THE EQUILIBRIUM AND THE STABILITY OF THE DARWIN ELLIPSOIDS

S. Chandrasekhar
University of Chicago
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ABSTRACT

Darwin's problem is concerned with the equilibrium and the stability of synchronously rotating homogeneous masses under their mutual gravitational and tidal interactions. The problem is solved consistently, in a method of approximation due to Jeans, in two special cases: the case when one of the two components is of infinitesimal mass compared to the other and the case when the two components are of equal mass and congruent. In the former case, the problem insofar as the equilibrium and the stability of the infinitesimal mass is concerned, is hardly distinguishable from Roche's simpler problem in which the distorting mass is treated as a rigid sphere. However, in Darwin's formulation, the distorting mass (in the case considered) is a Maclaurin spheroid; and a principal result is that Darwin's problem has no solution when the eccentricity of the spheroid exceeds a certain maximum value (=0.40504).

In the case of the congruent components, the maximum angular velocity of orbital rotation, the distance of closest approach, and the Roche limit (where the equilibrium ellipsoid can be deformed into a neighboring equilibrium ellipsoid by a quasi-static, infinitesimal, solenoidal, ellipsoidal displacement), all occur at different points along the sequence; and instability, by a mode of natural oscillation of either component by itself, sets in at a still different point. It appears, moreover, that of the two figures of equilibrium one obtains (at each separation) those with the greater elongations overlap; all the physically realizable equilibrium ellipsoids are therefore stable with respect to their individual natural oscillations. The bearing of these results on the concepts of "limiting stability" and "partial stability" due to Darwin and Jeans is briefly examined.

I. INTRODUCTION

The classical investigations (and to a large extent the only extant investigations) on the equilibrium and the stability of double-star systems are those of Darwin (1906). Darwin's investigations are summarized by Jeans, in an original account, in his *Problems of Cosmogony and Stellar Dynamics* (1919; see pp. 55–64, 85, 86, and 134–136). However, the conclusions to which Darwin and Jeans arrived are vitiated by too loose a usage of the terms "stability" and "instability." Thus, a recent re-examination (Chandrasekhar 1963*c;* this paper[1] will be referred to hereafter as "Paper I") of the more elementary problem of Roche (in which the tidally distorting secondary is treated as a rigid sphere) has shown that, along a Roche sequence, instability in the strict sense (that the configuration is characterized by at least one normal mode of oscillation with an imaginary, or a complex, frequency) does not arise, as had commonly been supposed, at the distance of closest approach compatible with equilibrium. In Darwin's problem, in which (in contrast to Roche's problem) allowance is made for the centrifugal and tidal distortion of both components, the ambiguities preventing the application of intuitive considerations are even greater: along a Darwin sequence, the maximum angular velocity of orbital rotation does not occur at the distance of closest approach; at neither of the two extremes does the configuration allow an infinitesimal, solenoidal, ellipsoidal displacement which will deform the equilibrium ellipsoid into a neighboring equilibrium ellipsoid; and, finally, a displacement of the kind described is allowed at a still different point.[2] On these accounts we shall reconsider in this paper Darwin's problem and com-

[1] The following misprints in Paper I may be noted here: the headings for the second columns in Tables 2, 4, and 5 (on pp. 1193, 1199, and 1205) *should read* ϕ *instead of* θ. Also in the (1, 3)-element of the 3×3-determinant in equation (78) on p. 1200 *read* $\mathrm{B}_{13} - \mathrm{B}_{23}$ *instead of* $\mathrm{B}_{13} + \mathrm{B}_{23}$.

[2] As a further example of the ambiguities present in the discussions of Darwin and Jeans, we may refer to the illustration on p. 86 of Jeans's *Problems of Cosmogony and Stellar Dynamics*. In this illustration it is not made clear (and it is not also clarified in the text) whether the dashed line (separating the domains of "stability" and "instability") is intended to pass through S or end on a point on the line SB (representing the Maclaurin sequence); and, in either case, what does the "instability" of the Maclaurin spheroid, prior to the point of bifurcation at B, mean in this context? In view of all these ambiguities and

plete the present re-examination of the stability of the ellipsoidal configurations of homogeneous masses (see the various papers by Chandrasekhar and by Chandrasekhar and Lebovitz in the *Astrophysical Journal* of the past three years).

II. DARWIN'S PROBLEM

Darwin's problem is concerned with the equilibrium and the stability of two homogeneous bodies rotating about one another in a manner that maintains their relative dispositions.[3]

Let the masses of the "primary" and the "secondary" be M and M', respectively;[4] and let the distance between their centers of mass be R. We shall refer the system in a coordinate frame which is rotating uniformly with an angular velocity $\mathbf{\Omega}$ about their common center of mass. Let the origin of the chosen frame be at the center of mass of the primary, the x_1-axis point to the center of mass of the secondary, and the x_3-axis be parallel to the direction of $\mathbf{\Omega}$. In the frame of reference so chosen, the equation of motion governing the fluid elements of the primary is (cf. Paper I, eq. [4])

$$\rho \frac{d u_i}{d t} = -\frac{\partial p}{\partial x_i} + \rho \frac{\partial}{\partial x_i}\left[\mathfrak{B} + \mathfrak{B}' + \tfrac{1}{2}\Omega^2(x_1^2 + x_2^2) - \frac{M'R}{M+M'}\Omega^2 x_1\right] + 2\rho\Omega\epsilon_{il3}u_l\,, \quad (1)$$

where $\mathfrak{B}$ is the self-gravitational potential and $\mathfrak{B}'$ is the tidal potential due to M'.

In Roche's problem, the secondary is treated as a rigid sphere and the variation of the potential $\mathfrak{B}'$ over the primary is expanded in a Taylor series in x_i and only the terms linear and quadratic in x_i are retained. In Darwin's problem, while an exactly analogous procedure is followed, the attempt is made to allow, in a consistent manner, the distortion of the secondary as well. In allowing for this distortion of the secondary, we shall follow the method described by Jeans (1919).

Consider the tidal potential $\mathfrak{B}'$ in a coordinate system (X_1, X_2, X_3) whose origin is at the center of mass of M'. The transformation to the coordinate system (x_1, x_2, x_3) is, clearly,

$$x_1 = -(X_1 - R)\,, \qquad x_2 = -X_2\,, \qquad x_3 = X_3\,. \quad (2)$$

We now expand $\mathfrak{B}'(X_1, X_2, X_3)$ as a Taylor series about $(R, 0, 0)$; and in writing the Taylor expansion, we shall suppose that the secondary, as well as the primary, have triplanar symmetry in their respective coordinate systems. Then,

$$\begin{aligned}\mathfrak{B}'(X_1, X_2, X_3) &= \mathfrak{B}'(R, 0, 0) + (X_1 - R)\left(\frac{\partial \mathfrak{B}'}{\partial X_1}\right)_{R\,0,0} \\ &+ \tfrac{1}{2}(X_1 - R)^2\left(\frac{\partial^2 \mathfrak{B}'}{\partial X_1^2}\right)_{R,0,0} + \tfrac{1}{2}X_2^2\left(\frac{\partial^2 \mathfrak{B}'}{\partial X_2^2}\right)_{R,0,0} + \tfrac{1}{2}X_3^2\left(\frac{\partial^2 \mathfrak{B}'}{\partial X_3^2}\right)_{R\,0,0} + \dots\,;\end{aligned} \quad (3)$$

and we shall not consider the terms beyond those we have written.

Laplace's equation which governs the potential exterior to a body requires that

$$\sum_{j=0}^{3}\left(\frac{\partial^2 \mathfrak{B}'}{\partial X_j^2}\right)_{R,0,0} \equiv 0\,. \quad (4)$$

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uncertainties, it is not surprising to find Milne (1952 p. 112) describing these results by the beautifully uninformative statement: "it must suffice to say that only the earlier parts of the tidal series and the various double-star series are stable."

[3] It is not to be assumed that this will always be possible (see the remarks following eq. [9] below).

[4] Again, as in Paper I, we are not implying by the usage of the terms "primary" and "secondary" that $M > M'$; only that our attention is *primarily* on M.

Inserting the expansion (3) in equation (1), we have (in view of eqs. [2])

$$\rho \frac{du_i}{dt} = -\frac{\partial p}{\partial x_i} + \rho \frac{\partial}{\partial x_i} \Big\{ \mathfrak{B} + \tfrac{1}{2} \sum_{j=1}^{3} x_j^2 \left(\frac{\partial^2 \mathfrak{B}'}{\partial X_j^2}\right)_{R,0,0} + \tfrac{1}{2}\Omega^2 (x_1^2 + x_2^2) - x_1 \left[\left(\frac{\partial \mathfrak{B}'}{\partial X_1}\right)_{R,0,0} + \frac{M'R}{M+M'}\Omega^2\right] \Big\} + 2\rho\Omega\epsilon_{il3}u_l . \tag{5}$$

If we now specify Ω^2 by requiring that

$$\Omega^2 = -\frac{M+M'}{M'R}\left(\frac{\partial \mathfrak{B}'}{\partial X_1}\right)_{R,0,0}, \tag{6}$$

equation (5) reduces to the form

$$\rho \frac{du_i}{dt} = -\frac{\partial p}{\partial x_i} + \rho \frac{\partial}{\partial x_i}\left(\mathfrak{B} + \tfrac{1}{2}\sum_{k=1}^{3} \beta_k x_k^2\right) + 2\rho\Omega\epsilon_{il3}u_l , \tag{7}$$

where

$$\beta_1 = \Omega^2 + \left(\frac{\partial^2 \mathfrak{B}'}{\partial X_1^2}\right)_{R,0,0}, \qquad \beta_2 = \Omega^2 + \left(\frac{\partial^2 \mathfrak{B}'}{\partial X_2^2}\right)_{R,0,0}, \tag{8}$$

and

$$\beta_3 = \left(\frac{\partial^2 \mathfrak{B}'}{\partial X_3^2}\right)_{R,0,0} = -\left(\frac{\partial^2 \mathfrak{B}'}{\partial X_1^2} + \frac{\partial^2 \mathfrak{B}'}{\partial X_2^2}\right)_{R,0,0}. \tag{9}$$

It is manifest (and we shall presently verify) that equation (7) allows stationary solutions (i.e., solutions with $u_l = 0$) leading to ellipsoidal figures of equilibrium. On the other hand, since the treatment has to be symmetric with respect to both components, it is clear that stationary solutions derived from equation (7) can be considered valid only if the initial choice of the frame of reference, which enabled the reduction of equation (1) to equation (7) (via eq. [6]), is independent of the particular component which we may be considering momentarily. But this is clearly not the case: expression (6) for Ω^2 is not symmetric in M and M'. Indeed, it is apparent that the determination of Ω^2 by equation (6) is strictly admissible only in two cases: the case when $M = M'$ and the two components are, in addition, congruent; and the "singular" case $M/M' \to 0$ when the distortion of the secondary by the tidal effects of the primary can be ignored. In all other cases, the values of Ω^2 which will eliminate the "unwanted" terms in x_1 and X_1, in equation (5) for M and in the analogous equation for M', are different. Confronted with this situation, Darwin and Jeans adopt the artifice of taking the average of the two different values of Ω^2 and simply "deleting" the unwanted terms from the respective equations. We shall avoid recourse to this artifice by simply restricting ourselves to the two cases in which the method can be carried through consistently.

III. THE APPLICATION OF THE SECOND-ORDER VIRIAL THEOREM TO DARWIN'S PROBLEM

In treating Darwin's problem, we shall follow Paper I and use the same methods based on the virial theorem and its extensions. In this section we shall assemble the necessary formulae.

By multiplying equation (7) by x_j and integrating over the volume V occupied by the fluid, we obtain, in the usual manner, the second-order virial equation (cf. Paper I, eq. [10])

$$\frac{d}{dt}\int_V \rho u_i x_j d\mathbf{x} = 2\mathfrak{T}_{ij} + \mathfrak{W}_{ij} + [\![\beta_i I_{ij}]\!] + \delta_{ij}\Pi + 2\Omega\epsilon_{il3}\int_V \rho u_l x_j d\mathbf{x} , \tag{10}$$

where $[\![\ ]\!]$ signifies that the quantity enclosed is *not* to be summed over the repeated index. In equation (10)

$$\Pi = \int_V p\,d\boldsymbol{x}\,, \tag{11}$$

and $\mathfrak{T}_{ij}$, $\mathfrak{W}_{ij}$, and I_{ij} are the kinetic energy, the potential energy, and the moment of inertia tensors defined in the standard way.

a) The Virial Equations Governing Equilibrium

When no motions are present in the frame of reference considered and hydrostatic equilibrium prevails, equation (10) becomes

$$\mathfrak{W}_{ij} + [\![\beta_i I_{ij}]\!] = -\Pi\delta_{ij}\,. \tag{12}$$

The diagonal elements of this relation give

$$\mathfrak{W}_{11} + \beta_1 I_{11} = \mathfrak{W}_{22} + \beta_2 I_{22} = \mathfrak{W}_{33} + \beta_3 I_{33} = -\Pi\,; \tag{13}$$

while the non-diagonal elements give

$$\begin{aligned} \mathfrak{W}_{12} + \beta_1 I_{12} &= \mathfrak{W}_{21} + \beta_2 I_{21} = 0\,, \\ \mathfrak{W}_{23} + \beta_2 I_{23} &= \mathfrak{W}_{32} + \beta_3 I_{32} = 0\,, \\ \mathfrak{W}_{31} + \beta_3 I_{31} &= \mathfrak{W}_{13} + \beta_1 I_{13} = 0\,. \end{aligned} \tag{14}$$

In view of the symmetry of the tensors I_{ij} and $\mathfrak{W}_{ij}$, it follows from the foregoing equations that so long as the β_j's are finite and unequal,

$$\mathfrak{W}_{ij} = 0 \qquad \text{and} \qquad I_{ij} = 0\,. \qquad (i \neq j) \tag{15}$$

Therefore, *in the chosen coordinate system, the tensors* I_{ij} *and* $\mathfrak{W}_{ij}$ *are necessarily diagonal.*

Equations (13) and (14) are entirely general: they do not depend on any constitutive relations that may exist.

Now it can be shown quite readily that if the configuration is homogeneous (by assumption or by virtue of incompressibility), then, an ellipsoidal figure is consistent with the equations of hydrostatic equilibrium as well as the condition which requires the pressure to be constant over the bounding surface; indeed, the satisfaction of these conditions leads to precisely the same equations (13) and (14).

b) The Second-Order Virial Equations Governing Small Oscillations about Equilibrium

Suppose that an equilibrium ellipsoid, determined consistently with equations (13), is slightly perturbed; and, further, that the ensuing motions are described by a Lagrangian displacement of the form

$$\boldsymbol{\xi}(\boldsymbol{x})e^{\lambda t}\,, \tag{16}$$

where λ is a parameter whose characteristic values are to be determined. To the first order in $\boldsymbol{\xi}$, the virial equation (10) gives

$$\lambda^2 V_{i;j} - 2\lambda\Omega\epsilon_{il3}V_{l;j} = \delta\mathfrak{W}_{ij} + [\![\beta_i\delta I_{ij}]\!] + \delta_{ij}\delta\Pi\,, \tag{17}$$

where

$$V_{i;j} = \int_V \rho\,\xi_i x_j\,d\boldsymbol{x} \tag{18}$$

denotes the second-order unsymmetrized virial and $\delta\Pi$, $\delta\mathfrak{W}_{ij}$, and δI_{ij} are the first variations of Π, $\mathfrak{W}_{ij}$, and I_{ij} due to the deformation of the ellipsoid caused by the displacement $\boldsymbol{\xi}$.

It is known (cf. Paper I, eqs. [37]–[40]) that δI_{ij} and $\delta\mathfrak{W}_{ij}$ can be expressed in terms of the symmetrized virial

$$V_{ij} = V_{i;j} + V_{j;i}\,. \tag{19}$$

Thus,

$$\delta I_{ij} = V_{ij} , \tag{20}$$

and

$$\delta \mathfrak{W}_{ij} = -2B_{ij}V_{ij} , \qquad (i \neq j) \tag{21}$$

and

$$\delta \mathfrak{W}_{ii} = -(2B_{ii} - a_i^2 A_{ii}) V_{ii} + a_i^2 \sum_{l \neq i} A_{il} V_{ll} , \tag{22}$$

where

$$B_{ij} = A_i - a_j^2 A_{ij} = A_j - a_i^2 A_{ij} , \tag{23}$$

(no summation over repeated indices in eqs. [21]–[23])

and A_i and A_{ij} are the one- and the two-index symbols defined in an earlier paper (Chandrasekhar and Lebovitz 1962). (*Note that in writing eqs.* [*21*] *and* [*22*], *a common factor* $\pi G\rho a_1 a_2 a_3$ *has been suppressed.*)

Replacing δI_{ij} by V_{ij} (in accordance with eq. [20]), we can rewrite equation (17) in the form

$$\lambda^2 V_{i;j} - 2\lambda\Omega\epsilon_{il3}V_{l;j} = \delta\mathfrak{W}_{ij} + [\![\beta_i V_{ij}]\!] + \delta_{ij}\delta\Pi . \tag{24}$$

Equation (24) represents a total of nine equations for the nine virials $V_{i;j}$. These nine equations fall into two non-combining groups of four and five equations, respectively, distinguished by their parity with respect to the index 3. The odd equations are:

$$\lambda^2 V_{3;1} = \delta\mathfrak{W}_{31} + \beta_3 V_{31} = -(2B_{31} - \beta_3)V_{31} , \tag{25}$$

$$\lambda^2 V_{3;2} = \delta\mathfrak{W}_{32} + \beta_3 V_{32} = -(2B_{23} - \beta_3)V_{23} , \tag{26}$$

$$\lambda^2 V_{1;3} - 2\lambda\Omega V_{2;3} = \delta\mathfrak{W}_{13} + \beta_1 V_{13} = -(2B_{31} - \beta_1)V_{31} , \tag{27}$$

$$\lambda^2 V_{2;3} + 2\lambda\Omega V_{1;3} = \delta\mathfrak{W}_{23} + \beta_2 V_{23} = -(2B_{23} - \beta_2)V_{23} , \tag{28}$$

where we have substituted for $\delta\mathfrak{W}_{ij}$ $(i \neq j)$ in accordance with equation (21). And the even equations are:

$$\lambda^2 V_{3;3} = \delta\mathfrak{W}_{33} + \beta_3 V_{33} + \delta\Pi , \tag{29}$$

$$\lambda^2 V_{1;1} - 2\lambda\Omega V_{2;1} = \delta\mathfrak{W}_{11} + \beta_1 V_{11} + \delta\Pi , \tag{30}$$

$$\lambda^2 V_{2;2} + 2\lambda\Omega V_{1;2} = \delta\mathfrak{W}_{22} + \beta_2 V_{22} + \delta\Pi , \tag{31}$$

$$\lambda^2 V_{1;2} - 2\lambda\Omega V_{2;2} = \delta\mathfrak{W}_{12} + \beta_1 V_{12} = -(2B_{12} - \beta_1)V_{12} , \tag{32}$$

$$\lambda^2 V_{2;1} + 2\lambda\Omega V_{1;1} = \delta\mathfrak{W}_{21} + \beta_2 V_{21} = -(2B_{12} - \beta_2)V_{12} . \tag{33}$$

c) The Characteristic Equation for the Odd Modes

Adding equations (25) and (27) and similarly equations (26) and (28), we obtain

$$(\lambda^2 + 4B_{31} - \beta_1 - \beta_3)V_{31} - 2\lambda\Omega V_{23} + 2\lambda\Omega V_{3;2} = 0 \tag{34}$$

and

$$(\lambda^2 + 4B_{23} - \beta_2 - \beta_3)V_{23} + 2\lambda\Omega V_{31} - 2\lambda\Omega V_{3;1} = 0 . \tag{35}$$

Eliminating $V_{3;1}$ and $V_{3;2}$ from the foregoing equations with the aid of equations (25) and (26), we obtain

$$\lambda(\lambda^2 + 4B_{31} - \beta_1 - \beta_3)V_{31} - 2\Omega(\lambda^2 + 2B_{23} - \beta_3)V_{23} = 0 \tag{36}$$

and

$$\lambda(\lambda^2 + 4B_{23} - \beta_2 - \beta_3)V_{23} + 2\Omega(\lambda^2 + 2B_{31} - \beta_3)V_{31} = 0 ; \tag{37}$$

and these two equations lead to the characteristic equation

$$\lambda^2(\lambda^2 + 4B_{13} - \beta_1 - \beta_3)(\lambda^2 + 4B_{23} - \beta_2 - \beta_3) + 4\Omega^2(\lambda^2 + 2B_{13} - \beta_3)(\lambda^2 + 2B_{23} - \beta_3) = 0 . \tag{38}$$

d) The Characteristic Equation for the Even Modes

Turning to the even equations (29)–(33), we can combine them to give the following four equations in which $\delta\Pi$ no longer appears:

$$(\lambda^2 + 4B_{12} - \beta_1 - \beta_2)V_{12} + \lambda\Omega(V_{11} - V_{22}) = 0 , \tag{39}$$

$$\lambda^2(V_{1;2} - V_{2;1}) = \lambda\Omega(V_{11} + V_{22}) + (\beta_1 - \beta_2)V_{12} , \tag{40}$$

$$\tfrac{1}{2}\lambda^2(V_{11} - V_{22}) - 2\lambda\Omega V_{12} = \delta\mathfrak{W}_{11} - \delta\mathfrak{W}_{22} + \beta_1 V_{11} - \beta_2 V_{22} , \tag{41}$$

$$\tfrac{1}{2}\lambda^2(V_{11} + V_{22}) + 2\lambda\Omega(V_{1;2} - V_{2;1}) - \lambda^2 V_{33} = \delta\mathfrak{W}_{11} + \delta\mathfrak{W}_{22} - 2\delta\mathfrak{W}_{33} + \beta_1 V_{11} + \beta_2 V_{22} - 2\beta_3 V_{33} . \tag{42}$$

Rearranging equation (41) and eliminating $(V_{1;2} - V_{2;1})$ from equation (42) with the aid of equation (40) (and rearranging), we obtain the pair of equations

$$(\tfrac{1}{2}\lambda^2 - \beta_1)V_{11} - (\tfrac{1}{2}\lambda^2 - \beta_2)V_{22} - 2\lambda\Omega V_{12} = \delta\mathfrak{W}_{11} - \delta\mathfrak{W}_{22} , \tag{43}$$

$$(\tfrac{1}{2}\lambda^2 + 2\Omega^2 - \beta_1)V_{11} + (\tfrac{1}{2}\lambda^2 + 2\Omega^2 - \beta_2)V_{22} - (\lambda^2 - 2\beta_3)V_{33} + 2(\beta_1 - \beta_2)\frac{\Omega}{\lambda}V_{12} = \delta\mathfrak{W}_{11} + \delta\mathfrak{W}_{22} - 2\delta\mathfrak{W}_{33} . \tag{44}$$

From equations (21) and (22), we obtain (cf. Paper I, eqs. [41] and [42])

$$\delta\mathfrak{W}_{11} - \delta\mathfrak{W}_{22} = -(3B_{11} - B_{12})V_{11} + (3B_{22} - B_{12})V_{22} + (B_{23} - B_{13})V_{33} \tag{45}$$

and

$$\delta\mathfrak{W}_{11} + \delta\mathfrak{W}_{22} - 2\delta\mathfrak{W}_{33} = -(3B_{11} + B_{12} - 2B_{13})V_{11} - (3B_{22} + B_{21} - 2B_{23})V_{22} + (6B_{33} - B_{13} - B_{23})V_{33} . \tag{46}$$

Now substituting these relations in equations (43) and (44) and regrouping the terms, we find

$$(\tfrac{1}{2}\lambda^2 + 3B_{11} - B_{12} - \beta_1)V_{11} - (\tfrac{1}{2}\lambda^2 + 3B_{22} - B_{21} - \beta_2)V_{22} + (B_{13} - B_{23})V_{33} - 2\lambda\Omega V_{12} = 0 \tag{47}$$

and

$$(\tfrac{1}{2}\lambda^2 + 3B_{11} + B_{12} - 2B_{13} + 2\Omega^2 - \beta_1)V_{11} + (\tfrac{1}{2}\lambda^2 + 3B_{22} + B_{12} - 2B_{23} + 2\Omega^2 - \beta_2)V_{22} - (\lambda^2 + 6B_{33} - B_{13} - B_{23} - 2\beta_3)V_{33} + 2(\beta_1 - \beta_2)\frac{\Omega}{\lambda}V_{12} = 0 . \tag{48}$$

Equations (39), (47), and (48) provide three relations among the four virials V_{11}, V_{22}, V_{33}, and V_{12}. A fourth relation is obtained by making use of the solenoidal character of the Lagrangian displacement. In the present context, the relation which expresses this requirement is (cf. Lebovitz 1961, eq. [83])

$$\frac{V_{11}}{a_1^2} + \frac{V_{22}}{a_2^2} + \frac{V_{33}}{a_3^2} = 0 . \tag{49}$$

The characteristic equation governing the even modes now follows from setting the determinant of equations (39) and (47)–(49) equal to zero. We thus obtain

$$\left\Vert \begin{matrix} \tfrac{1}{2}\lambda^2+3B_{11}-B_{12}-\beta_1 & -\tfrac{1}{2}\lambda^2-3B_{22}+B_{21}+\beta_2 & B_{13}-B_{23} & -2\lambda\Omega \\ \tfrac{1}{2}\lambda^2+3B_{11}+B_{12}-2B_{13}+2\Omega^2-\beta_1 & +\tfrac{1}{2}\lambda^2+3B_{22}+B_{21}-2B_{23}+2\Omega^2-\beta_2 & -\lambda^2-6B_{33}+B_{31}+B_{32}+2\beta_3 & 2(\beta_1-\beta_2)\Omega/\lambda \\ \lambda\Omega & -\lambda\Omega & 0 & \lambda^2+4B_{12}-\beta_1-\beta_2 \\ \dfrac{1}{a_1^2} & \dfrac{1}{a_2^2} & \dfrac{1}{a_3^2} & 0 \end{matrix} \right\Vert = 0\,. \quad (50)$$

After some elementary transformations, equation (50) can be brought to the following somewhat simpler form:

$$(\lambda^2+4B_{12}-\beta_1-\beta_2)\left\Vert \begin{matrix} \tfrac{1}{2}\lambda^2+3B_{11}-B_{12}-\beta_1 & B_{23}-B_{13} & 3B_{11}-3B_{22}+B_{13}-B_{23}+\beta_2-\beta_1 \\ \Omega^2+B_{12}-B_{13} & \tfrac{1}{2}\lambda^2+3B_{33}-B_{32}-\beta_3 & 3B_{22}-3B_{33}+B_{12}-B_{13}+2\Omega^2-\beta_2+\beta_3 \\ \dfrac{1}{a_1^2} & -\dfrac{1}{a_3^2} & \dfrac{1}{a_1^2}+\dfrac{1}{a_2^2}+\dfrac{1}{a_3^2} \end{matrix} \right\Vert$$

$$+\Omega^2\left\Vert \begin{matrix} 2\lambda^2 & B_{23}-B_{13} & 3B_{11}-3B_{22}+B_{13}-B_{23}+\beta_2-\beta_1 \\ -(\lambda^2+\beta_1-\beta_2) & \tfrac{1}{2}\lambda^2+3B_{33}-B_{32}-\beta_3 & 3B_{22}-3B_{33}+B_{12}-B_{13}+2\Omega^2-\beta_2+\beta_3 \\ 0 & -\dfrac{1}{a_3^2} & \dfrac{1}{a_1^2}+\dfrac{1}{a_2^2}+\dfrac{1}{a_3^2} \end{matrix} \right\Vert = 0\,. \quad (51)$$

e) The Roche Limit

Along a sequence of equilibrium ellipsoids, there is generally a neutral point where the equilibrium ellipsoid can be deformed into a neighboring equilibrium ellipsoid by the application of a quasi-static, solenoidal, purely ellipsoidal, infinitesimal displacement. Along the sequence of the Maclaurin spheroids, such a neutral point occurs where Ω^2 attains its maximum value; and along a Roche sequence it occurs where Ω^2, similarly, attains its maximum value (cf. Paper I, Sec. V; also Chandrasekhar 1963*a*, *b;* Chandrasekhar and Lebovitz 1963). But, as we shall presently verify in the case of the Darwin ellipsoids, the occurrence of a neutral point of the kind described need not require that some particular parameter labeling the sequence attain an extreme value at that point.

The requirements on the displacement, expressed in terms of the symmetrized virials, are

$$V_{12} = V_{23} = V_{31} = 0\,, \qquad V_{11} \neq 0\,, \qquad V_{22} \neq 0\,, \qquad V_{33} \neq 0\,, \tag{52}$$

and

$$\frac{V_{11}}{a_1^2} + \frac{V_{22}}{a_2^2} + \frac{V_{33}}{a_3^2} = 0\,. \tag{53}$$

The conditions (52) express the required purely ellipsoidal character of the displacement; and the condition (53) expresses the required solenoidal character. The additional requirement, that the displacement is not to affect the equilibrium of the ellipsoid, implies that equations (25)–(33) (or equivalently, eqs. [36], [37], and [39]–[42]), with λ set equal to zero, are also satisfied. The equations to be satisfied, besides equation (53), are, therefore,

$$\delta\mathfrak{W}_{11} - \delta\mathfrak{W}_{22} + \beta_1 V_{11} - \beta_2 V_{22} = 0 \tag{54}$$

and

$$\delta\mathfrak{W}_{11} + \delta\mathfrak{W}_{22} - 2\delta\mathfrak{W}_{33} + \beta_1 V_{11} + \beta_2 V_{22} - 2\beta_3 V_{33} = 0\,. \tag{55}$$

Combined with equations (45) and (46), the foregoing equations give

$$(\beta_1 - 3B_{11} + B_{12})V_{11} - (\beta_2 - 3B_{22} + B_{12})V_{22} + (B_{23} - B_{13})V_{33} = 0 \tag{56}$$

and

$$\begin{aligned}(\beta_1 - 3B_{11} - B_{12} + 2B_{13})V_{11} &+ (\beta_2 - 3B_{22} - B_{12} + 2B_{23})V_{22} \\ &+ (-2\beta_3 + 6B_{33} - B_{13} - B_{23})V_{33} = 0\,.\end{aligned} \tag{57}$$

The condition for the occurrence of a neutral point of the kind described is that the determinant of equations (53), (56), and (57) vanishes; i.e.,

$$\begin{Vmatrix} \beta_1 - 3B_{11} + B_{12} & -\beta_2 + 3B_{22} - B_{12} & B_{23} - B_{13} \\ \beta_1 - 3B_{11} - B_{12} + 2B_{13} & +\beta_2 - 3B_{22} - B_{12} + 2B_{23} & -2\beta_3 + 6B_{33} - B_{13} - B_{23} \\ \dfrac{1}{a_1^2} & \dfrac{1}{a_2^2} & \dfrac{1}{a_3^2} \end{Vmatrix} = 0\,. \tag{58}$$

Equation (58) can be simplified to the form

$$\begin{Vmatrix} \beta_1 - 3B_{11} + B_{13} & B_{23} - B_{12} & -\beta_3 + 3B_{33} - B_{13} \\ B_{13} - B_{12} & \beta_2 - 3B_{22} + B_{23} & -\beta_3 + 3B_{33} - B_{23} \\ \dfrac{1}{a_1^2} & \dfrac{1}{a_2^2} & \dfrac{1}{a_3^2} \end{Vmatrix} = 0\,. \tag{59}$$

We shall define the point along a sequence of ellipsoids, where equation (59) is satisfied, as the Roche limit. It should be emphasized that the condition for the occurrence of the Roche limit has no bearing on the onset of instability: the point along the sequence, where instability sets in, is determined by equation (51) with λ set equal to zero, i.e., by the equation

$$\begin{Vmatrix} 3B_{11}-B_{12}-\beta_1 & B_{23}-B_{13} & 3B_{11}-3B_{22}+B_{13}-B_{23}+\beta_2-\beta_1 \\ B_{12}-B_{13}+\Omega^2 & 3B_{33}-B_{23}-\beta_3 & 3B_{22}-3B_{33}+B_{12}-B_{13}+2\Omega^2-\beta_2+\beta_3 \\ \dfrac{1}{a_1^2} & -\dfrac{1}{a_3^2} & \dfrac{1}{a_1^2}+\dfrac{1}{a_2^2}+\dfrac{1}{a_3^2} \end{Vmatrix} + \frac{\Omega^2(\beta_1-\beta_2)}{4B_{12}-\beta_1-\beta_2}\begin{Vmatrix} B_{23}-B_{13} & 3B_{11}-3B_{22}+B_{13}-B_{23}+\beta_2-\beta_1 \\ -\dfrac{1}{a_3^2} & \dfrac{1}{a_1^2}+\dfrac{1}{a_2^2}+\dfrac{1}{a_3^2} \end{Vmatrix} = 0\,; \tag{60}$$

and this equation is quite different from equation (59). (For a further clarification of these matters see Lebovitz 1963.)

IV. THE CASE $M/M'=0$

In this case, the primary is of infinitesimal mass compared to the secondary; and we may consider the secondary as unaffected by the presence of the primary. The departure of the secondary from sphericity is, then, determined solely by the centrifugal potential. The secondary is, accordingly, a Maclaurin spheroid or a Jacobi ellipsoid. However, it will appear that solutions for Darwin's problem exist only for members of the Maclaurin sequence with eccentricities less than 0.40504. Consequently, we need not consider the Jacobian form for the secondary.

Consider, then, a Maclaurin spheroid whose meridional section has an eccentricity e. The angular velocity of rotation which we must associate with this eccentricity is (cf. Chandrasekhar and Lebovitz 1962, eq. [77])

$$\frac{\Omega^2}{\pi G\rho_{\mathrm{Mc}}}=\frac{2(1-e^2)^{1/2}}{e^3}(3-2e^2)\sin^{-1}e-\frac{6}{e^2}(1-e^2), \tag{61}$$

where ρ_{Mc} is the density of the spheroid.

In order now that the "satellite" we are considering may rotate synchronously with the Maclaurin spheroid, it is necessary that it describe a circular orbit about it with the same angular velocity Ω. The orbit must accordingly be described at such a distance R that the dynamical condition (6) leads to the same value of Ω^2 as equation (61). The equation which expresses this equality can be obtained as follows.

The gravitational potential, in the equatorial plane of a homogeneous spheroid, at a distance ϖ from the center is given by

$$\frac{\mathfrak{B}'}{\pi G\rho_{\mathrm{Mc}}}=-\frac{(1-e^2)^{1/2}}{e}a^2_{\mathrm{Mc}}\left[\left(\frac{\varpi^2}{e^2a^2_{\mathrm{Mc}}}-2\right)\sin^{-1}\left(\frac{e\,a_{\mathrm{Mc}}}{\varpi}\right)-\frac{\varpi}{e\,a_{\mathrm{Mc}}}\left(1-\frac{e^2a^2_{\mathrm{Mc}}}{\varpi^2}\right)^{1/2}\right], \tag{62}$$

where a_{Mc} denotes the semimajor axes. From equation (62) we readily find that

$$\frac{1}{\pi G\rho_{\mathrm{Mc}}}\left(\frac{\partial\mathfrak{B}'}{\partial X_1}\right)_{\varpi=R}=-\frac{2(1-e^2)^{1/2}}{e^2}a_{\mathrm{Mc}}\left[\frac{R}{e\,a_{\mathrm{Mc}}}\sin^{-1}\left(\frac{e\,a_{\mathrm{Mc}}}{R}\right)-\left(1-\frac{e^2a^2_{\mathrm{Mc}}}{R^2}\right)^{1/2}\right]. \tag{63}$$

Equation (6) now gives

$$\frac{\Omega^2}{\pi G\rho_{\mathrm{Mc}}}=\frac{2(1-e^2)^{1/2}}{e^3}\left[\sin^{-1}\left(\frac{e\,a_{\mathrm{Mc}}}{R}\right)-\frac{e\,a_{\mathrm{Mc}}}{R}\left(1-\frac{e^2a^2_{\mathrm{Mc}}}{R^2}\right)^{1/2}\right]. \tag{64}$$

Since Ω^2 given by equations (61) and (64) must agree, we must have

$$\sin^{-1}\left(\frac{e\,a_{\mathrm{Mc}}}{R}\right)-\frac{e\,a_{\mathrm{Mc}}}{R}\left(1-\frac{e^2a^2_{\mathrm{Mc}}}{R^2}\right)^{1/2}=(3-2e^2)\sin^{-1}e-3e(1-e^2)^{1/2}. \tag{65}$$

Equation (65) determines R/a_{Mc} along the Maclaurin sequence. It should be noted in this connection that, if the mass of the spheroid is specified (as it is in the present context), a_{Mc} will vary along the sequence. What is constant is $a^3_{\mathrm{Mc}}\sqrt{(1-e^2)}$. Accordingly, a convenient unit in which we may measure R is

$$\bar{a}_{\mathrm{Mc}}=a_{\mathrm{Mc}}(1-e^2)^{1/6}. \tag{66}$$

Returning to equation (62) and evaluating its second derivatives with respect to X_1 and X_2, we find

$$\frac{1}{\pi G\rho_{\mathrm{Mc}}}\left(\frac{\partial^2\mathfrak{B}'}{\partial X_1^2}\right)_{\varpi=R}=4\left(\frac{a_{\mathrm{Mc}}}{R}\right)^3\left(\frac{1-e^2}{1-e^2a^2_{\mathrm{Mc}}/R^2}\right)^{1/2}-\frac{\Omega^2}{\pi G\rho_{\mathrm{Mc}}}, \tag{67}$$

$$\left(\frac{\partial^2\mathfrak{B}'}{\partial X_2^2}\right)_{\varpi=R}=-\Omega^2, \tag{68}$$

and

$$\frac{1}{\pi G\rho_{\mathrm{Mc}}}\left(\frac{\partial^2\mathfrak{B}'}{\partial X_3^2}\right)_{\varpi=R}=\frac{2\Omega^2}{\pi G\rho_{\mathrm{Mc}}}-4\left(\frac{a_{\mathrm{Mc}}}{R}\right)^3\left(\frac{1-e^2}{1-e^2a^2_{\mathrm{Mc}}/R^2}\right)^{1/2}. \tag{69}$$

Letting

$$4\left(\frac{a_{\mathrm{Mc}}}{R}\right)^3\left(\frac{1-e^2}{1-e^2a^2_{\mathrm{Mc}}/R^2}\right)^{1/2}=q\,\frac{\Omega^2}{\pi G\rho_{\mathrm{Mc}}} \tag{70}$$

(where q, by this definition, is a dimensionless constant depending only on the eccentricity of the Maclaurin spheroid), we find that the coefficients β_j defined in Section II (eqs. [8] and [9]) have now the values

$$\beta_1=q\Omega^2\,,\qquad \beta_2=0\,,\qquad \text{and}\qquad \beta_3=-(q-2)\Omega^2\,. \tag{71}$$

It might be noted here that the value q appropriate to Roche's problem is 3.

a) The Properties of the Equilibrium Ellipsoids

With the β's given by equation (71), the virial equation (13) gives

$$\mathfrak{W}_{11}+q\Omega^2I_{11}=\mathfrak{W}_{22}=\mathfrak{W}_{33}-(q-2)\Omega^2I_{33}\,, \tag{72}$$

or, alternatively,

$$\Omega^2[qI_{11}+(q-2)I_{33}]=\mathfrak{W}_{33}-\mathfrak{W}_{11} \tag{73}$$

and

$$\Omega^2(q-2)I_{33}=\mathfrak{W}_{33}-\mathfrak{W}_{22}\,. \tag{74}$$

We shall find it convenient to consider equations (73) and (74) in the forms

$$1+\frac{q}{q-2}\,\frac{I_{11}}{I_{33}}=\frac{\mathfrak{W}_{33}-\mathfrak{W}_{11}}{\mathfrak{W}_{33}-\mathfrak{W}_{22}} \tag{75}$$

and

$$\Omega^2=\frac{\mathfrak{W}_{33}-\mathfrak{W}_{22}}{(q-2)I_{33}}=\frac{\mathfrak{W}_{33}-\mathfrak{W}_{11}}{qI_{11}+(q-2)I_{33}}\,. \tag{76}$$

Expressions for the various tensors describing the properties of homogeneous ellipsoids have been given in an earlier paper (Chandrasekhar and Lebovitz 1962, eqs. [57] and [58]); in particular

$$\mathfrak{W}_{ii} = -2\pi G\rho a_1 a_2 a_3 A_i I_{ii} \qquad \text{and} \qquad I_{ii} = \tfrac{1}{5} M a_i^2 , \tag{77}$$
(no summation over repeated indices)

where ρ is the density (not necessarily the same as ρ_{Mc}), a_1, a_2, and a_3 are the semi-axes of the ellipsoid, and the A_i's are the one-index symbols defined in the same paper.

Inserting the expressions for $\mathfrak{W}_{ii}$ and I_{ii} in equations (75) and (76), we obtain

$$1 + \frac{q a_1^2}{(q-2) a_3^2} = \frac{A_1 a_1^2 - A_3 a_3^2}{A_2 a_2^2 - A_3 a_3^2} \tag{78}$$

and

$$\frac{\Omega^2}{\pi G \rho} = 2\, \frac{A_1 a_1^2 - A_3 a_3^2}{q a_1^2 + (q-2) a_3^2} = 2\, \frac{A_2 a_2^2 - A_3 a_3^2}{(q-2) a_3^2} , \tag{79}$$

where it may be recalled that A_1, A_2, and A_3 are expressed in terms of the elliptic integrals, $E(\theta, \phi)$ and $F(\theta, \phi)$, of the two kinds with the arguments

$$\theta = \sin^{-1} \sqrt{\frac{a_1^2 - a_2^2}{a_1^2 - a_3^2}} \qquad \text{and} \qquad \phi = \cos^{-1} \frac{a_3}{a_1} . \tag{80}$$

A convenient procedure for solving equations (78) and (79) and determining the figures of the equilibrium ellipsoids is the following:

We start with a Maclaurin spheroid of some assigned eccentricity e. The angular velocity of rotation appropriate to this eccentricity is known from equation (61); and the distance R at which the satellite must circulate, in order that it may rotate synchronously with the spheroid, follows from equation (65). Equation (70) then determines the constant q which appears in all the subsequent formulae. Next, we consider equation (78) with the value of q which has been determined. For an assigned value of the ratio a_3/a_1 (or, more precisely, in the calculations, an assigned value of the argument ϕ) equation (78) requires that a_2/a_1 (or, more precisely, in the calculations, the argument θ) have a determinate value. A pair of values (ϕ, θ) compatible with equation (78) will not, in general, determine (in accordance with eq. [79]) a value of Ω^2 appropriate for the Maclaurin spheroid considered. The problem, then, is to determine a pair of values (ϕ, θ) which will be compatible with equation (78) and which will yield at the same time a value of Ω^2 (in accordance with eq. [79]) in agreement with equation (61). It was found that in practice the problem can be solved (to an accuracy sufficient for our purposes) by simple interpolation among four pairs of values (ϕ, θ) compatible with equation (78). Table 1 illustrates a sample calculation. It will be observed that for the chosen value of e there are two solutions.

Calculations similar to that illustrated in Table 1 have been carried out for other values of e and the principal results are summarized in Tables 2 and 3. And in Figures 1 and 2 these results for the Darwin ellipsoids are compared with those for the corresponding Roche ellipsoids of Paper I. It will be observed that the results for the two sets of ellipsoids are very nearly the same. This near identity of the results in the two cases is due to the fact that the values of q for the Maclaurin spheroids, in the range of eccentricity of interest, differ from 3 (the value appropriate for a rigid sphere) by an amount which hardly exceeds $\frac{1}{2}$ per cent (see Table 2). A fact of greater interest in the present context is that, by the manner of its formulation, Darwin's problem does not allow solutions for Maclaurin spheroids with eccentricities exceeding the value

$$e_{\max} = 0.405034 ; \tag{81}$$

whereas, in Roche's problem, in the limit $p = M/M' \to \infty$, we have solutions for the entire (combined) Maclaurin-Jacobi sequence (though, in this limit, the Jacobian part of the sequence must be considered as unstable; see Paper I, Secs. VIII and X). This difference in the two problems arises from the requirements, in Darwin's problem, that the secondary (now of infinitesimal mass) also be in equilibrium and rotate synchronously with the primary, requirements which are absent in Roche's problem since, in this

TABLE 1

SAMPLE CALCULATION FOR THE CASE $\rho = \rho_{\mathrm{Mc}}$, $e = 0.40$, Ω^2 (IN THE UNIT $\pi G\rho$) = 0.08727, $R/a_{\mathrm{Mc}} = 2.4170$, AND $q = 3.01679$

ϕ	θ	$\Omega^2(\phi, \theta)$	$\Omega^2_{\mathrm{Mc}} - \Omega^2(\phi, \theta)$	ϕ	θ	$\Omega^2(\phi, \theta)$	$\Omega^2_{\mathrm{Mc}} - \Omega^2(\phi, \theta)$
55° .	75°568	0 08569	+0 0158	64°	80°319	0 08860	−0 0133
56	76 089	08679	+ 0048	65	80 845	08778	− 0051
57..	76 613	08772	− 0045	66	81 368	08670	+ 0057
58.	77 140	0 08848	−0 0121	67	81 888	0 08534	+0 0194

Interpolated solutions:

$\phi = 56°490$, $\theta = 76°345$, $a_2/a_1 = 0.5861$, $a_3/a_1 = 0.5521$,

$\phi = 65°508$, $\theta = 81°111$, $a_2/a_1 = 0.4378$, $a_3/a_1 = 0.4146$.

TABLE 2

THE CONSTANTS OF THE DARWIN SEQUENCE FOR $M/M' = 0$

e	$\Omega^2/\pi G\rho_{\mathrm{Mc}}$	R/a_{Mc}	$R/\bar{a}_{\mathrm{Mc}}$	q
0 390	0 08287	2 4623	2 5309	3 0153
.400	.08727	2 4170	2 4883	3 0168
.402. . ..	.08816	2.4081	2 4799	3 0171
404	.08906	2 3994	2 4717	3.0174
0.405	0 08951	2 3950	2.4676	3 0175

TABLE 3

THE PROPERTIES OF THE DARWIN ELLIPSOIDS

($M/M' = 0$; $\rho = \rho_{\mathrm{Mc}}$)

$\phi = \cos^{-1} a_3/a_1$; $\theta = \sin^{-1}\sqrt{[(a_1^2 - a_2^2)/(a_1^2 - a_3^2)]}$; $A_i^* = a_1a_2a_3A_i$; $\bar{a}_i = a_i/(a_1a_2a_3)^{1/3}$

e	ϕ	θ	a_2/a_1	a_3/a_1	A_1^*	A_2^*	A_3^*	$\bar{a}_1$	$\bar{a}_2$	$\bar{a}_3$
0.390	52°857	74°471	0 64041	0.60380	0 435093	0 756253	0 808655	1 3726	0 8790	0 8288
.400..	56 490	76 345	58612	.55208	397593	774184	.828223	1 4566	.8537	8042
402..	57 569	76 911	56934	.53629	385819	779969	834212	1 4851	8455	7964
404 .	59 082	77 709	54531	.51382	.368813	.788445	842740	1 5282	.8333	7852
.405.	60 784	78 612	.51764	.48811	.349012	798498	852490	1 5818	.8188	7721
.405..	61 533	79 010	.50526	47665	.340077	803094	.856828	1 6073	.8121	7661
.404 .	63 182	79.884	.47762	.45116	319954	.813574	.866471	1 6680	.7967	.7525
402	64 564	80 615	45405	.42950	.302615	822751	874634	1 7244	.7830	7406
0.400.	65 508	81 111	0 43776	0 41457	0 290539	0 829214	0 880245	1 7663	0 7732	0 7322

problem, the secondary is considered as a rigid sphere and synchronism (in the sense that the major axes of the two ellipsoidal components remain collinear) loses its meaning. While the mathematical basis for the difference in the two problems is manifest, it is hard to believe that equilibrium along the combined Maclaurin-Jacobi sequence can be affected by considering the attendant infinitesimal secondary as a rigid sphere or as a liquid mass in equilibrium.

The constants describing the Darwin ellipsoid and the Maclaurin spheroid at $e_{\max}$ are listed in Table 4.

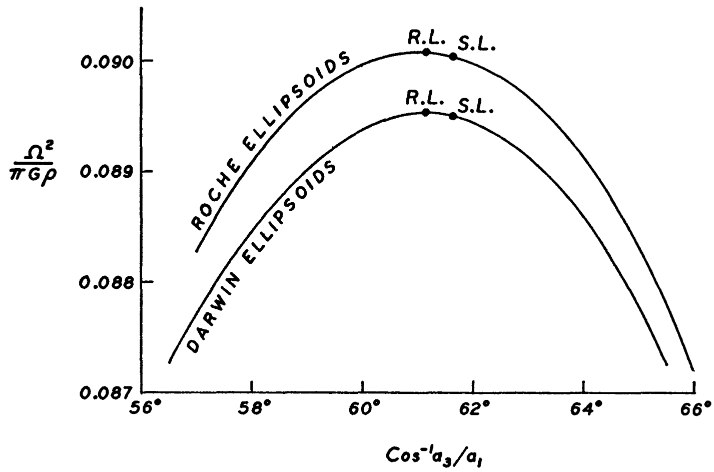

FIG. 1.—The variation of Ω^2 along the Darwin and the Roche sequences for the case $M/M' = 0$. The points R L. and S L. denote the Roche limit and the stability limit, respectively. Along both sequences the Roche limit occurs at the point where Ω^2 attains its maximum.

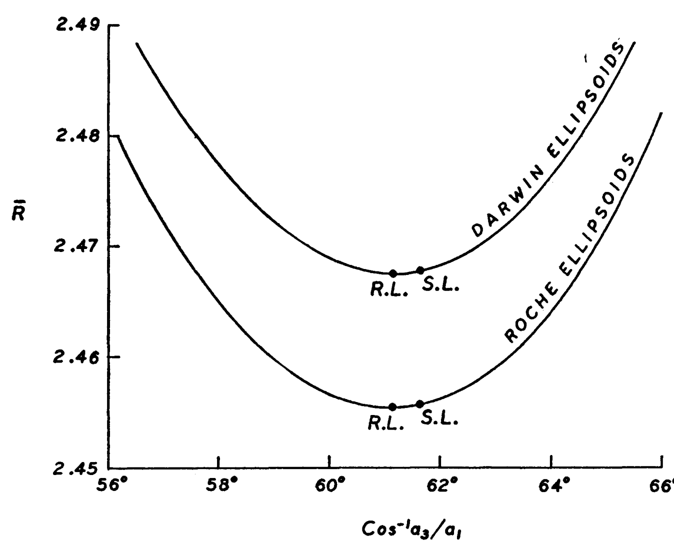

FIG. 2.—The variation of the distance between the centers of mass of the two components in the Roche and in the Darwin problems for the case $M/M' = 0$. Along both sequences the distance of separation is measured in the units of the mean radius of the central body. (In the Darwin problem the central body is a Maclaurin spheroid.) The points R L. and S L. have the same meanings as in Fig. 1. Along both sequences the Roche limit occurs at the distance of closest approach.

0 390	1 636	0 9636	+0 1635	1 647	0 7219	0 1110
400	1 688	8888	+ 0938	1 700	6728	1216
402	1 704	8643	+ 0735	1 716	6568	1243
404	1 726	8282	+ 0455	1 738	6331	1277

member can be deformed into a neighboring one by the application of an infinitesimal solenoidal ellipsoidal displacement without affecting equilibrium and where eq. [59] is satisfied) occurs at the same place.

c) The Characteristic Frequencies of Oscillation Belonging to the Second Harmonics: The Point at Which Instability Sets in along the Sequence

With the values of Ω^2 and q listed in Table 2, the coefficients β_j follow from equation (71); and equations (38) and (51) can be solved for their characteristic roots. The roots σ^2 $(= -\lambda^2)$ obtained in this manner are listed in Table 5. An examination of these roots shows that these Darwin ellipsoids (like the corresponding Roche ellipsoids) become unstable by a mode of oscillation belonging to the second harmonics at a point subsequent to the Roche limit. By interpolation among the roots σ_3^2 listed in Table 5, we find that instability sets in for the ellipsoid with the greater elongation at

$$e = e_s = 0.40498 . \tag{82}$$

TABLE 4

THE CONSTANTS OF THE DARWIN ELLIPSOIDS AND THE MACLAURIN SPHEROIDS AT THE ROCHE LIMIT (AT $e = e_{max} = 0.405034$) AND AT THE POINT OF ONSET OF INSTABILITY (AT $e = e_s = 0.40498$)

	Maclaurin Spheroid			Darwin Ellipsoid				
e	$\bar{a}_{Mc}$	$\Omega^2/\pi G\rho$	$R/\bar{a}_{Mc}$	ϕ	θ	$\bar{a}_1$	$\bar{a}_2$	$\bar{a}_3$
e_{max}	1 03032	0 089529	2 46745	61°16	78°81	1 5943	0 81549	0 76914
e_s	1 03031	0 089504	2 46767	61 64	79 07	1 6110	0 81114	0 76528

TABLE 5

THE SQUARES OF THE CHARACTERISTIC FREQUENCIES BELONGING TO THE SECOND HARMONICS

(σ^2 Is Listed in the Unit $\pi G\rho$)

	Even Modes			Odd Modes		
e	σ_1^2	σ_2^2	σ_3^2	σ_4^2	σ_5^2	σ_6^2

The constants describing the Darwin ellipsoid and the Maclaurin spheroid at $e = e_s$ are included in Table 4.

V. THE CASE OF CONGRUENT COMPONENTS

We now consider the case when the masses and the densities of the two components are the same. The components of the system will then be congruent and the reduction to a stationary solution in a common rotating frame of reference can be accomplished without any ambiguity. The problem then is to determine the geometry of the ellipsoidal figures of equilibrium (in the framework of the approximation described in Sec. II) when the centers of mass of the two components are at a distance R apart and they are rotating about each other with an angular velocity Ω consistent with their figures and their separation. The solution of this problem can be accomplished as follows.

Consider a homogeneous ellipsoid of mass M and semi-axes a_j. The gravitational potential $\mathfrak{B}$, at an external point x_j, is given by

$$\mathfrak{B}(x_j) = -\tfrac{3}{4}GM\int_{\lambda}^{\infty}\left(\sum_{j=1}^{3}\frac{x_j^2}{a_j^2+u}-1\right)\frac{du}{\Delta(u)}, \tag{83}$$

where

$$\Delta^2(u) = (a_1^2+u)(a_2^2+u)(a_3^2+u), \tag{84}$$

and λ is the positive root of the equation

$$\sum_{j=1}^{3}\frac{x_j^2}{a_j^2+\lambda} = 1. \tag{85}$$

For the application of the general method described in Section II, we need the first and the second derivatives of $\mathfrak{B}$ with respect to the spatial coordinates at $(R, 0, 0)$. For such a point

$$\lambda = R^2 - a_1^2 \tag{86}$$

and

$$\Delta(\lambda) = R[(R^2+a_2^2-a_1^2)(R^2+a_3^2-a_1^2)]^{1/2}. \tag{87}$$

Also, in evaluating the derivatives of $\mathfrak{B}$ given by equation (83), we must allow for the fact that λ, as determined by equation (85), is, implicitly, a function of the coordinates and that

$$\frac{\partial\lambda}{\partial x_i} = \frac{2x_i}{a_i^2+\lambda}\left[\sum_{j=1}^{3}\frac{x_j^2}{(a_j^2+\lambda)^2}\right]^{-1}. \tag{88}$$

By straightforward calculations we now find:

$$\left(\frac{\partial\mathfrak{B}}{\partial x_1}\right)_{R,0,0} = -\alpha_1 R, \quad \left(\frac{\partial\mathfrak{B}}{\partial x_2}\right)_{R,0,0} = \left(\frac{\partial\mathfrak{B}}{\partial x_3}\right)_{R,0,0} = 0, \tag{89}$$

$$\left(\frac{\partial^2\mathfrak{B}}{\partial x_1^2}\right)_{R,0,0} = -\alpha_1 + 3\frac{GM}{\Delta(\lambda)}. \tag{90}$$

$$\left(\frac{\partial^2\mathfrak{B}}{\partial x_2^2}\right)_{R,0,0} = -\alpha_2, \quad \text{and} \quad \left(\frac{\partial^2\mathfrak{B}}{\partial x_3^2}\right)_{R,0,0} = -\alpha_3, \tag{91}$$

where

$$\alpha_j = \tfrac{3}{2}GM\int_{R^2-a_1^2}^{\infty}\frac{du}{(a_j^2+u)\Delta(u)}. \tag{92}$$

It follows from equations (90) and (91), and it can also be verified directly from the definitions of the α_j's, that

$$\alpha_1 + \alpha_2 + \alpha_3 = 3 \frac{GM}{\Delta(\lambda)} . \tag{93}$$

Making use of the foregoing expressions, we can write for the tidal potential, acting on the component whose equilibrium we are considering by the other component (which is congruent to it), the expansion

$$\mathfrak{B}' = \mathfrak{B}'(R, 0, 0) + \alpha_1 R x_1 + \tfrac{1}{2}[(\alpha_2 + \alpha_3)x_1^2 - \alpha_2 x_2^2 - \alpha_3 x_3^2] + \ldots ; \tag{94}$$

and the equation of motion (1), governing the fluid elements of M, takes the form (since, now, $M = M'$)

$$\rho \frac{du_i}{dt} = -\frac{\partial p}{\partial x_i} + \rho \frac{\partial}{\partial x_i} \{ \mathfrak{B} + \tfrac{1}{2}\Omega^2(x_1^2 + x_2^2) + \tfrac{1}{2}[(\alpha_2 + \alpha_3)x_1^2 - \alpha_2 x_2^2 - \alpha_3 x_3^2] - x_1 R(\tfrac{1}{2}\Omega^2 - \alpha_1)\} + 2\rho\Omega\epsilon_{il3}u_l . \tag{95}$$

Hence, by the choice

$$\Omega^2 = 2\alpha_1 , \tag{96}$$

equation (95) reduces to the standard form (7), with the coefficients β_j having the values

$$\beta_1 = 2\alpha_1 + \alpha_2 + \alpha_3 , \qquad \beta_2 = 2\alpha_1 - \alpha_2 , \qquad \text{and} \qquad \beta_3 = -\alpha_3 . \tag{97}$$

The coefficients α_j are expressible in terms of the standard elliptic integrals, $E(\theta, \phi)$ and $F(\theta, \phi)$, of the two kinds with the arguments (cf. eq. [80])

$$\theta = \sin^{-1}\sqrt{\frac{a_1^2 - a_2^2}{a_1^2 - a_3^2}} \qquad \text{and} \qquad \phi_R = \sin^{-1}\sqrt{\frac{a_1^2 - a_3^2}{R^2}} . \tag{98}$$

Thus,

$$\alpha_1 = \frac{4}{(a_1^2 - a_3^2)^{3/2}} \frac{1}{\sin^2\theta} [F(\theta, \phi_R) - E(\theta, \phi_R)] , \tag{99}$$

$$\alpha_2 = \frac{4}{(a_1^2 - a_3^2)^{3/2}} \frac{1}{\sin^2\theta \cos^2\theta} \times \left[E(\theta, \phi_R) - F(\theta, \phi_R)\cos^2\theta - \frac{\sin^2\theta \sin\phi_R \cos\phi_R}{\sqrt{(1 - \sin^2\theta \sin^2\phi_R)}}\right] , \tag{100}$$

$$\alpha_3 = \frac{4}{(a_1^2 - a_3^2)^{3/2}} \frac{1}{\cos^2\varphi} [\tan\phi_R \sqrt{(1 - \sin^2\theta \sin^2\phi_R)} - E(\theta, \phi_R)] , \tag{101}$$

where a common factor $\pi G\rho a_1 a_2 a_3$ $(= 3GM/4)$, in the expressions for the α's, has been suppressed.

a) The Properties of the Equilibrium Ellipsoids

With the coefficients β_j given in equations (97), the virial equations (13) now give

$$\frac{2(A_1 - A_3 a_3^2/a_1^2)}{2 + (\alpha_2/\alpha_1) + (1 + a_3^2/a_1^2)(\alpha_3/\alpha_1)} = \frac{2[(A_2 a_2^2/a_1^2) - (A_3 a_3^2/a_1^2)]}{(2 - \alpha_2/\alpha_1)(a_2^2/a_1^2) + (\alpha_3 a_3^2)/(\alpha_1 a_1^2)} = \alpha_1 , \tag{102}$$

where it may be recalled that the coefficients A_1, A_2, and A_3 are also expressible in terms of the elliptic integrals of the two kinds with, however, the arguments $\phi = \cos^{-1} a_3/a_1$

(which is different from ϕ_R) and θ (which is the same as in eqs. [99]–[101] for the α's).

A convenient procedure for solving equations (102) and determining the figures of the equilibrium ellipsoids is the following.

We first assign an angle ϕ_R. Then for a chosen ϕ, we determine, by a method of trial and error, the angle θ (which occurs in the expressions for both the A's and the α's) such that the first of the two equalities in (102) is satisfied. But for an arbitrarily chosen ϕ, the value of α_1, which the second equality in (102) requires, will not agree with the value which the assigned ϕ_R and the determined θ will require according to equation (99). The problem, then, is to "adjust" ϕ such that, for the assigned ϕ_R, the values of α_1 required by equations (99) and (102) agree. Table 6 illustrates a sample calculation of how this agreement may be accomplished.

In Table 7 we give the results of the calculations for various initially assigned values of ϕ_R. The results are further illustrated in Figures 3 and 4.

TABLE 6

A Sample Calculation: $\phi_R = 20°$

ϕ	θ	α_1*(by eq [102])	α_1 (by eq [100])	$\alpha_1^* - \alpha$
48°	66°522	0 05696	0 06810	−0 01113
49	67 167	05840	06261	− 00421
50	67 823	05978	05757	+ 00221
51	68 488	0 06107	0 05292	+0 00815

Interpolated solution:

$\phi = 49°65$, $\theta = 67°59$, $\alpha_1 = 0.05930$.

TABLE 7

The Constants of the Darwin Ellipsoids for the Case of Congruent Components

$\phi_R = \sin^{-1}\sqrt{[(a_1^2 - a_3^2)/R^2]}$; $\phi = \cos^{-1} a_3/a_1$;

$\theta = \sin^{-1}\sqrt{[(a_1^2 - a_2^2)/(a_1^2 - a_3^2)]}$; $A_i^* = (a_1a_2a_3)A_i$; $\alpha_i^* = (a_1a_2a_3)\alpha_i$

ϕ_R	ϕ	θ	A_1^*	A_2^*	A_3^*	α_1^*	α_2^*	α_3^*
10°	33°047	57°912	0 589264	0 681901	0 728836	0 032510	0 032941	0 033113
14	40 296	61 630	546740	694712	758548	.045185	.046462	046852
17	45 154	64 579	511346	707511	781143	053141	055504	.056072
20	49 647	67 590	473159	723240	803601	059300	063186	.063908
25.	56 511	72 562	404072	.755948	839979	064760	072082	072914
29	61 566	76 320	344941	787365	.867694	064360	074908	075666
30	62 780	77 212	329774	795825	874400	063587	.074944	075664
40	73 922	84 621	0 177895	0 886727	0 935377	0 043609	0.060124	0 060340

$\bar{a}_i = a_i/(a_1a_2a_3)^{1/3}$; $\bar{R} = R/(a_1a_2a_3)^{1/3}$

ϕ_R	$\Omega^2/\pi G\rho$	$\bar{R}$	a_2/a_1	a_3/a_1	$\bar{a}_1$	$\bar{a}_2$	$\bar{a}_3$
10°. . .	0 06502	3 4667	0 88687	0 83822	1.1039	0 97901	0.92531
14	.09037	3 1231	.82229	.76271	1 1683	.96065	89104
17 .	10628	2 9749	76808	70520	1 2268	.94225	.86512
20 .	.11860	2 8874	.70969	.64750	1 2959	.91967	.83908
25	.12952	2 8436	.60574	.55178	1 4410	.87284	.79509
29	.12873	2 8894	.51958	.47614	1 5930	.82767	.75847
30	12717	2 9122	49796	.45741	1 6374	81538	.74898
40 . .	0.08722	3 4599	0.29123	0.27695	2.3145	0 67405	0.64100

We observe that along the Darwin sequence, the place where Ω^2 attains its maximum value is different from the place where R attains its minimum value. And it is also seen that at a certain point along the sequence $R = 2a_1$; at this point the two components are in contact; and beyond this point $R < 2a_1$ and the two components are overlapping. From Figure 4 it would appear that contact occurs very nearly where R attains its minimum value; but a more careful examination reveals (see Table 8) that at the distance of closest approach the components are almost, but not quite, in contact. It follows from these results that, of the two congruent figures of equilibrium which one obtains as solutions of the relevant equations (at each separation), those with the greater elongations overlap; consequently, these second solutions, "although satisfying the mathematical equations, are physically impossible."

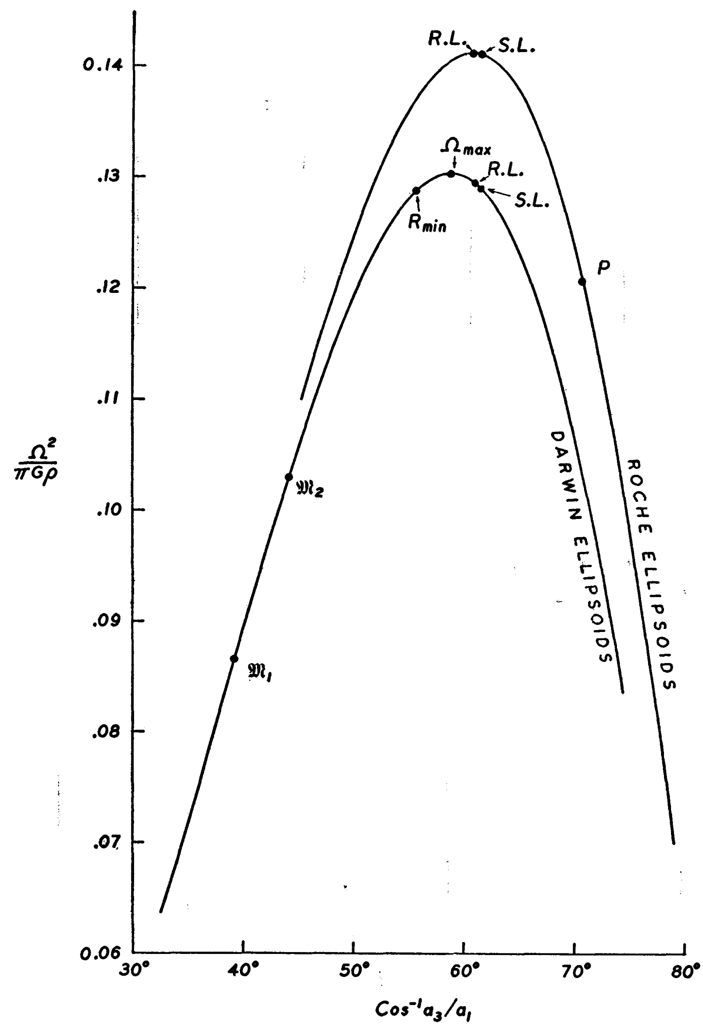

FIG. 3.—The variation of Ω^2 along the Darwin sequence of congruent components of equal mass. Along the sequence we distinguish the points $\mathfrak{M}_1$, $\mathfrak{M}_2$, R_{min}, and Ω_{max}, where the angular momenta $\mathfrak{M}_1$ and $\mathfrak{M}_2$ (see eq. [105]), the distance ($\bar{R}$) between the centers of mass of the two components, and the angular velocity Ω attain their extreme values. The Roche limit occurs at *R.L.*, and instability by a mode of oscillation belonging to the second harmonics sets in at *S.L.* The results for the Roche sequence for the case $M/M' = 1$ are included for comparison; along this sequence, besides the Roche limit (*R.L.*) and stability limit (*S.L.*), we also have the point (P) where instability by a mode of oscillation belonging to the third harmonics sets in.

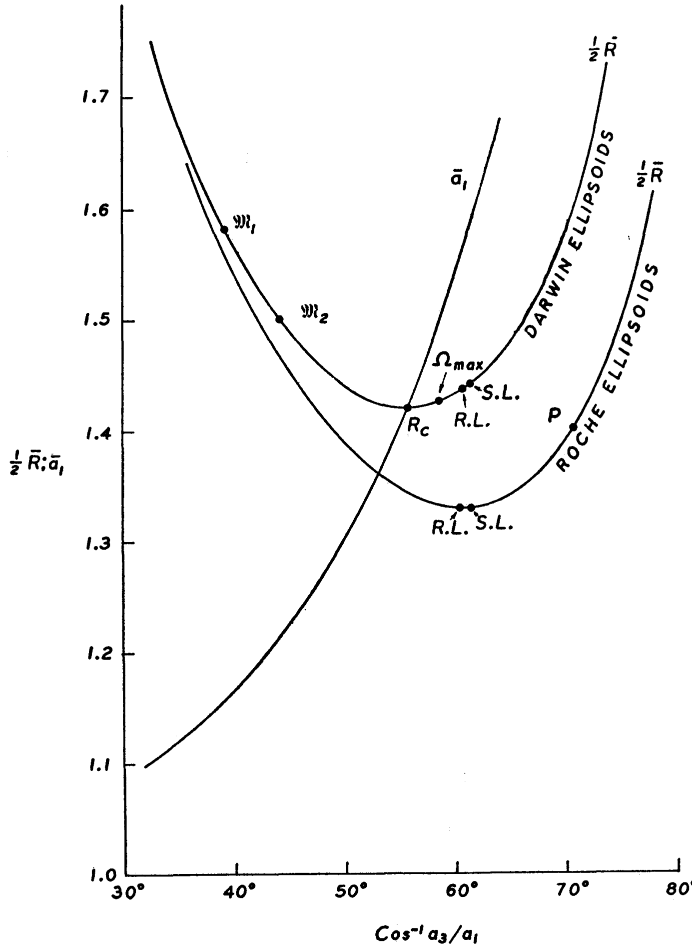

FIG. 4.—The variation of the distance ($\bar{R}$) between the centers of mass of the two components and of the semimajor axis ($\bar{a}_1$) along the Darwin sequence of congruent components. The points indicated by $\mathfrak{M}_1$, $\mathfrak{M}_2$, Ω_{max}, R $L.$, and $S.L.$ have the same meanings as in Fig. 3. At R_c, where the curves for $\frac{1}{2}\bar{R}$ and $\bar{a}_1$ cross, the two components are in contact. It will be observed that R_c occurs very close to the distance of closest approach The results for the Roche sequence for the case $M/M' = 1$ are included for comparison; and the various points along this sequence have the same meanings as in Fig. 3.

TABLE 8

THE CONSTANTS OF THE ELLIPSOIDS AT THE VARIOUS CRITICAL POINTS*

Critical Point	ϕ_R	ϕ	$\bar{R}$	$\Omega^2/\pi G\rho$	$\bar{a}_1$	$\bar{a}_2$	$\bar{a}_3$
$\mathfrak{M}_1$ (min)	13°34	39°17	3 1667	0 08648	1 1567	0 96414	0 89670
$\mathfrak{M}_2$ (min)	16 33	44 10	3 0017	10301	1 2128	94671	87092
R (min)	24 34	55 64	2 8426	12883	1 4192	87966	80098
$R=2a_1$.	24 40	55 72	2 8426	12890	1 4212	87902	80043
Ω^2 (max)	26 64	58 62	2 8536	13025	1 4986	85511	78030
Roche limit	28 36	60 77	2 8771	12944	1 5658	83537	76450
Stability limit	28 89	61 43	2 8871	0 12887	1 5882	0 82902	0 75953

* Basing himself on Darwin's calculation, Jeans (1919) quotes values for the configurations $\mathfrak{M}_1$ (min), $\mathfrak{M}_2$ (min), R (min), and $R = 2a_1$; but they are all substantially different from the ones obtained in this paper Thus, for the configurations enumerated, according to Jeans, $\bar{R}$ = 3 324, 3 167, 2 952, and 2 843, respectively; and these values should be contrasted with the ones given in the table

And finally, we may also distinguish along the Darwin sequence a Roche limit where equation (59) is satisfied. It is found that this point is again different from the points where Ω^2 and R attain their extremes. Along the Roche sequences these three points coincide; but along the Darwin sequence they are all distinct.

The constants of the ellipsoids at the different "critical" points we have enumerated (and others besides) are listed in Table 8.

b) The Characteristic Frequencies of Oscillation Belonging to the Second Harmonics: The Point of Onset of Instability

With the values of a_j listed in Table 7, the coefficients β_j follow from equation (97); and equations (38) and (51) can be solved for their characteristic roots. The roots σ^2 $(= -\lambda^2)$ obtained in this manner are listed in Table 9. An examination of the roots listed in this table shows that the Darwin ellipsoids (like the Roche ellipsoids) become unstable by a mode of oscillation belonging to the second harmonics. The point beyond which the Darwin ellipsoids are unstable was determined by interpolation among the roots σ_3^2 listed in Table 9; and the constants of the corresponding ellipsoid are included in Table 8.

TABLE 9

THE SQUARES OF THE CHARACTERISTIC FREQUENCIES BELONGING TO THE SECOND HARMONICS

(σ^2 Is Listed in the Unit $\pi G\rho$)

ϕ_R	Even Modes			Odd Modes		
	σ_1^2	σ_2^2	σ_3^2	σ_4^2	σ_5^2	σ_6^2
10°.	1 469	1 1811	+0 4608	1 4243	0 8611	0 0715
14	1 525	1 1877	+ 3306	1.5182	.8200	.1040
17.	1 569	1 1621	+ 2475	1 5820	7864	1267
20.	1.617	1 1100	+ .1741	1 6412	7479	.1467
25.	1.703	0 9748	+ .0690	1 7305	6688	.1712
29.	1 770	0 8370	− 0018	1 7939	5910	1802
30	1 786	0 7999	− .0177	1.8084	5696	.1807
40. .. .	1.887	0.4254	−0 1244	1 9264	0 3299	0.1457

We observe that the point at which instability sets in is subsequent to *all* the points considered in Section V*a* above. In particular, since instability first occurs among the more elongated of the ellipsoids, it is clear that *all the physically realizable ellipsoids are stable with respect to their own natural oscillations.*

c) The Variation of Angular Momenta along the Sequence: The Criteria of Darwin and Jeans

In the earlier discussions of the problem of the stability of double-star systems by Darwin and Jeans, it is argued that the discriminant, relevant for distinguishing stability along a sequence, is the total angular momentum of the system; and that, in the context, the relevant quantity is

$$\mathfrak{M}_1 = \left[(I_{11} + I_{22})_M + (I_{11} + I_{22})_{M'} + \frac{MM'}{M+M'} R^2 \right] \Omega \,, \tag{103}$$

where the subscripts M and M' signify that the quantity enclosed refers to that particular component. Precisely, the assertion of Darwin and Jeans is that the minimum of $\mathfrak{M}_1$

along the sequence determines the "*configuration of limiting stability*" (Jeans). In addition to this concept of "limiting stability," Darwin introduces the concept of "*partial stability*" to describe a system which "is stable except for tidal friction arising from the tides in the primary" (Jeans). And, further, according to Jeans, "Darwin believes that the limit of partial stability of a series of configurations . . . can be found by discovering the value at which

$$\mathfrak{M}_2 = \left[(I_{11} + I_{22})_M + \frac{MM'}{M+M'} R^2 \right] \Omega \tag{104}$$ [5]

is a minimum, the value of $\mathfrak{M}_2$ representing all that part of the moment of momentum which is liable to variation when tides cannot be raised in M'."

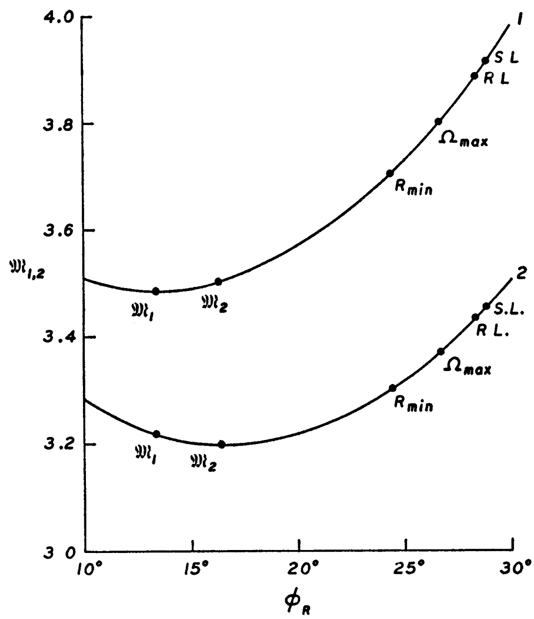

FIG. 5.—The variation of the angular momenta $\mathfrak{M}_1$ (curve labeled "1") and $\mathfrak{M}_2$ (curve labeled "2") (see eq [105]) along the Darwin sequence of congruent components. The various critical points indicated on the curves have the same meanings as in Fig 3.

For the case of congruent components considered in this section,

$$\mathfrak{M}_1 = \tfrac{1}{2} M \bar{a}^2 [\tfrac{4}{5}(\bar{a}_1^2 + \bar{a}_2^2) + \bar{R}^2] \Omega$$

and

$$\mathfrak{M}_2 = \tfrac{1}{2} M \bar{a}^2 [\tfrac{2}{5}(\bar{a}_1^2 + \bar{a}_2^2) + \bar{R}^2] \Omega \,, \tag{105}$$

where $\bar{a}(= \sqrt[3]{[a_1 a_2 a_3]})$ is the mean radius of either component and is a constant along the sequence. The variation of $\mathfrak{M}_1$ and $\mathfrak{M}_2$ along the sequence we have constructed is exhibited in Table 10 and Figure 5. We observe that $\mathfrak{M}_1$ and $\mathfrak{M}_2$ do show minima along the sequence. We find that

$$\mathfrak{M}_1(\min) = 3.481 \frac{M\bar{a}^2}{2\pi G\rho} \quad \text{and} \quad \mathfrak{M}_2(\min) = 3.196 \frac{M\bar{a}^2}{2\pi G\rho} \,. \tag{106}$$

The further constants describing the ellipsoid at $\mathfrak{M}_1$ (min) and $\mathfrak{M}_2$ (min) are included in Table 8.

[5] In writing this expression, we have departed slightly from Jeans's notation.

If we trust the arguments of Darwin and Jeans, we should conclude that "instability unconditionally" sets in at $\mathfrak{M}_1$ (min) and "presumably" sets in at $\mathfrak{M}_2$ (min). But instability, in the strict sense we are using that term, certainly does not set in at either point by any natural mode of oscillation of either component by itself. The question remains whether the tidal coupling between the two components can induce a further instability or at least a neutral mode of oscillation. No treatment of such coupled oscillations exists at the present time. And it would appear that only by such a treatment can criteria, similar to those of Darwin and Jeans, emerge.

TABLE 10

THE VARIATION OF THE ANGULAR MOMENTA $\mathfrak{M}_1$ AND $\mathfrak{M}_2$ ALONG THE DARWIN SEQUENCE OF CONGRUENT COMPONENTS

[$\mathfrak{M}_1$ and $\mathfrak{M}_2$ Are Listed in the Unit $M\bar{a}^2/(2\pi G\rho)$]

ϕ_R	$\mathfrak{M}_1$	$\mathfrak{M}_2$	ϕ_R	$\mathfrak{M}_1$	$\mathfrak{M}_2$
10°	3 509	3 286	25°	3 727	3 319
14	3 482	3 207	29	3 920	3 458
17	3 509	3 197	30	3 979	3 502
20	3 567	3 219	40	4 908	4 222

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