NON-RADIAL OSCILLATIONS OF GASEOUS MASSES

S. Chandrasekhar and Norman R. Lebovitz
University of Chicago
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ABSTRACT

In this paper the fundamental frequencies of non-radial oscillations of polytropic gas spheres, belonging to spherical harmonics of orders $l = 1$ and 2, are evaluated, in a "second approximation," by a variational method. Also, the value of the ratio of the specific heats γ is determined for which an accidental degeneracy occurs between the fundamental modes of radial oscillation and non-radial oscillation belonging to $l = 2$; it is found that this value of γ varies from 1 6 for a homogeneous compressible sphere to 1 5719 for a polytrope of index $n = 3$.

I. INTRODUCTION

In a recent paper (Chandrasekhar and Lebovitz 1963; this paper[1] will be referred to hereafter as "Paper I") the non-radial oscillations of gaseous masses belonging to the spherical harmonics of orders $l = 1$ (in a "second approximation") and $l = 3$ (in a "first approximation") were considered on the basis of the virial equations of orders 1 and 3 and trial functions of suitable forms for the Lagrangian displacement. The modes of oscillation belonging to $l = 2$ (in a first approximation) had been considered earlier (Chandrasekhar and Lebovitz 1962*a*, *c*); since their evaluation in a second approximation by the method of Paper I would have required the use of the virial equations of order 4, as well, it was not attempted. However, soon afterward, a general variational principle governing the non-radial oscillations of gaseous masses and belonging to the spherical harmonics of the different orders was established (Chandrasekhar 1964; this paper will be referred to hereafter as "Paper II"). In this paper the variational principle will be used to evaluate the modes belonging to $l = 2$ in a second approximation with the principal object of determining, more precisely than hitherto, the value of the ratio of the specific heats γ for which an accidental degeneracy occurs between the fundamental modes of radial oscillation and non-radial oscillation belonging to $l = 2$. The facts that such a degeneracy must occur and, further, that it may lie at the base of the beat phenomenon exhibited by the β Canis Majoris stars have been pointed out (Chandrasekhar and Lebovitz 1962*a*, *b*, and *d*); and the value $\gamma = 1.6$ (independently of the structure of the configuration) for the occurrence of such a degeneracy was deduced by equating the first approximations to the characteristic frequencies given by the two formulae (Ledoux 1945; Chandrasekhar and Lebovitz 1962*a*)

$$\sigma_R{}^2 = (3\gamma - 4)\frac{|\mathfrak{W}|}{I} \qquad \text{and} \qquad \sigma_S{}^2 = \tfrac{4}{5}\frac{|\mathfrak{W}|}{I}, \tag{1}$$

where $\mathfrak{W}$ denotes the gravitational potential energy and I the moment of inertia of the configuration. By restricting ourselves to polytropic gas spheres and evaluating $\sigma_R{}^2$ and $\sigma_S{}^2$ in a second approximation, we shall determine the dependence of γ (for which $\sigma_R{}^2 = \sigma_S{}^2$) on the density distribution in the configuration.

II. THE CHARACTERISTIC EQUATION IN THE SECOND APPROXIMATION

As shown in Paper II (Sec. III) the non-radial oscillations of gaseous masses, belonging to spherical harmonics of a particular order l, are determined by two radial functions

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[1] The following errata for this paper may be noted here: In eq. (66) the entry opposite 3.5 should be 0.054251 (in place of 0.054834); the last of the equations in (87) should read $\sigma^4 - 0.12097 (\gamma - 1.04641)\sigma^2 + 0.0011820 (\gamma - 1.2784) = 0$; and finally in Table 1 on p. 199 the entry opposite $\int \theta^n \xi^6 d\xi$ in the last column should read 160 393 (in place of 158.689).

ψ and χ. And the variational principle, which expresses σ^2 as a functional of ψ and χ, requires that σ^2 be stationary with respect to arbitrary, infinitesimal variations of ψ and χ compatible with the boundary conditions.

Now it can be deduced from Pekeris' (1939) exact treatment of the non-radial oscillations of a homogeneous compressible sphere that the proper solutions for ψ and χ, in that case, are of the forms

$$\psi = A r^{l+1} + B r^{l+3} \quad \text{and} \quad \chi = A r^{l+1} + C r^{l+3} . \tag{2}$$

We shall accordingly assume for ψ and χ these forms and treat the constants A, B, and C, as variational parameters.

It will be observed that we have made the coefficients of r^{l+1} equal in the assumed forms of ψ and χ. This equality in the coefficients follows, as it has been pointed out to us by Professor Paul H. Roberts, from the requirement that div $\boldsymbol{\xi}$, as given by equation (33) of Paper II, should behave like r^l at the origin[2]—a fact which becomes apparent when we examine the behavior at the origin of the expression for $\delta\rho$ given by equation (35) of Paper II.

By substituting for ψ and χ the chosen forms in Paper II, equation (41), we find that the characteristic equation for σ^2 becomes

$$|\boldsymbol{M}| = 0 , \tag{3}$$

where $\boldsymbol{M}$ is a symmetric matrix whose elements are

$$M_{11} = \sigma^2 \frac{2l+1}{l} \rho_{2l} - 2(2l-1)(l+1)p_{2l-2} - V_{2l+2}$$
$$+ 4\pi G \int_0^R r^{2l}[F_{l+1} - (l+1)f]^2 dr ,$$

$$M_{12} = \sigma^2 \rho_{2l+2} - (2l+1)(l+1)p_{2l} - V_{2l+4}$$
$$+ 4\pi G \int_0^R r^{2l}F_{l+3}[F_{l+1} - (l+1)f] dr ,$$

$$M_{13} = \sigma^2 \frac{l+3}{l} \rho_{2l+2} - (2l+1)(l+3)p_{2l} \tag{4}$$
$$- 4\pi G(l+3) \int_0^R r^{2l} g[F_{l+1} - (l+1)f] dr ,$$

$$M_{22} = \sigma^2 \rho_{2l+4} - \gamma(l+3)^2 p_{2l+2} - V_{2l+6} + 4\pi G \int_0^R r^{2l}F^2_{l+3} dr ,$$

$$M_{23} = (l+3)[(l+3)\gamma - (2l+3)]p_{2l+2} - 4\pi G(l+3) \int_0^R r^{2l}F_{l+3}g\, dr ,$$

$$M_{33} = \sigma^2 \frac{(l+3)^2}{l(l+1)} \rho_{2l+4} - \gamma(l+3)^2 p_{2l+2} + 4\pi G(l+3)^2 \int_0^R r^{2l} g^2 dr ,$$

[2] This circumstance, that $\psi - \chi$ must behave like r^{l+3} as $r \to 0$, makes the reduction of the variational principle given by eq. (41) of Paper II to the form of a determinant of a 2×2 matrix (by varying ψ and χ independently of each other) strictly incorrect. However, it appears that the determinantal form retains a certain usefulness in providing an adequate approximation (see n. 4 on p. 1522) and clarifying the origin of the two physically distinct classes of modes.

and the following abbreviations have been used:

$$\rho_m = \int_0^R \rho\, r^m d r, \qquad p_m = \int_0^R p\, r^m d r, \tag{5}$$

$$V_m = \int_0^R \frac{1}{\rho}\frac{dp}{dr}\frac{d}{dr}(\rho\, r^m)\frac{dr}{r^2} = \int_0^R \frac{d\mathfrak{B}}{dr}\frac{d}{dr}(\rho\, r^m)\frac{dr}{r^2} \tag{6}$$

$$f(r) = \int_r^R \frac{d s}{s}\rho(s), \qquad g(r) = \int_r^R d s\, s\rho(s), \tag{7}$$

and

$$F_{l+1}(r) = \int_r^R \frac{d s}{s^{l+1}}\frac{d}{d s}(\rho\, s^{l+1}) = -\rho + (l+1)f(r),$$

$$F_{l+3}(r) = \int_r^R \frac{d s}{s^{l+1}}\frac{d}{d s}(\rho\, s^{l+3}) = -\rho r^2 + (l+1)g(r). \tag{8}$$

The integrals which are explicitly written out in equations (4) can be reduced by integrations by parts and expressed in terms of the moments,

$$f_m = \int_0^R f\rho\, r^m d r, \qquad g_m = \int_0^R g\rho\, r^m d r. \tag{9}$$

We find

$$\int_0^R r^{2l} f^2 d r = \frac{2}{2l+1}f_{2l},$$

$$\int_0^R r^{2l} f g\, d r = \frac{1}{2l+1}(f_{2l+2} + g_{2l}),$$

$$\int_0^R r^{2l} g^2 d r = \frac{2}{2l+1}g_{2l+2},$$

$$\int_0^R r^{2l} F_{l+1} f\, d r = \frac{1}{2l+1}f_{2l},$$

$$\int_0^R r^{2l} F_{l+1} g\, d r = \frac{1}{2l+1}[(l+1)f_{2l+2} - l g_{2l}],$$

$$\int_0^R r^{2l} F_{l+3} f\, d r = \frac{1}{2l+1}[(l+1)g_{2l} - l f_{2l+2}], \tag{10}$$

$$\int_0^R r^{2l} F_{l+3} g\, d r = \frac{1}{2l+1}g_{2l+2},$$

$$\int_0^R r^{2l} F^2_{l+1} d r = \int_0^R r^{2l}\rho^2 d r - \frac{2l(l+1)}{2l+1}f_{2l},$$

$$\int_0^R r^{2l} F_{l+1} F_{l+3} d r = \int_0^R r^{2l+2}\rho^2 d r - \frac{l(l+1)}{2l+1}(f_{2l+2} + g_{2l}),$$

$$\int_0^R r^{2l} F^2_{l+3} d r = \int_0^R r^{2l+4}\rho^2 d r - \frac{2l(l+1)}{2l+1}g_{2l+2}.$$

The integral defining V_m (eq. [6]) can also be reduced by integrations by parts and by making use of Poisson's equation and the equation governing equilibrium. Thus,

$$V_m = \int_0^R \frac{1}{r^4}\left(r^2 \frac{d\mathfrak{V}}{dr}\right)\frac{d}{dr}(\rho\, r^m)\, dr$$

$$= -\int_0^R \rho\, r^m \left[\frac{1}{r^4}\frac{d}{dr}\left(r^2\frac{d\mathfrak{V}}{dr}\right) - \frac{4}{r^3}\frac{d\mathfrak{V}}{dr}\right] dr$$

$$= \int_0^R \rho\, r^m \left(\frac{4\pi G\rho}{r^2} + \frac{4}{r^3}\frac{1}{\rho}\frac{dp}{dr}\right) dr$$

$$= 4\pi G \int_0^R \rho^2 r^{m-2} dr - 4(m-3)p_{m-4} \qquad (m \geq 4). \tag{11}$$

Using equations (10) and (11), we find that the expressions for the matrix elements of $\boldsymbol{M}$ become

$$M_{11} = \sigma^2 \frac{2l+1}{l}\, \rho_{2l} - 2(2l-1)(l-1)p_{2l-2},$$

$$M_{12} = \sigma^2 \rho_{2l+2} - (2l+1)(l-3)p_{2l} - 4\pi G(l+1)g_{2l},$$

$$M_{13} = \sigma^2 \frac{l+3}{l}\, \rho_{2l+2} - (2l+1)(l+3)p_{2l} + 4\pi G(l+3)g_{2l},$$

$$M_{22} = \sigma^2 \rho_{2l+4} - [\gamma(l+3)^2 - 4(2l+3)]p_{2l+2} - 8\pi G \frac{l(l+1)}{2l+1} g_{2l+2}, \tag{12}$$

$$M_{23} = (l+3)[(l+3)\gamma - (2l+3)]p_{2l+2} - 4\pi G \frac{l+3}{2l+1} g_{2l+2},$$

$$M_{33} = \sigma^2 \frac{(l+3)^2}{l(l+1)}\, \rho_{2l+4} - \gamma(l+3)^2 p_{2l+2} + 8\pi G \frac{(l+3)^2}{2l+1} g_{2l+2}.$$

The approximation considered by Cowling (1942) and others of neglecting the variations in the gravitational potential during the oscillations is equivalent, in the present context, to ignoring the integrals over F_{l+1}, F_{l+3}, f, and g in the expressions for the matrix elements of $\boldsymbol{M}$ given in equations (4). Ignoring, then, these terms and substituting for the remaining terms in V_m in accordance with equation (11), we obtain the simpler expressions

$$M_{11}{}^{(a)} = \sigma^2 \frac{2l+1}{l}\, \rho_{2l} - 2(2l-1)(l-1)p_{2l-2} - 4\pi G \int_0^R \rho^2 r^{2l} dr,$$

$$M_{12}{}^{(a)} = \sigma^2 \rho_{2l+2} - (2l+1)(l-3)p_{2l} - 4\pi G \int_0^R \rho^2 r^{2l+2} dr,$$

$$M_{13}{}^{(a)} = \sigma^2 \frac{l+3}{l}\, \rho_{2l+2} - (2l+1)(l+3)p_{2l},$$

$$M_{22}{}^{(a)} = \sigma^2 \rho_{2l+4} - [\gamma(l+3)^2 - 4(2l+3)]p_{2l+2} - 4\pi G \int_0^R \rho^2 r^{2l+4} dr, \tag{13}$$

$$M_{23}{}^{(a)} = (l+3)[(l+3)\gamma - (2l+3)]p_{2l+2}.$$

$$M_{33}{}^{(a)} = \sigma^2 \frac{(l+3)^2}{l(l+1)}\, \rho_{2l+4} - \gamma(l+3)^2 p_{2l+2},$$

The Characteristic Equation for the Radial Mode of Oscillation in the Second Approximation

The variational expression for the characteristic frequencies of radial oscillation is given in Paper II (eq. [49]). With the trial function

$$\psi = ar^3 + br^5 \tag{14}$$

with the two variational parameters a and b, we obtain the characteristic equation

$$\left\| \begin{array}{cc} \sigma^2\rho_4 - 3(3\gamma - 4)p_2 & \sigma^2\rho_6 - 5(3\gamma - 4)p_4 \; . \\ \sigma^2\rho_6 - 5(3\gamma - 4)p_4 & \sigma^2\rho_8 - (25\gamma - 28)p_6 \end{array} \right\| = 0 \; , \tag{15}$$

where ρ_m and p_m continue to have the same meanings as in equations (4). It is known (cf. Ledoux and Walraven 1958) that this "second approximation" gives the frequency of the fundamental mode of radial oscillation to well within 5 per cent in most cases of physical interest.

III. APPLICATION TO POLYTROPES

We shall use the characteristic equations derived in the preceding section to evaluate the frequencies of the fundamental modes of the radial and the non-radial oscillations of the polytropes.

When r, ρ, and p are expressed in terms of the usual Emden variables ξ and θ (cf. Chandrasekhar and Lebovitz 1962c, eq. [8]) and σ^2 is measured in the unit $4\pi G\rho_c/(n + 1)$ (where n is the polytropic index), it can be readily verified that the elements of the secular matrix continue to be given by equations (12) if we replace $4\pi G$, wherever it occurs, by $n + 1$ and define ρ_m, p_m, and g_m in terms of the corresponding dimensionless variables as follows:

$$\rho_m = \int_0^{\xi_1} \theta^n \xi^m d\xi \; , \qquad p_m = \int_0^{\xi_1} \theta^{n+1} \xi^m d\xi \; ,$$

and
$$\tag{16}$$

$$g_m = \int_0^{\xi_1} g \, \theta^n \xi^m d\xi = \int_0^{\xi_1} d\xi \, \xi^m \theta^n \int_\xi^{\xi_1} d\eta \, \eta \, \theta^n(\eta) \; ,$$

where ξ_1 is the first zero of θ.

In the case under consideration, g_m can be expressed directly in terms of ρ_m and p_m; thus,

$$g(\xi) = \int_\xi^{\xi_1} \theta^n(\eta) \, \eta \, d\eta = -\int_\xi^{\xi_1} \left(\eta \frac{d^2\theta}{d\eta^2} + 2 \frac{d\theta}{d\eta} \right) d\eta = \xi \frac{d\theta}{d\xi} + \theta + \xi_1 | \theta_1' | \; , \tag{17}$$

where θ_1' is the value of $d\theta/d\xi$ at ξ_1, and

$$g_m = \xi_1 | \theta_1' | \rho_m + p_m + \int_0^{\xi_1} \theta^n \frac{d\theta}{d\xi} \xi^{m+1} d\xi = \xi_1 | \theta_1' | \rho_m + \frac{n - m}{n + 1} p_m \; . \tag{18}$$

Also, it may be noted that

$$\rho_2 = \xi_1^2 | \theta_1' | \qquad \text{and} \qquad p_2 = \frac{n + 1}{5 - n} \xi_1^3 | \theta_1' |^2 \; . \tag{19}$$

In Table 1, we list the values of the various integrals which are needed for the determination (in the second approximation) of the lowest modes belonging to $l = 0, 1$, and 2.

a) The Convective Instability of the Polytropes for
$\gamma < 1 + 1/n$ by Modes Belonging to $l = 1$

We have already shown in Paper I, by an application of the virial equations, that the manifestation of the convective instability of the polytropes for $\gamma < 1 + 1/n$, by modes

belonging to $l = 1$, can be explicitly demonstrated: the critical value of γ for marginal stability predicted by the (approximate) theory differs from $1 + 1/n$ by less than 0.7 per cent for $n \leq 3.5$. It is evident that the present characteristic equation $|M| = 0$ for $l = 1$ must predict the *same* critical values of γ (for the different polytropes) as were derived in Paper I: for the trial function assumed for the Lagrangian displacement in both the treatments are the same; the characteristic equations which follow cannot, therefore, be different.[3] And the fact that the present characteristic equation $|M| = 0$ for $l = 1$ leads to the same critical values for γ as were derived in Paper I can, indeed, be verified directly.[4]

TABLE 1

A TABLE OF INTEGRALS

INTEGRAL	n				
	1 0	1 5	2 0	3 0	3 5
ρ_2	3 14159	2 71406	2 41105	2 01824	1 89056
ρ_4 .	12 1567	11 1197	10 6110	10 8516	11 7454
ρ_6 .	62 8853	64 9770	71 7372	109 748	160 39
ρ_8	379 112	460 821	617 802	1625 30	3585 89
p_0 .	1 41815	1 34001	1 27421	1 16855	1 12515
p_2	1 57080	1 44002	1 33547	1 18120	1 12446
p_4	4 38231	4 23148	4 17017	4 31761	4 56828
p_6	17 4550	18 7887	21 0149	30 4548	40 9726
g_2	2 35619	1 72803	1 33547	0 885900	0 749639
g_4	5 58325	4 02839	3 09731	2 09611	1 82105
g_6	19 2477	14 4461	11 7151	9 2748	9 0368

b) The "Kelvin Modes" Belonging to $1 = 2$; and the Modes Exhibiting Convective Instability

Using the same matrix equation $|M| = 0$ for $l = 2$, we have determined, with the aid of the integrals listed in Table 1, the lowest characteristic values for polytropes with the indices 1, 1.5, 2, 3, and 3.5. The calculated characteristic values together with the constants of the corresponding characteristic vector $(1, B/A, C/A)$ are listed in Table 2.

We notice the remarkably slight dependence of σ^2 on γ for $n \leq 2$. We also observe that for moderate central condensations $(n \leq 1.5)$ the approximation $\psi \simeq \chi \simeq r^{l+1}$

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[3] The fact that the two methods, the variational and the virial, must lead to the same characteristic equation (albeit by different routes) can be seen as follows: in the variational method, the characteristic equation for σ^2 appropriate to a trial function of the form

$$\xi_i = L_{i;jk}x_jx_k + L_i$$

(assumed in Paper I) follows from inserting this expression for ξ in a certain Hermitian form $(\xi, H\xi)$, where $H\xi$ gives the time-dependence of ξ for small departures from equilibrium in accordance with the linearized equations of motion, and making the result stationary for small variations of the constants $L_{i;jk}$ and L_i; whereas, in the virial method, we start with the three zero-order and the eighteen second-order x_jx_k-moments of the exact non-linear equations of motion, linearize these moment equations for small departures from equilibrium, and *then* make the same assumption for the Lagrangian displacement. The two sets of homogeneous equations which follow for the constants $L_{i;jk}$ and L_i, clearly, cannot differ (cf. Clement [1964] where the analysis exhibiting this equivalence is set out in full in another context).

[4] We find, for example, that the equation $|M| = 0$ for $l = 1$ leads to the values $\gamma = 1\ 996,\ 1\ 662,$ 1.494, 1.327, and 1.279 for the marginal stability of the polytropes $n = 1, 1\ 5, 2, 3$, and 3.5, respectively; and these values should be compared with 1.995, 1.661, 1.493, 1 326, and 1.278 derived in Paper I (eq [88]).

provides a satisfactory representation.[5] As we have seen in Paper II (Sec. IVb) the *assumption* $\psi = \chi = r^{l+1}$ leads to an expression for σ^2 which is *exactly* analogous to Kelvin's formula for the non-radial modes of an incompressible sphere. On this account, it would seem proper to describe the modes obtained in this section as the "Kelvin modes."

TABLE 2

THE SQUARES OF THE CHARACTERISTIC FREQUENCIES AND RELATED CONSTANTS BELONGING TO THE FUNDAMENTAL MODES OF RADIAL AND NON-RADIAL ($l = 2$) OSCILLATIONS

(σ^2 Is Listed in the Unit $4\pi G\rho_c/[n + 1]$)

n	γ	RADIAL		NON-RADIAL ($l=2$)		
		σ^2	a_3/a_1	σ^2	B/A	C/A
1 0	1 55	0 25002	0 01594	0 30305	0 04938	+0 02537
	1 60	30726	01928	30332	04604	+ 02287
	1 65	36437	02252	30357	04293	+ 02055
	5/3	38338	02357	30365	04194	+ 01982
1 5	1 55	24711	02615	29302	07114	+ 02754
	1 60	30287	03206	29390	06429	+ 02284
	1 65	35825	03790	29475	05810	+ 01859
	5/3	37663	03983	29503	05616	+ 01727
2 0	1 55	23357	03925	26848	09219	+ 02462
	1 60	28473	04910	27062	08041	+ 01753
	1 65	33509	05922	27271	07020	+ 01143
	5/3	35171	06265	27339	06709	+ 00959
3 0	1 55	17088	09509	18184	14326	+ 01203
	1 60	20195	13147	18845	11701	+ 00195
	1 65	23125	17596	19503	09691	− 00538
	5/3	24069	19299	19721	09123	− 00738
3 5	1 55	11740	20435	12214	18433	+ 00533
	1 60	13442	34067	12990	14854	− 00457
	1 65	15028	59891	13780	12296	− 01097
	5/3	0 15538	0 74444	0 14044	0 11597	−0 01259

To test how good the approximation is of ignoring the variations in the gravitational potential during the oscillation, we have evaluated σ^2 from the simplified equation $|M^{(a)}| = 0$ (cf. eq. [13]) for the case $n = 3$, $l = 2$, and $\gamma = 1.6$ and $\frac{5}{3}$. We find

$$\sigma^2 = \begin{cases} 0.2264 & (\gamma = 1.6), \\ 0.2398 & (\gamma = \frac{5}{3}); \end{cases} \tag{20}$$

and these values should be compared with $\sigma^2 = 0.1884$ and 0.1972 listed in Table 2. It would not appear from this comparison that the approximation is a very good one under the circumstances considered.

In addition to the Kelvin modes we have considered, there exist also modes which exhibit the convective instability of the polytropes for $\gamma < 1 + 1/n$. Indeed, we find

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[5] If we had varied ψ and χ independently of each other, then for the same assumed form of the trial functions we should have found $\sigma^2 = 0.1800, 0.1868, 0$ 1935, and 0.1957 for the case $n = 3$ and $\gamma = 1.55$, 1.60, 1 65, and $\frac{5}{3}$, respectively; and these values should be contrasted with the values $\sigma^2 = 0.1818, 0.1885$, 0.1950, and 0.1972 listed in Table 2 derived after satisfying the proper boundary condition, $\psi - \chi \to$ constant r^{l+3} as $r \to 0$.

from the same secular equation $|M| = 0$ and $l = 2$ that a neutral mode occurs when (cf. Paper I, eq. [88]; also n. 3 on p. 1522)

$$\gamma = \begin{cases} 1.9946 & \text{for} & n = 1.0, \\ 1.6588 & \text{for} & n = 1.5, \\ 1.4899 & \text{for} & n = 2.0, \\ 1.3188 & \text{for} & n = 3.0, \\ 1.2686 & \text{for} & n = 3.5. \end{cases} \quad (21)$$

The departures of these values from $1 + 1/n$ is a measure of the accuracy attained by the present manner of application of the variational method.

c) The Radial Modes

The characteristic equation (15) governing the radial modes has also been solved for the same values of γ and n for which the Kelvin modes for $l = 2$ have been determined. The results of the calculations are included in Table 2.

TABLE 3

THE VALUES OF γ FOR WHICH THE ACCIDENTAL DEGENERACY OCCURS AND THE CONSTANTS OF THE CORRESPONDING MODES

n	γ	RADIAL		NON-RADIAL $(l=2)$		
		σ^2	a_3/a_1	σ^2	B/A	C/A
0 .	1 6000	0 26667	0	0 26667	0	0
1 0	1 5965	30330	0 01906	30330	0 04626	+0 02304
1 5	1 5918	29376	03109	29376	06537	+ 02358
2 0	1 5855	27001	04622	27001	08364	+ 01947
3 0	1 5719	18473	1101	18473	13240	+ 00790
3 5	1 5745	0 12592	0 2619	0 12591	0 16526	−0 00005

d) The Value of γ for Which the Accidental Degenercy Occurs

By interpolating among the values of σ^2 listed in Table 2, the values of γ, for which σ^2 for the radial mode and the non-radial mode belonging to $l = 2$ are equal, were determined. The results of this interpolation are given in Table 3.

We observe that, in contrast to what follows from the solutions in the first approximation, the value of γ for which the accidental degeneracy occurs depends on the density distribution in the configuration. However, the dependence is not very pronounced.

For the polytrope $n = 3$, the accidental degeneracy occurs for $\gamma = 1.572$. While this value of γ differs from 1.6 by only 0.028, the stellar requirements for the coincidence of the two frequencies are appreciably different: on the assumption that differences in γ arise from different admixtures of a monatomic gas and radiation, the change in the effective ratio of the specific heats from 1.6 to 1.572 means that $1 - \beta =$ (radiation pressure/total pressure) changes from 0.0532 to 0.0868; and this change in $(1 - \beta)$ implies, on the standard model, a change in $(M\mu^2/\odot)$ from 4.62 to 6.36. It would therefore appear that the suggested interpretation of the beat phenomenon exhibited by the β Canis Majoris stars will not be found inadequate on this account: its validity will have to be tested on other grounds.

We are greatly indebted to Miss Donna D. Elbert for her assistance with all the numerical work in connection with the preparation of this paper.

APPENDIX

AN ALTERNATIVE FORMULATION OF THE VARIATIONAL PRINCIPLE

In deriving the variational principle in Paper II, the assumption was made that the density vanishes on the boundary; and this assumption was explicitly used in the further reductions Nevertheless, it was found, after suitable transformations of the basic equations, that the variational principle gives correctly the characteristic frequencies of the Kelvin modes of an incompressible sphere (cf. Paper II, Sec. IVb, eq. [53]). The fact that such "suitable transformations" are possible suggests that there is an alternative formulation of the variational principle which is valid without the assumption that the density vanishes on the surface. We shall show that such an alternative formulation exists which, moreover, permits the density to be discontinuous in the interior.

It is convenient for our present purposes to adopt the Lagrangian, instead of the Eulerian, formulation of the perturbation equations.

If $\Delta F(x)$ and $\delta F(x)$ denote, respectively, the Lagrangian and the Eulerian changes in a variable $F(x)$, then (cf. Lebovitz 1961, eq. [21])

$$\Delta F(x) = \delta F(x) + \xi \cdot \mathrm{grad}\, F(x), \tag{A.1}$$

where $\xi(x)$ is the Lagrangian displacement. Also, it may be verified that the Lagrangian change in the gradient of $F(x)$ is given by

$$\Delta\left(\frac{\partial F}{\partial x_i}\right) = \frac{\partial \Delta F}{\partial x_i} - \frac{\partial F}{\partial x_k}\frac{\partial \xi_k}{\partial x_i}. \tag{A.2}$$

Using the foregoing equations, we find that the Lagrangian form of the equations governing the perturbations are (cf. Paper I, eqs. [9]–[13])

$$\sigma^2 \rho\, \xi_i = \frac{\partial \Delta p}{\partial x_i} - \Delta\rho\, \frac{\partial \mathfrak{B}}{\partial x_i} - \rho\, \frac{\partial \Delta \mathfrak{B}}{\partial x_i}, \tag{A.3}$$

$$\Delta\rho = -\rho\, \mathrm{div}\, \xi, \qquad \Delta p = -\gamma p\, \mathrm{div}\, \xi, \tag{A.4}$$

and

$$\Delta \mathfrak{B} = \delta \mathfrak{B} + \xi \cdot \mathrm{grad}\, \mathfrak{B}, \tag{A.5}$$

where

$$\delta \mathfrak{B}(x) = G \int_V \rho(x')\, \xi_i(x')\, \frac{\partial}{\partial x_i'}\, \frac{1}{|x - x'|}\, dx'; \tag{A.6}$$

and the boundary condition is that Δp vanish on the boundary.

Multiplying equation (A.3) by ξ_i, contracting, and integrating over the volume occupied by the fluid, we obtain

$$\sigma^2 \int_V \rho\, |\xi|^2 dx = \int_V \gamma p\, (\mathrm{div}\, \xi)^2 dx + \int_V \xi_i\, \frac{\partial p}{\partial x_i}\, \mathrm{div}\, \xi\, dx$$

$$- \int_V \rho\, \xi_i\, \frac{\partial}{\partial x_i}\left(\xi_k\, \frac{\partial \mathfrak{B}}{\partial x_k}\right) dx - \int_V \rho\, \xi_i\, \frac{\partial \delta \mathfrak{B}}{\partial x_i}\, dx, \tag{A.7}$$

where we have performed an integration by parts in the first term on the right-hand side of equation (A.7) and have used equations (A.4) and (A.5) as well as the equilibrium condition $\mathrm{grad}\, p = \rho\, \mathrm{grad}\, \mathfrak{B}$.

It can be verified that equation (A.7) provides a variational base for determining the characteristic frequencies. The verification is sufficiently similar to that given in Paper II (Sec. II) that it is omitted here.

We shall now specialize equation (A.7) to the case when the equilibrium configuration is spherically symmetric. One can, without loss of generality, restrict consideration to a single spherical harmonic $Y_l^m(\vartheta, \varphi)$. The Lagrangian displacement may then be written (cf. Paper II, eq. [31]; see also eqs. [32] and [33])

$$\xi_r = \frac{\psi(r)}{r^2} Y_l^m(\vartheta, \varphi), \qquad \xi_\vartheta = \frac{1}{l(l+1)r} \frac{d\chi(r)}{dr} \frac{\partial Y_l^m(\vartheta, \varphi)}{\partial \vartheta},$$

and

$$\xi_\varphi = \frac{1}{l(l+1)r \sin \vartheta} \frac{d\chi(r)}{dr} \frac{\partial Y_l^m(\vartheta, \varphi)}{\partial \varphi}. \tag{A.8}$$

These expressions for the components of ξ must be substituted into equation (A.7) and the integrations over the angles performed. This reduction has already been performed for the left-hand side and the first two terms on the right-hand side of equation (A.7) (Paper II, eq. [38]) with the result

$$\sigma^2 \int_0^R \left[\frac{\psi^2}{r^2} + \frac{1}{l(l+1)} \left(\frac{d\chi}{dr} \right)^2 \right] dr = \int_0^R \gamma p \left[\frac{d}{dr}(\psi - \chi) \right]^2 \frac{dr}{r^2}$$

$$+ \int_0^R \frac{dp}{dr} \psi \frac{d}{dr}(\psi - \chi) \frac{dr}{r^2} - \int_V \rho \xi_i \frac{\partial}{\partial x_i} \left(\xi_k \frac{\partial \mathfrak{B}}{\partial x_k} \right) dx - \int_V \rho \xi_i \frac{\partial \delta \mathfrak{B}}{\partial x_i} dx, \tag{A.9}$$

where we have suppressed a factor $N_{lm} = 4\pi(l + |m|)!/(2l + 1)(l - |m|)!$ since it will ultimately appear in every term of equation (A.10) and hence may be canceled.

To facilitate evaluating the final two terms of equation (A.9), we state the following lemma: Let ξ be given by equation (A.8), $F(r)$ be arbitrary, and let S_1 denote the unit sphere and dS_1 the associated element of area. Then

$$\iint_{S_1} \xi \cdot \text{grad}\, [F(r) Y_\lambda^\mu(\vartheta, \varphi)]\, dS_1 = \frac{N_{lm}}{r^2} \left(\psi \frac{dF}{dr} + F \frac{d\chi}{dr} \right) \delta_{l\lambda} \delta_{m\mu}. \tag{A.10}$$

The proof is omitted, since it is an easy consequence of a known result (cf. Chandrasekhar 1961, p. 625).

Turning now to the third term on the right-hand side of equation (A.10), we find

$$-\int_V \rho \xi \cdot \text{grad}\left(\xi_r \frac{d\mathfrak{B}}{dr} \right) dx = -\int_0^R dr\, r^2 \rho(r) \iint_{S_1} dS_1 \xi \cdot \text{grad}\left(\frac{\psi}{r^2} \frac{d\mathfrak{B}}{dr} Y_l^m \right)$$

$$= -N_{lm} \int_0^R \rho(r) \left[\psi \frac{d}{dr}\left(\frac{\psi}{r^4} r^2 \frac{d\mathfrak{B}}{dr} \right) + \frac{\psi}{r^2} \frac{d\mathfrak{B}}{dr} \frac{d\chi}{dr} \right] dr$$

$$= -N_{lm} \int_0^R \rho(r) \left(-4 \frac{\psi^2}{r^3} \frac{d\mathfrak{B}}{dr} + \frac{\psi}{r^2} \frac{d\psi}{dr} \frac{d\mathfrak{B}}{dr} - 4\pi G\rho \frac{\psi^2}{r^2} + \frac{\psi}{r^2} \frac{d\mathfrak{B}}{dr} \frac{d\chi}{dr} \right) dr \tag{A.11}$$

$$= -N_{lm} \int_0^R \left[-4 \frac{\psi^2}{r^3} \frac{dp}{dr} + \frac{\psi}{r^2} \frac{d}{dr}(\psi + \chi) \frac{dp}{dr} - 4\pi G\rho^2 \frac{\psi^2}{r^2} \right] dr.$$

To reduce the final term in equation (A.9) we first find $\delta\mathfrak{V}$ with the aid of equation (A.7) and the known expansion of $|\boldsymbol{x}-\boldsymbol{x}'|^{-1}$ in spherical harmonics,

$$\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|} = \sum_{l=0}^{\infty} f_l(r,r') \sum_{m=-l}^{l} \frac{(l-|m|)!}{(l+|m|)!} Y_l{}^m(\vartheta,\varphi) Y_l{}^m(\vartheta',\varphi'), \quad \text{(A.12)}$$

where

$$f_l(r,r') = \begin{cases} r'^l/r^{l+1} & \text{if} \quad r'<r, \\ r^l/r'^{l+1} & \text{if} \quad r'>r. \end{cases} \quad \text{(A.13)}$$

Applying the lemma (eq [A.10]), we find

$$\begin{aligned}
\delta\mathfrak{V}(\boldsymbol{x}) &= G\int_V \rho(\boldsymbol{x}')\xi_i(\boldsymbol{x}')\frac{\partial}{\partial x_i'}\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}d\boldsymbol{x}' \\
&= G\int_0^R dr'r'^2\rho(r')\iint_{S_1}dS_1\xi_i(\boldsymbol{x}')\frac{\partial}{\partial x_i'}\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|} \\
&= GN_{lm}\frac{(l-|m|)!}{(l+|m|)!}Y_l{}^m(\vartheta,\varphi)\int_0^R \rho(r')\left[\psi(r')\frac{\partial f_l(r,r')}{\partial r'}\right. \\
&\quad\left.+\frac{d\chi(r')}{dr'}f_l(r,r')\right]dr' = \frac{4\pi G}{2l+1}Y_l{}^m(\vartheta,\varphi)\left[\frac{J_l(r)}{r^{l+1}}-r^lK_l(r)\right],
\end{aligned} \quad \text{(A.14)}$$

where, in virtue of equation (A.13),

$$J_l(r) = \int_0^r \rho(s)s^l\left[l\frac{\psi(s)}{s}+\frac{d\chi(s)}{ds}\right]ds$$

and

$$K_l(r) = \int_r^R \frac{\rho(s)}{s^{l+1}}\left[(l+1)\frac{\psi(s)}{s}-\frac{d\chi(s)}{ds}\right]ds. \quad \text{(A.15)}$$

We cannot eliminate both J_l and K_l from the final formula, but we can eliminate one of them, say J_l. For this purpose we need the following formula, which is an easy consequence of equations (A.15):

$$\frac{d}{dr}\left(\frac{J_l}{r^{l+1}}-r^lK_l\right) = -\frac{l+1}{r^{l+2}}J_l-lr^{l-1}K_l+(2l+1)\frac{\rho\psi}{r^2}. \quad \text{(A.16)}$$

Using equations (A 14) and (A 16) and the lemma (A.10), we now find

$$\begin{aligned}
\int\rho\boldsymbol{\xi}\cdot\operatorname{grad}\delta\mathfrak{V}d\boldsymbol{x} &= \frac{4\pi G}{2l+1}\int_0^R dr\rho r^2\iint_{S_1}dS_1\boldsymbol{\xi}\cdot\operatorname{grad}\left[\left(\frac{J_l}{r^{l+1}}-r^lK_l\right)Y_l{}^m\right] \\
&= \frac{4\pi GN_{lm}}{2l+1}\int_0^R \rho\left[\psi\frac{d}{dr}\left(\frac{J_l}{r^{l+1}}-r^lK_l\right)+\frac{d\chi}{dr}\left(\frac{J_l}{r^{l+1}}-r^lK_l\right)\right]dr \\
&= 4\pi GN_{lm}\int_0^R \rho^2\frac{\psi^2}{r^2}dr+\frac{4\pi GN_{lm}}{2l+1}\int_0^R\left(J_l\frac{dK_l}{dr}-K_l\frac{dJ_l}{dr}\right)dr \\
&= 4\pi GN_{lm}\int_0^R \rho^2\frac{\psi^2}{r^2}dr-\frac{8\pi GN_{lm}}{2l+1}\int_0^R K_l\frac{dJ_l}{dr}dr \\
&= 4\pi GN_{lm}\int_0^R \rho^2\frac{\psi^2}{r^2}dr-\frac{8\pi GN_{lm}}{2l+1}\int_0^R \rho K_l r^l\left(l\frac{\psi}{r}+\frac{d\chi}{dr}\right)dr.
\end{aligned} \quad \text{(A.17)}$$

On account of equations (A.11) and (A.17), equation (A.9) becomes

$$\sigma^2 \int_0^R \rho \left[\frac{\psi^2}{r^2} + \frac{1}{l(l+1)} \left(\frac{d\chi}{dr} \right)^2 \right] dr = \int_0^R \gamma p \left[\frac{d}{dr}(\psi - \chi) \right]^2 \frac{dr}{r^2}$$

$$+ 2 \int_0^R \frac{dp}{dr} \frac{\psi}{r^2} \left(2 \frac{\psi}{r} - \frac{d\chi}{dr} \right) dr + \frac{8\pi G}{2l+1} \int_0^R \rho K_l r^l \left(l \frac{\psi}{r} + \frac{d\chi}{dr} \right) dr ,$$

(A.18)

which is the desired formula.

It may be useful to note that the formula (A.18) can be deduced also from equation (41) of Paper II, if one *continues* to assume that $\rho(R) = 0$. Straightforward manipulations involving integration by parts and explicit use of the condition $\rho(R) = 0$ alter two of the terms of that equation as follows:

$$\int_0^R \frac{1}{\rho} \frac{dp}{dr} \frac{d}{dr}(\rho \psi^2) \frac{dr}{r^2} = \int_0^R \frac{d\mathfrak{B}}{dr} \frac{d}{dr}(\rho \psi^2) \frac{dr}{r^2}$$

$$= 4 \int_0^R \frac{dp}{dr} \frac{\psi^2}{r^3} dr + 4\pi G \int_0^R \rho^2 \frac{\psi^2}{r^2} dr$$

(A.19)

and

$$\int_0^R r^{2l} \left[\int_r^R \frac{\delta \rho(s)}{s^{l-1}} ds \right]^2 dr = \int_0^R \rho^2 \frac{\psi^2}{r^2} dr + \int_0^R r^{2l} K_l \left(K_l - 2 \frac{\rho \psi}{r^{l+1}} \right) dr$$

$$= \int_0^R \rho^2 \frac{\psi^2}{r^2} dr - \frac{2}{2l+1} \int_0^R \rho K_l r^l \left(l \frac{\psi}{r} + \frac{d\chi}{dr} \right) dr ,$$

(A.20)

where

$$\delta \rho(r) = \frac{1}{r^2} \left[\frac{d}{dr}(\rho \psi) - \rho \frac{d\chi}{dr} \right],$$

and we have made the observation that

$$\int_r^R \frac{\delta \rho(s)}{s^{l-1}} ds = -\frac{\rho \psi}{r^{l+1}} + K_l(r).$$

(A.21)

Using equations (A.19) and (A.20) in equation (41) of Paper II, we recover equation (A.18).

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