1}{2}\lambda^2 + 2B_{12}\right)V_{11} = \delta\mathfrak{W}_{11} + \delta\Pi \quad (49)$$

and

$$\left(\frac{1}{2}\lambda^2 + 2B_{12}\right)V_{22} = \delta\mathfrak{W}_{22} + \delta\Pi, \quad (50)$$

where we have again made use of equation (46). Next eliminating $\delta\Pi$ between equations (37), (49), and (50) we obtain the pair of equations

$$\begin{aligned} \left(\frac{1}{2}\lambda^2 + 2B_{12}\right)V_{11} - \frac{1}{2}\lambda^2 V_{33} &= \delta\mathfrak{W}_{11} - \delta\mathfrak{W}_{33} \\ &= -(3B_{11} - B_{13})V_{11} + (B_{23} - B_{12})V_{22} + (3B_{33} - B_{13})V_{33} \end{aligned} \quad (51)$$

and

$$\begin{aligned} \left(\frac{1}{2}\lambda^2 + 2B_{12}\right)V_{22} - \frac{1}{2}\lambda^2 V_{33} &= \delta\mathfrak{W}_{22} - \delta\mathfrak{W}_{33} \\ &= -(3B_{22} - B_{23})V_{22} + (B_{13} - B_{12})V_{11} + (3B_{33} - B_{23})V_{33}, \end{aligned} \quad (52)$$

where we have substituted for $\delta\mathfrak{W}_{11} - \delta\mathfrak{W}_{22}$ and $\delta\mathfrak{W}_{11} - \delta\mathfrak{W}_{33}$ in accordance with equation (24). Equations (51) and (52) must be supplemented by equation (35) which express the solenoidal condition on ξ ; and these three equations lead directly to the characteristic equation

$$\left\| \begin{array}{ccc} \frac{1}{2}\lambda^2 + 2B_{12} + 3B_{11} - B_{13} & B_{12} - B_{23} & -\frac{1}{2}\lambda^2 - 3B_{33} + B_{13} \\ B_{12} - B_{13} & \frac{1}{2}\lambda^2 + 2B_{12} + 3B_{22} - B_{23} & -\frac{1}{2}\lambda^2 - 3B_{33} + B_{23} \\ \frac{1}{a_1^2} & \frac{1}{a_2^2} & \frac{1}{a_3^2} \end{array} \right\| = 0. \quad (53)$$

A somewhat simpler form of equation (53) is

$$\left\| \begin{array}{ccc} \frac{1}{2}\lambda^2 + 2B_{12} + 3B_{11} - B_{13} & B_{12} - B_{23} & 3(B_{11} + B_{12} - B_{33}) - B_{23} \\ B_{12} - B_{13} & \frac{1}{2}\lambda^2 + 2B_{12} + 3B_{22} - B_{23} & 3(B_{22} + B_{12} - B_{33}) - B_{13} \\ \frac{1}{a_1^2} & \frac{1}{a_2^2} & \frac{1}{a_1^2} + \frac{1}{a_2^2} + \frac{1}{a_3^2} \end{array} \right\| = 0. \quad (54)$$

Remembering that for the Jacobi ellipsoid $\Omega^2 = 2B_{12}$ (in the present normalization), we observe that equation (54) is identical with the equation which governs the corresponding modes of the Jacobi ellipsoid (cf. Chandrasekhar and Lebovitz 1963*a*, eq. [AI, 3]).

Returning to equation (47) and the possibility of a non-trivial root $\lambda = 0$, we observe that we do indeed have such a root belonging to a proper solution associated with the possibility

$$V_{11} = V_{22} = V_{33} = 0 \quad \text{and} \quad V_{12} \neq 0. \quad (55)$$

In the existence of such a neutral mode, the Dedekind sequence has the same behavior as the Jacobian sequence.

Turning next to the odd equations (42)–(45), we can combine them so as to give the two pairs of equations

$$(\lambda^2 + 4B_{13})V_{13} - 2Q_1(\lambda V_{3;2} - Q_2V_{3;1}) = 0, \quad (56)$$

$$(\lambda^2 + 4B_{23})V_{23} - 2Q_2(\lambda V_{3;1} - Q_1V_{3;2}) = 0, \quad (57)$$

and

$$\lambda^2(V_{1;3} - V_{3;1}) + 2\lambda Q_1V_{3;2} = 0, \quad (58)$$

$$\lambda^2(V_{2;3} - V_{3;2}) + 2\lambda Q_2V_{3;1} = 0. \quad (59)$$

Excluding the possibility of a non-trivial root $\lambda = 0$, we can rewrite equations (58) and (59) in the forms

$$\lambda V_{13} = 2(\lambda V_{3;1} - Q_1V_{3;2}) \quad (60)$$

and

$$\lambda V_{23} = 2(\lambda V_{3;2} - Q_2V_{3;1}). \quad (61)$$

With the aid of these last two equations, equations (56) and (57) become

$$(\lambda^2 + 4B_{13})V_{13} - \lambda Q_1V_{23} = 0 \quad (62)$$

and

$$(\lambda^2 + 4B_{23})V_{23} - \lambda Q_2V_{13} = 0. \quad (63)$$

We first observe that equations (60)–(63) can be satisfied by setting

$$V_{13} = V_{1;3} + V_{3;1} = 0, \quad V_{23} = V_{2;3} + V_{3;2} = 0, \quad (64)$$

$$\lambda^2 = Q_1Q_2 = -2B_{12} \quad \text{and} \quad \frac{V_{3;1}}{V_{3;2}} = \frac{Q_1}{\lambda} = \frac{\lambda}{Q_2}. \quad (65)$$

Besides this root $\lambda^2 = Q_1Q_2$, we have also the roots of the characteristic equation

$$(\lambda^2 + 4B_{13})(\lambda^2 + 4B_{23}) + 2\lambda^2B_{12} = 0, \quad (66)$$

which follows from equations (62) and (63) if V_{13} and V_{23} do not vanish identically. The roots of equation (66) are

$$\lambda^2 = -(2B_{13} + 2B_{23} + B_{12}) \pm [(2B_{13} + 2B_{23} + B_{12})^2 - 16B_{13}B_{23}]^{1/2}; \quad (67)$$

and, again, this equation is identical with the equation which governs the corresponding odd modes of the Jacobi ellipsoid (cf. Chandrasekhar and Lebovitz 1963*a*, eq. [AI, 2]). Also, the root $\lambda^2 = Q_1Q_2$, which the Dedekind ellipsoid allows, coincides with the root $\lambda^2 = -\Omega^2$ which the Jacobi ellipsoid allows. Thus, *there is a complete identity of the characteristic frequencies of oscillation belonging to the second harmonics along the Dedekind and the Jacobian sequences.*

V. THE ISOLATION OF THE NEUTRAL POINT BELONGING TO THE THIRD HARMONICS ALONG THE DEDEKIND SEQUENCE

Poincaré's (1885) discovery of the neutral point along the Jacobian sequence, where the sequence of the pear-shaped equilibrium configurations branches off, has generally been considered as one of the major accomplishments in the theory of the ellipsoidal figures of equilibrium of liquid masses. Nevertheless, the question whether a similar neutral point occurs along the congruent Dedekind sequence does not appear to have

been considered, or indeed, even raised.² However, the identity of the characteristic frequencies of oscillation belonging to the second harmonics along the two sequences makes the question of more than academic interest: for the isolation of the neutral point belonging to the third harmonics along the Dedekind sequence will enable us to establish whether the physical difference between the two sequences is manifested among these higher modes of oscillation.

In this section, we shall show how the various integral properties provided by the third-order virial theorem enable us to isolate the neutral point belonging to the third harmonics along the Dedekind sequence; and it will appear that it occurs at a point different from where the pear-shaped sequence branches off along the (congruent) Jacobian sequence.

Now the third-order virial theorem gives

$$\frac{d}{dt} \int_V \rho u_i x_j x_k dx = 2(\mathfrak{T}_{ij;k} + \mathfrak{T}_{ik;j}) + \mathfrak{W}_{ij;k} + \mathfrak{W}_{ik;j} + \Pi_k \delta_{ij} + \Pi_j \delta_{ik}, \quad (68)$$

where

$$\mathfrak{T}_{ij;k} = \frac{1}{2} \int_V \rho u_i u_j x_k dx,$$

$$\mathfrak{W}_{ij;k} = -\frac{1}{2} \int_V \rho \mathfrak{B}_{ij} x_k dx, \quad (69)$$

and

$$\Pi_k = \int_V \dot{p} x_k dx.$$

Under conditions of equilibrium, equation (68) gives

$$2(\mathfrak{T}_{ij;k} + \mathfrak{T}_{ik;j}) + \mathfrak{W}_{ij;k} + \mathfrak{W}_{ik;j} = -\Pi_k \delta_{ij} - \Pi_j \delta_{ik}. \quad (70)$$

It has been explained on earlier occasions (cf. Chandrasekhar 1962, 1963*a*, *b*, and 1964) that the first variations of all the integral relations which follow from the virial equations of the various orders must vanish at a neutral point. In particular, at a neutral point belonging to the third harmonics, the first variations of the integral relations which follow from equation (70) must vanish for a Lagrangian displacement which leads to a set of third-order virials

$$V_{i;jk} = \int_V \rho \xi_i x_j x_k dx \quad (71)$$

which are not all zero. And as in the cases of the Maclaurin, the Jacobi, and the Roche sequences, the five relations which are odd in the index 1 and even in the indices 2 and 3 will suffice to isolate the neutral point; and the relations to be considered in this instance are

$$\mathfrak{W}_{12;2} + 2\mathfrak{T}_{12;2} = 0, \quad (72)$$

$$\mathfrak{W}_{13;3} + 2\mathfrak{T}_{13;3} = 0, \quad (73)$$

$$\mathfrak{W}_{11;1} + 2\mathfrak{T}_{11;1} = -\Pi_1, \quad (74)$$

$$\mathfrak{W}_{21;2} + \mathfrak{W}_{22;1} + 2(\mathfrak{T}_{21;2} + \mathfrak{T}_{22;1}) = -\Pi_1, \quad (75)$$

$$\mathfrak{W}_{31;3} + \mathfrak{W}_{33;1} + 2(\mathfrak{T}_{31;3} + \mathfrak{T}_{33;1}) = -\Pi_1. \quad (76)$$

² It is remarkable that no reference to the sequences of Dedekind and Riemann (1860; see also Basset 1888) is to be found in any of the writings of Poincaré, Darwin, or Jeans.

By suitably combining equations (72)–(76), we can obtain two relations, besides (72) and (73), which are independent of Π_1 . Thus, defining (cf. Chandrasekhar and Lebovitz 1963*a*, eq. [62])

$$S_{ijj} = -4\mathfrak{B}_{ij;j} - 2\mathfrak{B}_{jj;i} + 2\mathfrak{B}_{ii;i}$$

and

$$R_{ijj} = -4\mathfrak{T}_{ij;j} - 2\mathfrak{T}_{jj;i} + 2\mathfrak{T}_{ii;i},$$

(no summation over repeated indices)

we readily obtain the relations

$$S_{122} + 2R_{122} = 0 \quad \text{and} \quad S_{133} + 2R_{133} = 0. \quad (78)$$

And the conditions we seek to satisfy in isolating the neutral point are

$$J_1 = -2\delta\mathfrak{B}_{12;2} - 4\delta\mathfrak{T}_{12;2} = 0, \quad (79)$$

$$J_2 = -2\delta\mathfrak{B}_{13;3} - 4\delta\mathfrak{T}_{13;3} = 0, \quad (80)$$

$$J_3 = \delta S_{122} + 2\delta R_{122} = 0, \quad (81)$$

and

$$J_4 = \delta S_{133} + 2\delta R_{133} = 0, \quad (82)$$

where $\delta\mathfrak{B}_{12;2}$, etc., are the changes in the respective quantities induced by an appropriate Lagrangian displacement.

It has been shown in an earlier paper (Chandrasekhar and Lebovitz 1963*a*, Table 2) that the first variations of $\delta\mathfrak{B}_{12;2}$, $\delta\mathfrak{B}_{13;3}$, δS_{122} , and δS_{133} can all be expressed as linear combinations of the three symmetrized virials

$$V_{111} = 3V_{1\ 11} = 3 \int_V \rho \xi_1 x_1^2 dx,$$

$$V_{122} = V_{1;22} + 2V_{2\ 12} = \int_V \rho \xi_1 x_2^2 dx + 2 \int_V \rho \xi_2 x_1 x_2 dx, \quad (83)$$

and

$$V_{133} = V_{1;33} + 2V_{3;13} = \int_V \rho \xi_1 x_3^2 dx + 2 \int_V \rho \xi_3 x_1 x_3 dx.$$

We shall now show how the particular $\delta\mathfrak{T}_{ij;k}$'s which occur in the relations (79)–(82) can also be expressed in terms of these same five virials.

We first prove the following lemma.

LEMMA: *If the macroscopic fluid motions in an equilibrium configuration are of the form*

$$u_i = Q_{ik} x_k, \quad (84)$$

where Q is a certain constant matrix, then for a quasi-static Lagrangian displacement ξ

$$\int_V \rho \Delta u_i x_l x_m dx = -(Q_{ln} V_{i;mn} + Q_{mn} V_{i;ln}), \quad (85)$$

where Δu_i is the Lagrangian change in u_i resulting from the displacement ξ .

PROOF: For a general time-dependent Lagrangian displacement, it follows from the definition of the virial $V_{i;lm}$ that

$$\frac{dV_{i;lm}}{dt} = \int_V \rho \frac{D\xi_i}{Dt} x_l x_m dx + \int_V \rho \xi_i u_l x_m dx + \int_V \rho \xi_i x_l u_m dx. \quad (86)$$

Making use of equations (20) and (84), we obtain

$$\frac{dV_{i;lm}}{dt} = \int_V \rho \Delta u_i x_l x_m d\mathbf{x} + Q_{ln} \int_V \rho \xi_i x_m x_n d\mathbf{x} + Q_{mn} \int_V \rho \xi_i x_l x_n d\mathbf{x}, \quad (87)$$

or

$$\frac{dV_{i;lm}}{dt} = \int_V \rho \Delta u_i x_l x_m d\mathbf{x} + Q_{ln} V_{i;mn} + Q_{mn} V_{i;ln}. \quad (88)$$

For a quasi-static deformation $dV_{i;lm}/dt = 0$ and equation (88) reduces to the result stated.

With the aid of the lemma, we can express $\delta\mathfrak{T}_{ij;k}$ in terms of the virials. Thus,

$$\begin{aligned} 2\delta\mathfrak{T}_{ij;k} &= \int_V \rho \Delta u_i u_j x_k d\mathbf{x} + \int_V \rho \Delta u_j u_i x_k d\mathbf{x} + \int_V \rho u_i u_j \xi_k d\mathbf{x} \\ &= Q_{jl} \int_V \rho \Delta u_i x_l x_k d\mathbf{x} + Q_{il} \int_V \rho \Delta u_j x_l x_k d\mathbf{x} + Q_{im} Q_{jn} \int_V \rho \xi_k x_m x_n d\mathbf{x}. \end{aligned} \quad (89)$$

Now making use of equation (85), we have

$$2\delta\mathfrak{T}_{ij;k} = -Q_{jl}(Q_{ln} V_{i;kn} + Q_{kn} V_{i;ln}) - Q_{il}(Q_{ln} V_{j;kn} + Q_{kn} V_{j;ln}) + Q_{im} Q_{jn} V_{k;mn}. \quad (90)$$

It is to be particularly noted that, unlike the $\delta\mathfrak{B}_{ij;k}$'s, the $\delta\mathfrak{T}_{ij;k}$'s cannot be expressed in terms of the symmetrized virials only: the unsymmetrized virials occur explicitly.

For the case we are presently considering the only non-vanishing elements of \mathcal{Q} are (cf. eq. [36])

$$Q_{12} = Q_1 \quad \text{and} \quad Q_{21} = Q_2, \quad (91)$$

and we find from equation (90) that

$$2\delta\mathfrak{T}_{11;1} = -2Q_1 Q_2 V_{1;11} - Q_1^2 V_{1;22}, \quad (92)$$

$$2\delta\mathfrak{T}_{22;1} = -4Q_1 Q_2 V_{2;12} + Q_2^2 V_{1;11}, \quad (93)$$

$$2\delta\mathfrak{T}_{33;1} = 0, \quad (94)$$

$$2\delta\mathfrak{T}_{12;2} = -Q_1 Q_2 (V_{1;22} + V_{2;21}) - Q_2^2 V_{1;11}, \quad (95)$$

and

$$2\delta\mathfrak{T}_{13;3} = -Q_1 Q_2 V_{3;31}. \quad (96)$$

And combining the foregoing equations appropriately, we find

$$2\delta R_{122} = (2Q_2^2 - 4Q_1 Q_2) V_{1;11} + (4Q_1 Q_2 - 2Q_1^2) V_{1;22} + 12Q_1 Q_2 V_{2;12}, \quad (97)$$

and

$$2\delta R_{133} = -4Q_1 Q_2 V_{1;11} - 2Q_1^2 V_{1;22} + 4Q_1 Q_2 V_{3;31}. \quad (98)$$

Equations (95)–(98) together with the known expansions of $\delta\mathfrak{B}_{12;2}$, $\delta\mathfrak{B}_{13;3}$, δS_{122} , and δS_{133} (given in Chandrasekhar and Lebovitz 1963*a*, Table 2) enable us to express J_1, \dots, J_4 (defined in eqs. [79]–[82]) as linear combinations of the five virials $V_{1;11}$, $V_{1;22}$, $V_{2;12}$, $V_{1;33}$, and $V_{3;13}$. Thus, we may write

$$\begin{aligned} J_i &= \langle i|1;11\rangle V_{1;11} + \langle i|1;22\rangle V_{1;22} + \langle i|2;12\rangle V_{2;12} \\ &\quad + \langle i|1;33\rangle V_{1;33} + \langle i|3;13\rangle V_{3;13} \quad (i = 1, \dots, 4), \end{aligned} \quad (99)$$

where $\langle i|1;11\rangle$, etc., are certain "matrix elements" which are known. Equation (99) must be supplemented by the further condition

$$\frac{3}{a_1^2} V_{1;11} + \frac{1}{a_2^2} (V_{1;22} + 2V_{2;12}) + \frac{1}{a_3^2} (V_{1;33} + 2V_{3;13}) = 0, \tag{100}$$

which is an expression of the solenoidal character of ξ (cf. Chandrasekhar and Lebovitz 1963*a*, Sec. VI).

Equations (99) and (100) provide five linear homogeneous equations for the five virials; and a necessary condition for the occurrence of a neutral point is the vanishing of the determinant of equations (99) and (100) at that point. By evaluating the determinant with the constants of the congruent Jacobi ellipsoids provided in an earlier paper (Chandrasekhar 1962, Appendix I), the point where it will vanish was determined by interpolation. Then, the constants of the equilibrium ellipsoid, appropriate to the point so determined, were evaluated directly and the fact that all the conditions, requisite for the ellipsoid to be critical, were indeed satisfied, was explicitly verified.

TABLE 1
THE CONSTANTS OF THE DEDEKIND AND THE JACOBI ELLIPSOIDS
WHICH ARE NEUTRALLY STABLE*

| Parameter | Dedekind Ellipsoid | Jacobi Ellipsoid | Parameter | Dedekind Ellipsoid | Jacobi Ellipsoid |
|--|--------------------|------------------|--------------------------------|--------------------|------------------|
| $\text{Cos}^{-1} a_3/a_1$ | 69°4875 | 69°8166 | $2B_{12}$ | | |
| $\text{Sin}^{-1} [(a_1^2 - a_2^2) \div (a_1^2 - a_3^2)]^{1/2}$ | 73°3537 | 73°903 | ($= -Q_1 Q_2$ or Ω^2) | +0 287815 | 0 28400 |
| a_2/a_1 | 0 441331 | 0 432159 | Q_1 | -1 215604 | |
| a_3/a_1 | 0 350412 | 0 345026 | Q_2 | +0 236767 | |
| \bar{a}_1 | 1 86303 | 1 885826 | A_1^* | +0 2638224 | 0 258301 |
| \bar{a}_2 | 0 822212 | 0 814976 | A_2^* | +0 7595720 | 0 764728 |
| \bar{a}_3 | +0 652826 | 0 650659 | A_3^* | +0 9766054 | 0 976971 |

* The parameters \bar{a}_i and A_i^* have the following meanings: $\bar{a}_i = a_i / (a_1 a_2 a_3)^{1/3}$ and $A_i^* = a_1 a_2 a_3 A_i$

The constants of the critical Dedekind ellipsoid are listed in Table 1 and are contrasted with those of the Jacobi ellipsoid at the point of bifurcation. The two critical ellipsoids are seen to be distinct; and this distinctness is a manifestation of difference between the Dedekind and the Jacobian sequences.

The characteristic vector $(V_{1;11}, V_{1;22}, V_{2;12}, V_{1;33}, V_{3;13})$, which solves equations (99) and (100) when its determinant vanishes (as it does at the point we have determined), is found to be

$$\begin{aligned} V_{1;11} &= +10.2450, V_{1;22} = +2.49310, V_{2;12} = -5.01287, \\ V_{1;33} &= -1.0252, \text{ and } V_{3;13} = 1 \text{ (as arbitrarily set)}. \end{aligned} \tag{101}$$

To show that a neutral point along the Dedekind sequence does occur at the point we have determined (via the imposition of certain necessary conditions), we have only to verify that a Lagrangian displacement satisfying all the requirements of a proper solution does exist with the virials having the values (101). And this fact can be verified as follows.

First, we observe that the linearized third-order virial equations together with the appropriate solenoidal conditions on ξ suffice to determine all the characteristic frequencies belonging to the third harmonics uniquely. Therefore, the corresponding proper solu-

tions for ξ are uniquely determined by the specification of the eighteen third-order virials $V_{i;jk}$ and conversely. This unique determination of the proper solutions by the virials is possible only if the solutions for ξ are quadratic in the coordinates. And, moreover, it is clear that the conditions we have used to isolate the neutral point are precisely those which follow from the general virial equations which determine the characteristic frequencies.

With the knowledge that the proper solutions are quadratic in the coordinates, we can at once write down the form of ξ for which the only non-vanishing virials are those listed in (101). We must have

$$\begin{aligned}\xi_1 &= (a + \beta)x_1^2 + \gamma x_2^2 + \delta x_3^2 + \kappa, \\ \xi_2 &= -2\alpha x_1 x_2, \quad \text{and} \quad \xi_3 = -2\beta x_1 x_3,\end{aligned}\tag{102}$$

where $\alpha, \beta, \gamma, \delta$, and κ are five constants. (Note that the solenoidal requirement on ξ is satisfied by the chosen form.) For the displacement (102) we find

$$\begin{aligned}V_{1;11} &= a_1^2[3(a + \beta)a_1^2 + \gamma a_2^2 + \delta a_3^2 + 7\kappa], \\ V_{1;22} &= a_2^2[(a + \beta)a_1^2 + 3\gamma a_2^2 + \delta a_3^2 + 7\kappa], \\ V_{1;33} &= a_3^2[(a + \beta)a_1^2 + \gamma a_2^2 + 3\delta a_3^2 + 7\kappa], \\ V_{2;12} &= -2\alpha a_1^2 a_2^2, \quad \text{and} \quad V_{3;13} = -2\beta a_1^2 a_3^2,\end{aligned}\tag{103}$$

where a common factor $4\pi a_1 a_2 a_3 / 105$ has been suppressed. It should be noted here that the condition,

$$\langle \xi_1 \rangle = 7[(a + \beta)a_1^2 + \gamma a_2^2 + \delta a_3^2 + 5\kappa] = 0,\tag{104}$$

which follows from the requirement (which we may impose) that the center of mass of the ellipsoid does not move during the perturbation, is equivalent to the imposition of the solenoidal condition in the form (100).

Equations (103) now enable us to determine the constants $\alpha, \beta, \gamma, \delta$, and κ if these virials are to have the values listed in equations (101). We find

$$\alpha = 12.8685, \quad \beta = -4.07205, \quad \gamma = 51.7212, \quad \delta = -4.0782, \quad \text{and} \quad \kappa = -3.6739.\tag{105}$$

The corresponding proper solution for ξ is (with the normalization $\kappa = -1$)

$$\begin{aligned}\xi_1 &= 2.3943 x_1^2 + 14.078 x_2^2 - 1.1100 x_3^2 - 1, \\ \xi_2 &= -7.0053 x_1 x_2, \quad \text{and} \quad \xi_3 = +2.21674 x_1 x_3.\end{aligned}\tag{106}$$

With this explicit determination of ξ , the demonstration, that a neutral point does occur at the point isolated, is complete.

VI. CONCLUDING REMARKS

The remarkable behavior of the Dedekind ellipsoids, with respect to their modes of oscillation and manner of instability, suggests that the behavior of the more general ellipsoids of Riemann (which include the irrotational ellipsoids of particular interest; cf. Basset 1888) be similarly examined. We shall return to this examination in another paper.

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Note added December 7, 1964.—Since this paper was written, the properties of the associated Riemann sequences have been fully investigated; and the analysis discloses many interesting relationships between these sequences and those of Maclaurin and Jacobi. Further, analogues of Riemann's irrotational sequence exist for compressible masses and disclose new possibilities for the occurrence of genuine triaxial configurations.

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