# THE EQUILIBRIUM AND THE STABILITY OF THE ROCHE ELLIPSOIDS

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ABSTRACT

Roche's problem is concerned with the equilibrium and the stability of rotating homogeneous masses which are, further, distorted by the constant tidal action of an attendant rigid spherical mass. This ancient problem is reconsidered in this paper with the principal object of determining the stability of the equilibrium configurations (the ellipsoids of Roche) by a direct evaluation of their characteristic frequencies of oscillation belonging to the second harmonics The result of the evaluation is the demonstration that the Roche ellipsoid becomes unstable at a point subsequent to the Roche limit where the angular velocity of rotation, consistent with equilibrium, attains its maximum value. This result requires a revision of the current common view regarding the meaning that is to be attached to the Roche limit

Among related matters which are considered are the following: the relationships that exist between the sequences of Roche and those of Maclaurin, Jacobi, and Jeans; the exhibition and the isolation of the second neutral point (belonging to the third harmonics) along the Roche sequences; and the effect of compressibility on the stability of the Roche ellipsoids. A result which emerges from these considerations is the universal instability of the Jacobi ellipsoids under the least tidal action.

The methods used in this paper are those derived from the virial theorem and its various extensions. The principal results are summarized in Section X and are exhibited in Figures 1, 2, and 3.

## I. INTRODUCTION

Roche discovered in 1850 that no equilibrium configuration exists for an infinitesimal homogeneous satellite (of density $\rho$) rotating about a planet (of mass $M'$) in a circular Keplerian orbit (of radius $R'$), if the angular velocity ($\Omega$) of rotation exceeds the limit set by

$$\frac{\Omega^2}{\pi G\rho}=\frac{M'}{\pi\rho R'^3}\leq 0.090068\,, \tag{1}$$ [1]

where $G$ denotes the constant of gravitation. The lower limit to $R'$ set by this inequality is called the *Roche limit.* It is generally believed that the non-existence of equilibrium configurations below the Roche limit implies some sort of instability for the satellite. Thus, Darwin (to whom we owe the term "Roche limit") describes Roche's result as follows. "Now Roche showed that instability will set in when the elongation of figure of the satellite has reached a certain degree. In other words, at a certain stage of proximity, the satellite cannot hold together by the force of its own gravitation, and it will be torn apart by the tide generating force" (Darwin 1911, p. 340). And Darwin's description expresses the still prevalent common view (e.g., Struve 1961, p. 18). Nevertheless, if Darwin's statement is taken to mean (what it apparently says) that the satellite is capable of a certain normal mode of oscillation with respect to which it becomes unstable as the Roche limit is approached (in the direction of increasing $\Omega^2$), then, the statement (as we shall show in Sec. VIII below) is incorrect. However, other statements with other meanings have also been made concerning the nature of the "instability" which is supposed to set in at the Roche limit. Thus, Jeans (1919, pp. 52 and 53) has written: "We are dealing, it must be noted, with secular stability[2] only; the question means nothing except when dissipative forces are present . . . the instability is one of orbital motion only and

[1] Roche (1850) originally gave the value 0.092 for the constant on the right-hand side of this inequality. Darwin (1906) later gave the more precise value 0 09006 The value quoted, 0.090068, is that obtained in the present paper (see Table 2 in Sec. V below).

[2] For the meaning generally attached to this term see Chandrasekhar (1963*a*, p 1187; this paper will be referred to hereinafter as "Paper I").

not one of the configurations of the masses." And, again, as we shall see, much of the implications of this statement are also incorrect.

The conflicting statements and views (which we have, in part, quoted) concerning the role of the Roche limit for the stability of the satellite arise, mostly, from the incompleteness of the available analytical information on the Roche ellipsoids, and the lack of any specific investigation on their normal modes of oscillation.[3] On these accounts, it has appeared worthwhile to reconsider this ancient problem and provide as complete and as explicit a solution as seemed necessary to clarify the basic issues.

## II. ROCHE'S PROBLEM

Roche's problem is concerned with a particularly simple case in the equilibrium and the stability of a homogeneous body ("the primary") rotating about another ("the secondary") in a manner that their relative disposition remains the same.

Let the masses of the primary and the secondary be $M$ and $M'$, respectively.[4] Let the distance between their centers of mass be $R'$; and let the constant angular velocity of rotation about their common center of mass be $\Omega$.

Choose a coordinate system in which the origin is at the center of mass of $M$, the $x_1$-axis is pointing to the center of mass of $M'$, and the $x_3$-axis is parallel to the direction of $\Omega$. In this coordinate system, the equation of the axis of rotation is

$$x_1 = \frac{M'}{M+M'} R' \qquad \text{and} \qquad x_2 = 0 . \tag{2}$$

In this frame of reference, rotating uniformly with the angular velocity $\Omega$, the equation of motion governing the fluid elements of $M$ is

$$\rho \frac{du_i}{dt} = -\frac{\partial p}{\partial x_i} + \rho \frac{\partial}{\partial x_i} \left\{ \mathfrak{B} + \mathfrak{B}_T + \tfrac{1}{2}\Omega^2 \left[ \left( x_1 - \frac{M'R'}{M+M'} \right)^2 + x_2^2 \right] \right\} + 2\rho\Omega\epsilon_{il3} u_l , \tag{3}$$

where $\mathfrak{B}$ is the self-gravitational potential and $\mathfrak{B}_T$ is the tidal potential due to $M'$. Equation (3) can also be written as

$$\rho \frac{du_i}{dt} = -\frac{\partial p}{\partial x_i} + \rho \frac{\partial}{\partial x_i} \left[ \mathfrak{B} + \mathfrak{B}_T + \tfrac{1}{2}\Omega^2 ( x_1^2 + x_2^2 ) - \frac{M'R'}{M+M'} \Omega^2 x_1 \right] + 2\rho\Omega\epsilon_{il3} u_l . \tag{4}$$

In Roche's particular problem, the secondary is treated as a rigid sphere; then, over the primary, the tide-generating potential of $\mathfrak{B}_T$ can be expanded in the form

$$\mathfrak{B}_T = \frac{GM'}{R'} \left( 1 + \frac{x_1}{R'} + \frac{x_1^2 - \frac{1}{2} x_2^2 - \frac{1}{2} x_3^2}{R'^2} + \dots \right). \tag{5}$$

As is customary in the treatment of this problem, we shall retain in the expansion for $\mathfrak{B}_T$ only those terms which we have explicitly written out in equation (5). On this assumption concerning $\mathfrak{B}_T$, the equation of motion becomes

$$\rho \frac{du_i}{dt} = -\frac{\partial p}{\partial x_i} + \rho \frac{\partial}{\partial x_i} \left[ \mathfrak{B} + \tfrac{1}{2}\Omega^2 ( x_1^2 + x_2^2 ) + \mu ( x_1^2 - \tfrac{1}{2} x_2^2 - \tfrac{1}{2} x_3^2 ) + \left( \frac{GM'}{R'^2} - \frac{M'R'}{M+M'} \Omega^2 \right) x_1 \right] + 2\rho\Omega\epsilon_{il3} u_l , \tag{6}$$

[3] Thus, in Figs 7 and 15 in Jeans (1919, pp. 50 and 86) none of the lines representing the Roche sequences are based on any calculation; the same applies to Fig. 33 in Jeans (1929, p. 229). Milne's (1952, chap. ix; see particularly pp. 110–112) generous, but perceptive, account in his biography of Jeans brings out, by its accuracy, the inadequacy of the presently available information.

[4] By using the terms "primary" and "secondary" to describe the two bodies, we do not wish to imply that $M > M'$; indeed, in the important special case considered in Sec. I, $M \ll M'$.

where we have introduced the abbreviation

$$\mu=\frac{GM'}{R'^3}. \tag{7}$$

Now letting $\Omega^2$ have its "Keplerian value"

$$\Omega^2=\frac{G(M+M')}{R'^3}=\mu\left(1+\frac{M}{M'}\right), \tag{8}$$

we obtain the basic equation of this theory:

$$\rho\frac{du_i}{dt}=-\frac{\partial p}{\partial x_i}+\rho\frac{\partial}{\partial x_i}[\mathfrak{B}+\tfrac{1}{2}\Omega^2(x_1^2+x_2^2)+\mu(x_1^2-\tfrac{1}{2}x_2^2-\tfrac{1}{2}x_3^2)]+2\rho\Omega\epsilon_{il3}u_l; \tag{9}$$

and *the problem of Roche is that of the equilibrium and the stability of homogeneous configurations governed by equation* (9).

### a) *The Second-Order Virial Theorem Appropriate to Roche's Problem*

In treating Roche's problem, we shall use the methods, based on the virial theorem and its extensions, which have been developed recently (see the various papers by Chandrasekhar and by Chandrasekhar and Lebovitz in the *Astrophysical Journal* for the past three years).

By multiplying equation (9) by $x_j$ and integrating over the volume $V$ occupied by the fluid, we obtain in the usual manner the second-order tensor equation (cf. Chandrasekhar and Lebovitz 1963*b*, eq. [4]; this paper will be referred to hereafter as "Paper III")

$$\frac{d}{dt}\int_V\rho u_i x_j dx=2\mathfrak{T}_{ij}+\mathfrak{W}_{ij}+(\Omega^2-\mu)I_{ij}-\Omega^2\delta_{i3}I_{3j}+3\mu\delta_{i1}I_{1j}$$
$$+\delta_{ij}\Pi+2\Omega\int_V\rho\epsilon_{il3}u_l x_j dx, \tag{10}$$

where

$$\Pi=\int_V p dx, \tag{11}$$

and

$$\mathfrak{T}_{ij}=\tfrac{1}{2}\int_V\rho u_i u_j dx,\qquad \mathfrak{W}_{ij}=-\tfrac{1}{2}\int_V\rho\mathfrak{B}_{ij}dx,\qquad \text{an}\qquad I_{ij}=\int_V\rho x_i x_j dx \tag{12}$$

are the kinetic-energy, the potential-energy, and the moment of inertia tensors.

## III. PROPERTIES OF THE EQUILIBRIUM ELLIPSOIDS

When no motions are present in the frame of reference considered and hydrostatic equilibrium prevails, equation (10) becomes

$$\mathfrak{W}_{ij}+(\Omega^2-\mu)I_{ij}-\Omega^2\delta_{i3}I_{3j}+3\mu\delta_{i1}I_{1j}=-\Pi\delta_{ij}. \tag{13}$$

The diagonal elements of this relation give

$$\mathfrak{W}_{11}+(\Omega^2+2\mu)I_{11}=\mathfrak{W}_{22}+(\Omega^2-\mu)I_{22}=\mathfrak{W}_{33}-\mu I_{33}=-\Pi, \tag{14}$$

while the non-diagonal elements give:

$$\mathfrak{W}_{12}+(\Omega^2+2\mu)I_{12}=\mathfrak{W}_{21}+(\Omega^2-\mu)I_{21}=0,$$
$$\mathfrak{W}_{23}+(\Omega^2-\mu)I_{23}=\mathfrak{W}_{32}-\mu I_{32}=0, \tag{15}$$
$$\mathfrak{W}_{31}-\mu I_{31}=\mathfrak{W}_{13}+(\Omega^2+2\mu)I_{13}=0.$$

In view of the symmetry of the tensors $I_{ij}$ and $\mathfrak{W}_{ij}$, it follows from the foregoing equations that, so long as $\Omega^2$ and $\mu$ are finite,

$$\mathfrak{W}_{ij} = 0 \quad \text{and} \quad I_{ij} = 0 \quad (i \neq j) . \tag{16}$$

Therefore, *in the chosen coordinate system, the tensors $I_{ij}$ and $\mathfrak{W}_{ij}$ are, necessarily, diagonal.*

Equations (13)–(16) are entirely general: they do not depend on any constitutive relations that may exist.

Now it can be shown quite readily that if the configuration is homogeneous (by assumption or by virtue of incompressibility), then an ellipsoidal figure is consistent with the equations of hydrostatic equilibrium as well as the condition which requires the pressure to be constant over the bounding surface. Once this has been established, the virial equations suffice (as we shall presently see) to determine the geometry and the properties of the equilibrium ellipsoids.

Letting

$$p = \frac{M}{M'} \quad \text{so that} \quad \Omega^2 = (1 + p)\mu , \tag{17}$$

we can rewrite equations (14) in the form

$$\mathfrak{W}_{11} + (3 + p)\mu I_{11} = \mathfrak{W}_{22} + p\mu I_{22} = \mathfrak{W}_{33} - \mu I_{33} , \tag{18}$$

or, alternatively,

$$\mu[(3 + p)I_{11} + I_{33}] = \mathfrak{W}_{33} - \mathfrak{W}_{11} \tag{19}$$

and

$$\mu(pI_{22} + I_{33}) = \mathfrak{W}_{33} - \mathfrak{W}_{22} . \tag{20}$$

The geometry of the ellipsoids (for an assigned $p$) is, therefore, determined by the equation

$$\frac{(3 + p) I_{11} + I_{33}}{p I_{22} + I_{33}} = \frac{\mathfrak{W}_{33} - \mathfrak{W}_{11}}{\mathfrak{W}_{33} - \mathfrak{W}_{22}} . \tag{21}$$

Expressions for the various tensors describing the properties of homogeneous ellipsoids have been given in an earlier paper (Chandrasekhar and Lebovitz 1962*a*); in particular, we have

$$\mathfrak{W}_{ii} = -2\pi G\rho(a_1 a_2 a_3) A_i I_{ii} \quad \text{and} \quad I_{ii} = \tfrac{1}{5} M a_i^2 , \tag{22}$$

(no summation over repeated indices)

where $a_1$, $a_2$, and $a_3$ are the semi-axes of the ellipsoid and the $A_i$'s are the one-index symbols defined in the same paper.

Inserting the expressions for $\mathfrak{W}_{ii}$ and $I_{ii}$ in equation (21), we obtain

$$\frac{(3 + p) a_1^2 + a_3^2}{p a_2^2 + a_3^2} = \frac{A_1 a_1^2 - A_3 a_3^2}{A_2 a_2^2 - A_3 a_3^2} . \tag{23}$$

Using the known expressions for the constants $A_i$ (*loc. cit.*, eqs. [15]–[17]), we can reduce equation (23) to the following form which is convenient for numerical calculations:

$$\begin{aligned} &[(p + 3) a_1^4 (a_2^2 + a_3^2) + p a_2^4 (a_3^2 + a_1^2) - a_3^4 (a_1^2 + a_2^2) \\ &- 4(p+1) a_1^2 a_2^2 a_3^2] E(\theta, \phi) - (a_2^2 - a_3^2)[2(p+3) a_1^2 a_2^2 + a_3^2 (a_1^2 + a_2^2)] F(\theta, \phi) \\ &= \frac{a_2 a_3}{a_1} (a_1^2 - a_2^2)(a_1^2 - a_3^2)^{1/2} [2(p + 3) a_1^2 - p a_2^2 + a_3^2] . \end{aligned} \tag{24}$$[5]

[5] Notice that when $p \to \infty$, this equation tends to the one determining the geometry of the Jacobi ellipsoids (see Chandrasekhar 1962, Appendix I, eq. [AI, 5]).

In equation (24) $E(\theta, \phi)$ and $F(\theta, \phi)$ are the standard elliptic integrals of the two kinds with the arguments

$$\theta = \sin^{-1}\sqrt{\frac{a_1^2 - a_2^2}{a_1^2 - a_3^2}} \quad \text{and} \quad \phi = \cos^{-1}\frac{a_3}{a_1}. \tag{25}$$

For every pair of values $(a_2/a_1, a_3/a_1)$ determined consistently with equation (23) or (24), the associated values of $\Omega^2$ and $\mu$ follow from the equations

$$\frac{\mu}{\pi G\rho} = 2(a_1a_2a_3)\frac{A_1a_1^2 - A_3a_3^2}{(3+p)a_1^2 + a_3^2} \quad \text{and} \quad \Omega^2 = (1+p)\mu. \tag{26}$$

For a few given values of $p$ and different assigned values of $\phi$ (i.e., of $a_3/a_1$) equation (24) was solved for $\theta$ (i.e., for $a_2/a_1$) by a method of successive approximation; and the accuracy of the final solution was always tested against equation (23).

The solution of equations (23) and (24) has been carried out to determine adequately the Roche sequences for $p = 0, 1, 4, 20$, and 100. The results of the calculations are summarized in Table 1; in this table, in addition to the principal constants ($\theta$, $a_2/a_1$, $a_3/a_1$, $A_1$, $A_2$, $A_3$, $\Omega^2$, and $\mu$), the semi-axes of the ellipsoid, in the unit $(a_1a_2a_3)^{1/3}$, are also listed.

## IV. THE ARRANGEMENT OF THE SOLUTIONS

The solutions for the Roche sequences belonging to different values of $p$ and the relationships which exist between these sequences and those of Maclaurin, Jacobi, and Jeans are most clearly exhibited in a plane in which each equilibrium ellipsoid is represented by a point whose coordinates are

$$\bar{a}_1 = \frac{a_1}{(a_1a_2a_3)^{1/3}} \quad \text{and} \quad \bar{a}_2 = \frac{a_2}{(a_1a_2a_3)^{1/3}}. \tag{27}$$

The utility of this plane for the exhibition of these relationships seems to have been recognized, first, by Jeans.

Before we describe the arrangement of the solutions in the $(\bar{a}_1, \bar{a}_2)$-plane, it is important to observe that when $p = -1$, and $\Omega^2$ according to equation (17) is zero, we recover the pure tidally distorted spheroids of Jeans[6] (1917, 1919; see also Paper III); and, further, that when $p \to \infty$, and the term in $\mu$ in equation (9) becomes negligible, we similarly recover the pure rotationally distorted configurations of Maclaurin and Jacobi.

Now consider Figure 1 in which the solutions for all the sequences are exhibited. In this diagram, each equilibrium ellipsoid is represented by a point whose coordinates are $\bar{a}_1$ and $\bar{a}_2$. By the chosen normalization, the volume (and in view of the homogeneity, also the mass) of all the ellipsoids represented is unity. The undistorted sphere is, therefore, represented by the point $\bar{a}_1 = \bar{a}_2 = 1$; this is the point $S$ in Figure 1.

The Jeans spheroids, being prolate, are represented by the pseudo-hyperbolic locus

$$\bar{a}_1\bar{a}_2^2 = 1\,; \tag{28}$$

this is the curve $ST$.

The Maclaurin spheroids, being oblate, are represented by the straight line

$$\bar{a}_1 = \bar{a}_2(\geq 1)\,; \tag{29}$$

[6] The fact, that the spheroids of Jeans are obtained when $p$ is assigned the "unphysical" value $-1$, is the origin of the statement, that is sometimes made, that these spheroids are of "no physical interest." However, it will appear that for an understanding of "what happens" in the $(\bar{a}_1, \bar{a}_2)$-plane, it is essential that consideration is given to the Jeans spheroids.

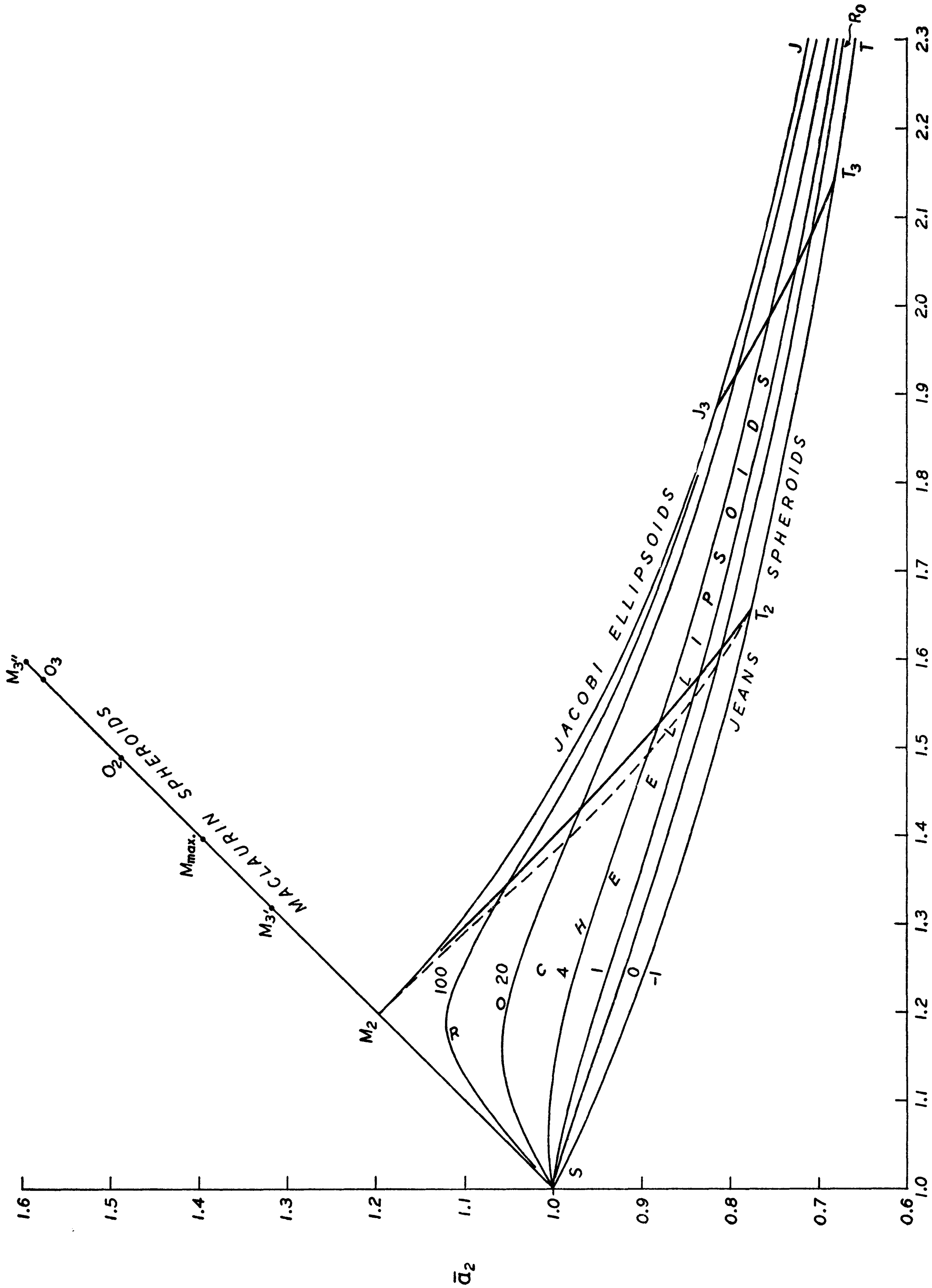
TABLE 1

THE PROPERTIES OF THE ROCHE ELLIPSOIDS†

$\phi = \cos^{-1} a_3/a_1$; $\theta = \sin^{-1}\sqrt{(a_1^2 - a_2^2)/(a_1^2 - a_3^2)}$; $A_i^* = a_1a_2a_3A_i$; $\bar{a}_i = a_i/(a_1a_2a_3)^{1/3}$

| $\phi$ | $\theta$ | $a_2/a_1$ | $a_3/a_1$ | $A_1^*$ | $A_2^*$ | $A_3^*$ | $\Omega^2$ | $\mu$ | $\bar{a}_1$ | $\bar{a}_2$ | $\bar{a}_3$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | $p=0$ | | | | | |
| 24° | 63°113 | 0 93188 | 0 91355 | 0 624238 | 0 679759 | 0 696004 | 0 022624 | 0 022624 | 1 0551 | 0 9832 | 0 9639 |
| 36 | 66 948 | 84112 | 80902 | 567104 | 700085 | 732811 | 047871 | 047871 | 1 1369 | 9563 | 9198 |
| 48 | 72 143 | 70687 | 66913 | 479968 | 736033 | 783999 | 074799 | 074799 | 1 2835 | 9072 | 8588 |
| 57 | 76 685 | 57787 | 54464 | 391944 | 777221 | 830835 | 088267 | 088267 | 1 4701 | 8495 | 8007 |
| 60 | 78 263 | 53013 | 50000 | 358101 | 794144 | 847756 | 089946 | 089946 | 1 5567 | 8253 | 7784 |
| 61 | 78 797 | 51373 | 48481 | 346293 | 800216 | 853492 | 090068 | 090068 | 1 5894 | 8165 | 7706 |
| 62 | 79 327 | 49714 | 46947 | 334272 | 806436 | 859292 | 089977 | 089977 | 1 6242 | 8074 | 7625 |
| 63 | 79 846 | 48040 | 45399 | 322096 | 812728 | 865177 | 089689 | 089689 | 1 6613 | 7981 | 7542 |
| 66 | 81 416 | 42898 | 40674 | 284086 | 832949 | 882963 | 087201 | 087201 | 1 7896 | 7677 | 7279 |
| 71 | 83 941 | 34052 | 32557 | 217086 | 869951 | 912963 | 077474 | 077474 | 2 0816 | 7088 | 6777 |
| 72 | 84 424 | 32254 | 30902 | 203288 | 877769 | 918942 | 074648 | 074648 | 2 1568 | 6957 | 6665 |
| 75 | 85 810 | 26827 | 25882 | 161540 | 901812 | 936647 | 064426 | 064426 | 2 4330 | 6527 | 6297 |
| 79 | 87 465 | 0 19569 | 0 19081 | 0 106446 | 0 934365 | 0 959192 | 0 047111 | 0 047111 | 2 9919 | 0 5855 | 0 5709 |
| | | | | | | $p=1$ | | | | | |
| 12° | 51°71 | 0 98660 | 0 97815 | 0 657194 | 0 667954 | 0 674850 | 0 009293 | 0 004647 | 1 0119 | 0 9984 | 0 9898 |
| 24 | 54 380 | 94376 | 91355 | 627500 | 672967 | 699532 | 036152 | 018076 | 1 0507 | 9916 | 9598 |
| 36 | 59 116 | 86345 | 80902 | 573517 | 685952 | 740530 | 076342 | 038171 | 1 1270 | 9731 | 9118 |
| 48 | 65 938 | 73454 | 66913 | 488555 | 715127 | 796317 | 118726 | 059363 | 1 2671 | 9308 | 8479 |
| 54 | 70 017 | 64956 | 58779 | 432108 | 739433 | 828458 | 134284 | 067142 | 1 3784 | 8954 | 8102 |
| 59 | 73 623 | 56892 | 51504 | 377406 | 766047 | 856546 | 140854 | 070427 | 1 5056 | 8566 | 7754 |
| 60 | 74 355 | 55186 | 50000 | 365639 | 772117 | 862243 | 141250 | 070625 | 1 5360 | 8477 | 7680 |
| 61 | 75 088 | 53451 | 48481 | 353605 | 778442 | 867954 | 141298 | 070649 | 1 5685 | 8384 | 7604 |
| 66 | 78 715 | 44429 | 40674 | 289716 | 813764 | 896520 | 135785 | 067892 | 1 7688 | 7859 | 7194 |
| 69 | 80 813 | 38813 | 35837 | 248826 | 837737 | 913437 | 127424 | 063712 | 1 9300 | 7491 | 6917 |
| 71 | 82 154 | 35022 | 32557 | 220788 | 854695 | 924516 | 119625 | 059812 | 2 0622 | 7222 | 6714 |
| 72 | 82 8038 | 33119 | 30902 | 206604 | 863429 | 929965 | 115054 | 057527 | 2 1379 | 7080 | 6606 |
| 73 | 83 437 | 31213 | 29237 | 192351 | 872298 | 935348 | 110044 | 055022 | 2 2211 | 6933 | 6494 |
| 75 | 84 648 | 27405 | 25882 | 163768 | 890351 | 945879 | 098753 | 049376 | 2 4158 | 6620 | 6252 |
| 78 | 86 304 | 21726 | 20791 | 121325 | 917769 | 960904 | 078934 | 039467 | 2 8079 | 6100 | 5838 |
| 81 | 87 728 | 0 16126 | 0 15643 | 0 080697 | 0 944630 | 0 974664 | 0 056499 | 0 028249 | 3 4097 | 0 5498 | 0 5334 |
| | | | | | | $p=4$ | | | | | |
| 12° | 38°67 | 0 99153 | 0 97815 | 0 658519 | 0 665285 | 0 676197 | 0 01452 | 0 002904 | 1 0103 | 1 0017 | 0 9882 |
| 24 | 41 492 | 96301 | 91355 | 632677 | 662197 | 705127 | 05642 | 011284 | 1 0436 | 1 0050 | 9534 |
| 36 | 46 922 | 90315 | 80902 | 584433 | 661930 | 753636 | 11911 | 023822 | 1 1103 | 1 0027 | 8982 |
| 48 | 55 906 | 78821 | 66913 | 504195 | 677213 | 818593 | 18486 | 036973 | 1 2377 | 0 9756 | 8282 |
| 58. | 66 148 | 63120 | 52992 | 403460 | 719650 | 876890 | 21593 | 043187 | 1 4406 | 0 9093 | 7634 |
| 60 | 68 353 | 59334 | 50000 | 379227 | 732740 | 888034 | 21685 | 043371 | 1 4994 | 0 8896 | 7497 |
| 61 | 69 456 | 57380 | 48481 | 366628 | 739884 | 893488 | 21648 | 043296 | 1 5319 | 0 8790 | 7427 |
| 70 | 78 845 | 38733 | 34202 | 241528 | 820128 | 938344 | 18514 | 037028 | 1 9616 | 0 7598 | 6709 |
| 72 | 80 682 | 34526 | 30902 | 211856 | 840934 | 947210 | 17110 | 034220 | 2 1084 | 0 7280 | 6515 |
| 74 | 82 390 | 30361 | 27564 | 182039 | 862282 | 955681 | 15465 | 030930 | 2 2862 | 0 6941 | 6302 |
| 75 | 83 194 | 28300 | 25882 | 167150 | 873109 | 959736 | 14555 | 029110 | 2 3900 | 0 6764 | 6186 |
| 80 | 86 639 | 0 18299 | 0 17365 | 0 095083 | 0 926613 | 0 978290 | 0 09329 | 0 018658 | 3 1572 | 0 5777 | 0 5482 |
| | | | | | | $p=20$ | | | | | |
| 36° | 27°969 | 0 96125 | 0 80902 | 0 599366 | 0 629135 | 0 771500 | 0 16763 | 0 007983 | 1 0874 | 1 0453 | 0 8797 |
| 48 | 38 306 | 88758 | 66913 | 530024 | 615046 | 854929 | 26374 | 012559 | 1 1897 | 1 0559 | 7960 |
| 58 | 55 238 | 71737 | 52992 | 428110 | 651851 | 920039 | 30624 | 014583 | 1 3804 | 0 9903 | 7315 |
| 60 | 59 136 | 66886 | 50000 | 401536 | 668988 | 929475 | 30559 | 014552 | 1 4407 | 0 9636 | 7203 |
| 66 | 69 927 | 51356 | 40674 | 313115 | 735445 | 951441 | 28232 | 013444 | 1 6854 | 0 8655 | 6855 |
| 70 | 75 865 | 41187 | 34202 | 250124 | 787015 | 962860 | 24980 | 011895 | 1 9219 | 0 7916 | 6573 |
| 72. | 78 422 | 36322 | 30902 | 218310 | 813640 | 968050 | 22890 | 010900 | 2 0731 | 0 7530 | 6406 |
| 74. | 80 711 | 31631 | 27564 | 186679 | 840313 | 973008 | 20522 | 009772 | 2 2552 | 0 7133 | 6216 |
| 75 | 81 759 | 0 29352 | 0 25882 | 0 171026 | 0 853573 | 0 975400 | 0 19243 | 0 009163 | 2 3611 | 0 6930 | 0 6111 |
| | | | | | | $p=100$ | | | | | |
| 48° | 21°399 | 0 96254 | 0 66913 | 0 547202 | 0 574019 | 0 878779 | 0 30021 | 0 0029724 | 1 1580 | 1 1146 | 0 7748 |
| 52 | 28 218 | 92799 | 61566 | 516745 | 567853 | 915402 | 33173 | 0032845 | 1 2051 | 1 1184 | 7420 |
| 56 | 39 753 | 84790 | 55919 | 473434 | 583718 | 942848 | 34922 | 0034576 | 1 2824 | 1 0874 | 7171 |
| 58 | 46 363 | 78950 | 52992 | 446235 | 602727 | 951038 | 35043 | 0034696 | 1 3370 | 1 0556 | 7085 |
| 60 | 52 610 | 72564 | 50000 | 416519 | 626800 | 956681 | 34697 | 0034353 | 1 4021 | 1 0174 | 7010 |
| 70 | 74 605 | 42333 | 34202 | 253984 | 772315 | 973702 | 27441 | 0027170 | 1 9044 | 0 8062 | 6513 |
| 75 | 81 228 | 29780 | 25882 | 172573 | 845842 | 981585 | 20935 | 0020728 | 2 3498 | 0 6998 | 6082 |
| 83 | 87 838 | 0 12749 | 0 12187 | 0 056598 | 0 949318 | 0 994088 | 0 08203 | 0 0008122 | 4 0075 | 0 5109 | 0 4884 |

† $\Omega^2$ and $\mu$ are listed in the unit $\pi G\rho$



JACOBI ELLIPSOIDS
JEANS SPHEROIDS
MACLAURIN SPHEROIDS
M2
M3'
M3''
Mmax.
O2
O3
S
J
J3
T
T2
T3
R0
100
20
4
1
0
-1
ā2
1.6
1.5
1.4
1.3
1.2
1.1
1.0
0.9
0.8
0.7
0.6
1.0
1.1
1.2
1.3
1.4
1.5
1.6
1.7
1.8
1.9
2.0
2.1
2.2
2.3

values, $\bar{a}_1 = a_1/(a_1a_2a_3)^{1/3}$ and $\bar{a}_2 = a_2/(a_1a_2a_3)^{1/3}$, of two of the principal axes of the equilibrium ellipsoid (or spheroid). The undistorted sphere is represented by $S$, the Maclaurin sequence is $SM_3''$, the Jacobian sequence by $M_2J$, the Jeans sequence by $ST$, and the Roche sequences (labeled by the values of $p = M/M'$ to which they belong) are confined to the domain bounded by the combined Maclaurin-Jacobi sequence, $SM_2J$, and the Roche sequence, $SR_0$, for $p = 0$. The first point of bifurcation along the Maclaurin sequences occurs at $M_2$; at $M_3'$ and $M_3''$ occur further neutral points belonging to the third harmonics. At the points $O_2$ and $O_3$ the Maclaurin spheroid becomes unstable by modes of overstable oscillation belonging to the second and the third harmonics; and at $M_{max}$, $\Omega^2$ attains its maximum values along $SM_3''$. The Jacobi ellipsoids become unstable by a mode of oscillation belonging to the third harmonics at $J_3$ where the pear-shaped sequence branches off; also, along the entire sequence $M_2J$, the Jacobi ellipsoids are characterized by a neutral mode of oscillation belonging to the second harmonics. At the points $T_2$ and $T_3$ the Jeans spheroids become unstable by modes of oscillation belonging to the second and the third harmonics. The Roche limit where $\Omega^2$ and $\mu$ attain their maxima along the different Roche sequences is represented by the dashed curve joining $M_2$ and $T_2$; and the locus of points where instability sets in by a mode of oscillation belonging to the second harmonics is shown by the heavy curve joining $M_2$ and $T_2$. The locus of the neutral point (belonging to the third harmonics) is shown by the curve joining $T_3$ and $J_3$. Note that the Jacobi ellipsoids are to be considered unstable in the limit $p \to \infty$.

this is the line $SM_{3''}$. The first point of bifurcation along the Maclaurin sequence occurs at $M_2$ where

$$\bar{a}_1 = \bar{a}_2 = 1.19723 \; ; \tag{30}$$

at this point $\Omega^2 = 0.37423 \, \pi G\rho$.

The Jacobian sequence branches off from the Maclaurin sequence at $M_2$; it is represented by the locus $M_2J$. Since the Jacobi ellipsoids eventually become prolate, the locus $M_2J$ becomes asymptotic to $ST$ as $a_1 \to \infty$. Also, the Jacobi ellipsoids are known to become unstable at the point where the sequence of the pear-shaped configurations branches off; this occurs at $J_3(\bar{a}_1 = 1.8858, \bar{a}_2 = 0.81498)$.

The Roche sequences for the different $p$'s are represented by a one-parameter family of loci in the domain bounded by $SM_2$, $M_2J$, and $ST$ (strictly, we should rather say $SM_2$, $M_2J$, and $SR_0$ the Roche sequence for $p = 0$). All these loci start at the point $S$ and all eventually become asymptotic to $ST$. It follows from our earlier remarks, and it is now apparent from Figure 1, that as $p \to \infty$, *the Roche sequence tends to the combined Maclaurin-Jacobi sequence represented by the broken curve $SM_2J$.*

## V. THE ROCHE LIMIT

Since $\Omega^2$ attains its maximum value along the Maclaurin sequence (at the point $M_{\max}$ in Fig. 1) subsequent to the point of bifurcation, and it decreases monotonically down the Jacobian sequence from the value ($=0.37423 \, \pi G\rho$) it has at the point of bifurcation ($M_2$), it is clear that along the combined Maclaurin-Jacobi sequence, $SM_2J$, $\Omega^2$ has its maximum value at $M_2$. It is also known that along the Jeans sequence, $\mu$ attains its maximum value at (cf. Paper III, eq. [26])

$$e = 0.883026 \qquad \text{where} \qquad \mu = \mu_{\max} = 0.125536 \, \pi G\rho \; ; \tag{31}$$

this point (denoted by $T_2$ in Fig. 1) occurs along $ST$ where

$$\bar{a}_1 = 1.65584 \qquad \text{and} \qquad \bar{a}_2 = 0.777125 \, . \tag{32}$$

We should accordingly expect that, along each of the Roche sequences, $\Omega^2$ and $\mu = \Omega^2/(1 + p)$ attain maxima, simultaneously, at some determinate point; that this is indeed the case is apparent from the results of Table 1 exhibited in Figure 2.

The place where $\Omega^2$ and $\mu$ attain their maxima along a Roche sequence can be determined as follows.

Since the structure of the Roche ellipsoids is uniquely determined by equations (14), it is clear that, when $\Omega^2$ and $\mu$ attain their maxima, not only these equations but also their first variations (with respect to a suitable infinitesimal solenoidal displacement which preserves the ellipsoidal shape) must be satisfied. Therefore, at the maximum, in addition to equations (14), the equations,

$$\delta\mathfrak{W}_{11} + (\Omega^2 + 2\mu)\delta I_{11} = \delta\mathfrak{W}_{22} + (\Omega^2 - \mu)\delta I_{22} = \delta\mathfrak{W}_{33} - \mu\delta I_{33} \, , \tag{33}$$

obtained by considering their first variations, must also be satisfied.[7] As two independent equations, equivalent to equations (33), we shall use

$$\delta\mathfrak{W}_{11} - \delta\mathfrak{W}_{22} + (\Omega^2 + 2\mu)\delta I_{11} - (\Omega^2 - \mu)\delta I_{22} = 0 \tag{34}$$

and

$$\delta\mathfrak{W}_{11} + \delta\mathfrak{W}_{22} - 2\delta\mathfrak{W}_{33} + (\Omega^2 + 2\mu)\delta I_{11} + (\Omega^2 - \mu)\delta I_{22} + 2\mu\delta I_{33} = 0 \, . \tag{35}$$

[7] Similar considerations can be applied equally to the location of the maximum of $\Omega^2$ along the Maclaurin sequence. The relevant analysis is included between equations (17) and (23) in Paper I; and this analysis must be supplemented by the corrigendum (Chandrasekhar 1963*b*) relative to the remark which follows equation (23).

Now, the first variations of the moment of inertia and the potential-energy tensors can all be expressed in terms of the symmetrized second-order virials

$$V_{ij} = \int_V \rho (\xi_i x_j + \xi_j x_i) dx , \tag{36}$$

where $\xi$ denotes the displacement in question. Thus

$$\delta I_{ij} = V_{ij} \tag{37}$$

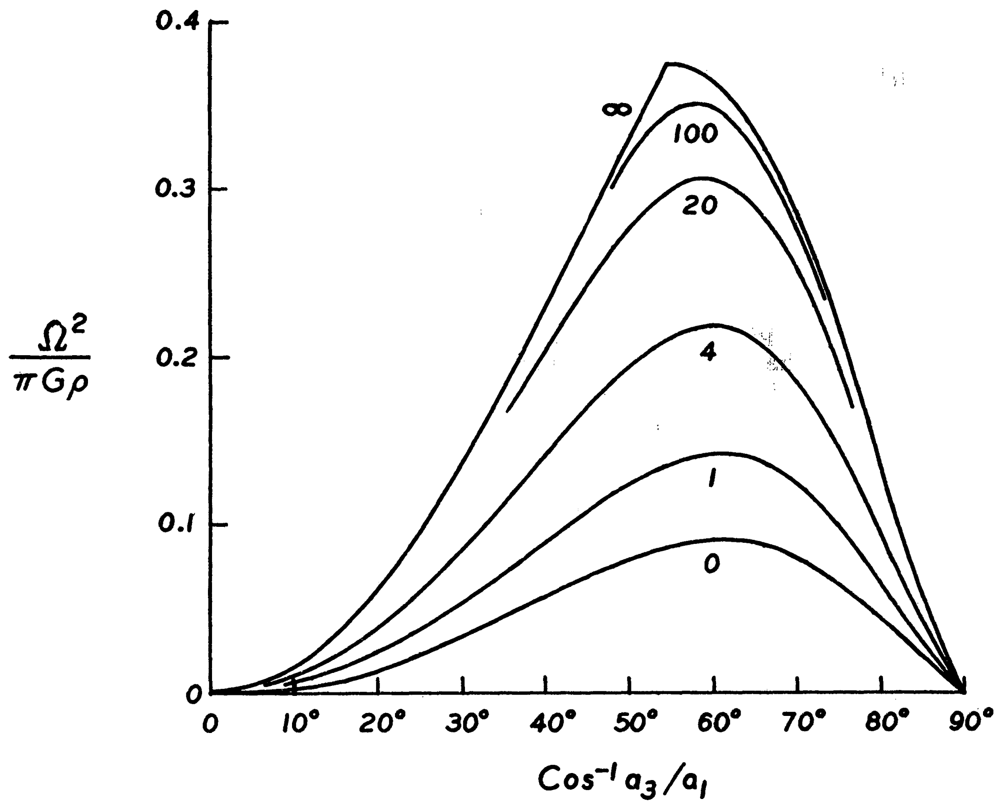



FIG. 2.—The variation of $\Omega^2$ along the Roche sequences. The curves are labeled by the values of $p$ to which they belong; and the curve labeled by $\infty$ belongs to the combined Maclaurin-Jacobi sequence. The maxima of the curves define the Roche limit.

and (cf. Paper III, eqs. [47] and [48])

$$\delta \mathfrak{W}_{ij} = -2B_{ij} V_{ij} \qquad (i \neq j) \tag{38}$$

and

$$\delta \mathfrak{W}_{ii} = -(2B_{ii} - a_i^2 A_{ii}) V_{ii} + a_i^2 \sum_{l \neq i} A_{il} V_{ll} , \tag{39}$$

where

$$B_{ij} = A_i - a_j^2 A_{ij} = A_j - a_i^2 A_{ij} , \tag{40}$$

(no summation over repeated indices in eqs. [38]–[40])

and $A_i$ and $A_{ij}$ are the one- and the two-index symbols defined in an earlier paper (Chandrasekhar and Lebovitz 1962*a*). (*Note that in writing eqs.* [38] *and* [39], *a common factor* $\pi G \rho a_1 a_2 a_3$ *has been suppressed.*)

From equations (38) and (39) we obtain, after some reductions in which use is made of equations (40),

$$\delta\mathfrak{W}_{11} - \delta\mathfrak{W}_{22} = -(3B_{11} - B_{12})V_{11} + (3B_{22} - B_{12})V_{22} + (B_{23} - B_{13})V_{33}, \tag{41}$$

and

$$\begin{aligned}\delta\mathfrak{W}_{11} + \delta\mathfrak{W}_{22} - 2\delta\mathfrak{W}_{33} = &-(3B_{11} + B_{12} - 2B_{13})V_{11} - (3B_{22} + B_{12} - 2B_{23})V_{22} \\ &+ (6B_{33} - B_{13} - B_{23})V_{33}.\end{aligned} \tag{42}$$

Now combining equations (34), (35), (37), (41), and (42), we obtain

$$\begin{aligned}[(\Omega^2 + 2\mu) - (3B_{11} - B_{12})]V_{11} - [(\Omega^2 - \mu) - (3B_{22} - B_{12})]V_{22} \\ + (B_{23} - B_{13})V_{33} = 0\end{aligned} \tag{43}$$

and

$$\begin{aligned}[(\Omega^2 + 2\mu) - (3B_{11} + B_{12} - 2B_{13})]V_{11} + [(\Omega^2 - \mu) - (3B_{22} + B_{12} - 2B_{23})]V_{22} \\ + [2\mu + (6B_{33} - B_{13} - B_{23})]V_{33} = 0.\end{aligned} \tag{44}$$

Since in writing the expressions for $\delta\mathfrak{W}_{ij}$ a common factor $\pi G\rho(a_1a_2a_3)$ was suppressed, it must be assumed that $\Omega^2$ and $\mu$ in equations (43) and (44) (and in similar equations in the sequel) are divided out by the same factor.

To equations (43) and (44) we must adjoin the condition (cf. Lebovitz 1961, eq. [83])

$$\frac{V_{11}}{a_1^2} + \frac{V_{22}}{a_2^2} + \frac{V_{33}}{a_3^2} = 0 \tag{45}$$

which expresses the solenoidal requirement on $\boldsymbol{\xi}$.

The condition for the occurrence of a maximum for $\Omega^2$ and $\mu$ is, then, the vanishing of the determinant of equations (43)–(45). We thus obtain

$$\left\|\begin{matrix} \Omega^2 + 2\mu - (3B_{11} - B_{12}) & -(\Omega^2 - \mu) + (3B_{22} - B_{12}) & B_{23} - B_{13} \\ \Omega^2 + 2\mu - (3B_{11} + B_{12} - 2B_{13}) & +(\Omega^2 - \mu) - (3B_{22} + B_{12} - 2B_{23}) & 2\mu + 6B_{33} - B_{13} - B_{23} \\ \frac{1}{a_1^2} & \frac{1}{a_2^2} & \frac{1}{a_3^2} \end{matrix}\right\| = 0. \tag{46}$$

By some elementary transformations, equation (46) can be brought to the somewhat simpler form

$$\left\|\begin{matrix} \Omega^2 + 2\mu - (3B_{11} - B_{13}) & B_{23} - B_{12} & \mu + 3B_{33} - B_{13} \\ B_{13} - B_{12} & \Omega^2 - \mu - (3B_{22} - B_{23}) & \mu + 3B_{33} - B_{23} \\ \frac{1}{a_1^2} & \frac{1}{a_2^2} & \frac{1}{a_3^2} \end{matrix}\right\| = 0, \tag{47}$$

By evaluating the determinant (47) along the different Roche sequences, the points listed in Table 2, where the maxima of $\Omega^2$ are attained, were determined by interpolation. The maxima located in this manner agree exactly (as they should) with those determined by a direct interpolation among the values of $\Omega^2$ given in Table 1.

In Figure 1 the locus of the points where the maxima of $\Omega^2$ (and, or $\mu$) are attained is shown by the dashed curve joining the points $T_2$ and $M_2$. This locus defines the Roche limit.

## VI. THE SECOND-ORDER VIRIAL EQUATIONS GOVERNING SMALL OSCILLATIONS ABOUT EQUILIBRIUM

As stated in the introduction, it is generally believed that the Roche ellipsoids develop some sort of instability at the Roche limit where $\Omega^2$ and $\mu$ attain their maxima. However, when we consider the following circumstances, it becomes clear that the matter requires a careful examination of the normal modes of oscillation of the Roche ellipsoids.

It is known (cf. Paper III, Sec. V) that the Jeans spheroid becomes unstable, by a mode of oscillation belonging to the second harmonics, when $\mu$ attains it maximum at $T_2$. And it is also known that the Maclaurin spheroid becomes neutral, with respect to the same mode of oscillation, at the point of bifurcation $M_2$; and that the Jacobi ellipsoid retains this neutrality along its entire sequence. The question which requires clarification

TABLE 2

THE ROCHE LIMIT AND THE CONSTANTS OF THE CRITICAL ELLIPSOID

($\Omega^2$ and $\mu$ Are Listed in the Unit $\pi G\rho$)

| $p$ | $\theta$ | $\Omega_{max}^2$ | $\mu_{max}$ | $\bar{a}_1$ | $\bar{a}_2$ | $\bar{a}_3$ |
|---|---|---|---|---|---|---|
| −1 | 62°009 | 0 | 0 125536 | 1 6558 | 0 7771 | 0 7771 |
| 0 | 61 156 | 0 090068 | 090068 | 1 5947 | 0 8151 | .7693 |
| 1 | 60 638 | 141322 | 070661 | 1 5565 | 0 8418 | .7632 |
| 4 | 59 840 | .216861 | .043372 | 1 4944 | 0 8913 | 7508 |
| 20 | 58 646 | .306396 | .014590 | 1 3989 | 0 9821 | 7279 |
| 100 | 57 499 | 350562 | 0 003471 | 1 3224 | 1 0642 | 7106 |
| ∞ . | 54 358 | 0 374230 | 0 | 1 1972 | 1 1972 | 0 6977 |

is: how does the mode, which for $p = -1$ is unstable beyond $\mu_{max}$ at $T_2$, become neutral, beyond $M_2$, along the entire Jacobian part of the combined Maclaurin-Jacobi sequence, when $p \to \infty$? Clearly, this question cannot be fully answered without a detailed analysis of the normal modes of oscillation of the Roche ellipsoids belonging to the second harmonics. We shall now show how the required analysis can be carried out with the aid of the linearized form of the virial equation (10).

Suppose, then, that the equilibrium ellipsoid considered in Sections III–V is slightly perturbed; and, further, that the ensuing motions are described by a Lagrangian displacement of the form

$$\xi(x)\, e^{\lambda t}\,, \tag{48}$$

where $\lambda$ is a parameter whose characteristic values are to be determined. To the first order in $\xi$, the virial equation (10) gives

$$\lambda^2 V_{i;j} - 2\lambda\Omega\epsilon_{il3}V_{l;j} = \delta\mathfrak{W}_{ij} + (\Omega^2 - \mu)\delta I_{ij} - \Omega^2\delta_{i3}\delta I_{3j} + 3\mu\delta_{i1}I_{1j} + \delta_{ij}\delta\Pi\,, \tag{49}$$

where

$$V_{i;j} = \int_V \rho\,\xi_i x_j d x \tag{50}$$

denotes the second-order (unsymmetrized) virial and $\delta\Pi$, $\delta\mathfrak{W}_{ij}$, and $\delta I_{ij}$ are the first variations of $\Pi$, $\mathfrak{W}_{ij}$, and $I_{ij}$ due to the deformation of the ellipsoid caused by the displacement $\xi$.

We have already seen (cf. eqs. [37]–[39]) how the $\delta\mathfrak{W}_{ij}$'s and $\delta I_{ij}$'s can be expressed in terms of the symmetrized virials

$$V_{ij} = V_{i;j} + V_{j;i}\,. \tag{51}$$

In particular, replacing $\delta I_{ij}$ by $V_{ij}$ (in accordance with eq. [37]), we can rewrite equation (49) in the form

$$\lambda^2 V_{i;j} - 2\lambda\Omega\epsilon_{il3}V_{l;j} = \delta\mathfrak{W}_{ij} + (\Omega^2 - \mu)V_{ij} - \Omega^2\delta_{i3}V_{3j} + 3\mu\delta_{i1}V_{1j} + \delta_{ij}\delta\Pi\,. \tag{52}$$

Equation (52) represents a total of nine equations for the nine virials $V_{i;j}$. These nine equations fall into two non-combining groups of four and five equations, respectively, distinguished by their parity (i.e., oddness or evenness) with respect to the index 3. It is convenient to have these equations written out explicitly. The odd equations are

$$\lambda^2 V_{3;1} = \delta\mathfrak{W}_{31} - \mu V_{13} = -(2B_{13} + \mu)V_{13}\,, \tag{53}$$

$$\lambda^2 V_{3;2} = \delta\mathfrak{W}_{32} - \mu V_{23} = -(2B_{23} + \mu)V_{23}\,, \tag{54}$$

$$\lambda^2 V_{1;3} - 2\lambda\Omega V_{2;3} = \delta\mathfrak{W}_{13} + (\Omega^2 + 2\mu)V_{13} = -(2B_{13} - \Omega^2 - 2\mu)V_{13}\,, \tag{55}$$

$$\lambda^2 V_{2;3} + 2\lambda\Omega V_{1;3} = \delta\mathfrak{W}_{23} + (\Omega^2 - \mu)V_{23} = -(2B_{23} - \Omega^2 + \mu)V_{23}\,, \tag{56}$$

where we have substituted for the $\delta\mathfrak{W}_{ij}$'s in accordance with equation (38). And, similarly, the even equations are

$$\lambda^2 V_{3;3} = \delta\mathfrak{W}_{33} - \mu V_{33} + \delta\Pi\,, \tag{57}$$

$$\lambda^2 V_{1;1} - 2\lambda\Omega V_{2;1} = \delta\mathfrak{W}_{11} + (\Omega^2 + 2\mu)V_{11} + \delta\Pi\,, \tag{58}$$

$$\lambda^2 V_{2;2} + 2\lambda\Omega V_{1;2} = \delta\mathfrak{W}_{22} + (\Omega^2 - \mu)V_{22} + \delta\Pi\,, \tag{59}$$

$$\lambda^2 V_{1;2} - 2\lambda\Omega V_{2;2} = \delta\mathfrak{W}_{12} + (\Omega^2 + 2\mu)V_{12} = -(2B_{12} - \Omega^2 - 2\mu)V_{12}\,, \tag{60}$$

$$\lambda^2 V_{2;1} + 2\lambda\Omega V_{1;1} = \delta\mathfrak{W}_{21} + (\Omega^2 - \mu)V_{12} = -(2B_{12} - \Omega^2 + \mu)V_{12}\,. \tag{61}$$

## VII. THE CHARACTERISTIC EQUATIONS

We shall now show how equations (53)–(61) can be used to determine the different characteristic frequencies of oscillation of the Roche ellipsoid belonging to the second harmonics.

### *a) The Characteristic Equation for the Odd Modes*

Adding equations (53) and (55) and similarly equations (54) and (56), we obtain

$$(\lambda^2 + 4B_{13} - \Omega^2 - \mu)V_{13} - 2\lambda\Omega V_{23} + 2\lambda\Omega V_{3;2} = 0 \tag{62}$$

and

$$(\lambda^2 + 4B_{23} - \Omega^2 + 2\mu)V_{23} + 2\lambda\Omega V_{13} - 2\lambda\Omega V_{3;1} = 0\,. \tag{63}$$

Eliminating $V_{3;1}$ and $V_{3;2}$ from the foregoing equations with the aid of equations (53) and (54), we have

$$\lambda(\lambda^2 + 4B_{13} - \Omega^2 - \mu)V_{13} - 2\Omega(\lambda^2 + 2B_{23} + \mu)V_{23} = 0 \tag{64}$$

and

$$\lambda(\lambda^2 + 4B_{23} - \Omega^2 + 2\mu)V_{23} + 2\Omega(\lambda^2 + 2B_{13} + \mu)V_{13} = 0\,; \tag{65}$$

and these two equations lead to the characteristic equation

$$\lambda^2(\lambda^2 + 4B_{13} - \Omega^2 - \mu)(\lambda^2 + 4B_{23} - \Omega^2 + 2\mu) + 4\Omega^2(\lambda^2 + 2B_{13} + \mu)(\lambda^2 + 2B_{23} + \mu) = 0 . \quad (66)$$

This is a cubic equation for $\lambda^2$; and it can be verified that, in the limits $\Omega^2 = 0$ and $\mu = 0$, the equation provides characteristic roots which are appropriate, respectively, for the Jeans spheroids and the Jacobi ellipsoids (or the Maclaurin spheroids if the indices 1 and 2 are not distinguished).

### b) *The Characteristic Equation for the Even Modes*

Turning next to the even equations (57)–(61), we can combine them to give the following four equations in which $\delta\Pi$ no longer appears:

$$(\lambda^2 + 4B_{12} - 2\Omega^2 - \mu)V_{12} + \lambda\Omega(V_{11} - V_{22}) = 0 , \quad (67)$$

$$\lambda^2(V_{1;2} - V_{2;1}) = \lambda\Omega(V_{11} + V_{22}) + 3\mu V_{12} , \quad (68)$$

$$\tfrac{1}{2}\lambda^2(V_{11} - V_{22}) - 2\lambda\Omega V_{12} = \delta\mathfrak{W}_{11} - \delta\mathfrak{W}_{22} + (\Omega^2 + 2\mu)V_{11} - (\Omega^2 - \mu)V_{22} , \quad (69)$$

$$\tfrac{1}{2}\lambda^2(V_{11} + V_{22}) + 2\lambda\Omega(V_{1;2} - V_{2;1}) - \lambda^2 V_{33} = \delta\mathfrak{W}_{11} + \delta\mathfrak{W}_{22} - 2\delta\mathfrak{W}_{33} + (\Omega^2 + 2\mu)V_{11} + (\Omega^2 - \mu)V_{22} + 2\mu V_{33} . \quad (70)$$

Rearranging equation (69) and eliminating $(V_{1;2} - V_{2;1})$ from equation (70) with the aid of equation (68) (and rearranging), we obtain the pair of equations:

$$(\tfrac{1}{2}\lambda^2 - \Omega^2 - 2\mu)V_{11} - (\tfrac{1}{2}\lambda^2 - \Omega^2 + \mu)V_{22} - 2\lambda\Omega V_{12} = \delta\mathfrak{W}_{11} - \delta\mathfrak{W}_{22} \quad (71)$$

$$(\tfrac{1}{2}\lambda^2 + \Omega^2 - 2\mu)V_{11} + (\tfrac{1}{2}\lambda^2 + \Omega^2 + \mu)V_{22} - (\lambda^2 + 2\mu)V_{33} + \frac{6\Omega\mu}{\lambda}V_{12} = \delta\mathfrak{W}_{11} + \delta\mathfrak{W}_{22} - 2\delta\mathfrak{W}_{33} . \quad (72)$$

Now substituting for $\delta\mathfrak{W}_{11} - \delta\mathfrak{W}_{22}$ and $\delta\mathfrak{W}_{11} + \delta\mathfrak{W}_{22} - 2\delta\mathfrak{W}_{33}$ their expansions (41) and (42) in terms of the virials, and regrouping the terms, we find

$$(\tfrac{1}{2}\lambda^2 - \Omega^2 - 2\mu + 3B_{11} - B_{12})V_{11} - (\tfrac{1}{2}\lambda^2 - \Omega^2 + \mu + 3B_{22} - B_{12})V_{22} + (B_{13} - B_{23})V_{33} - 2\lambda\Omega V_{12} = 0 \quad (73)$$

and

$$(\tfrac{1}{2}\lambda^2 + \Omega^2 - 2\mu + 3B_{11} + B_{12} - 2B_{13})V_{11} + (\tfrac{1}{2}\lambda^2 + \Omega^2 + \mu + 3B_{22} + B_{12} - 2B_{23})V_{22} - (\lambda^2 + 2\mu + 6B_{33} - B_{13} - B_{23})V_{33} + \frac{6\Omega\mu}{\lambda}V_{12} = 0 . \quad (74)$$

Equations (67), (73), and (74) provide three relations among the four virials $V_{11}$, $V_{12}$, $V_{22}$, and $V_{33}$. A fourth relation is obtained by making use of the solenoidal character of the Lagrangian displacement. In the present context, the relation which expresses this requirement is (cf. Lebovitz 1961, eq. [83])

$$\frac{V_{11}}{a_1^2} + \frac{V_{22}}{a_2^2} + \frac{V_{33}}{a_3^2} = 0 . \quad (75)$$

57) and equation governing the even modes now follows from setting the determinant of equations (6
in

$$
\begin{matrix}
3B_{11}-B_{12} & -\tfrac{1}{2}\lambda^2+\Omega^2-\mu-3B_{22}+B_{12} & B_{13}-B_{23} \\
6\Omega\ \ 3B_{11}+B_{12}-2B_{13} & +\tfrac{1}{2}\lambda^2+\Omega^2+\mu+3B_{22}+B_{12}-2B_{23} & -\lambda^2-2\mu-6B_{33}+B_{13}+B_{23} \\
\lambda^2+4B^2 & -\lambda\Omega & 0 \\
\tfrac{}{2} & \dfrac{1}{a_2^2} & \dfrac{1}{a_3^2}
\end{matrix}
$$

tary transformations, equation (76) can be brought to the following somewhat simpler form:

$$
\begin{matrix}
B_{13}-B & \tfrac{1}{2}\lambda^2-\Omega^2-2\mu+3B_{11}-B_{12} & -(B_{13}-B_{23}) & -3\mu+3(B_{11}-B_{22})+ \\
B_{12}-B_{2}-2\Omega^2-\mu) & \Omega^2+B_{12}-B_{13} & \tfrac{1}{2}\lambda^2+\mu+3B_{33}-B_{23} & \Omega^2+3(B_{22}-B_{33})+ \\
\dfrac{1}{a_3^2} & \dfrac{1}{a_1^2} & -\dfrac{1}{a_3^2} & \dfrac{1}{a_1^2}+\dfrac{1}{a_2^2}+
\end{matrix}
$$

$$
\begin{matrix}
B_{13}- & & 2\lambda^2 & -(B_{13}-B_{23}) & -3\mu+3(B_{11}-B_{22})+ \\
B_{12}- & +\Omega^2 & -(\lambda^2+3\mu) & \tfrac{1}{2}\lambda^2+\mu+3B_{33}-B_{23} & \Omega^2+3(B_{22}-B_{33})+ \\
\dfrac{1}{a_3^2} & & 0 & -\dfrac{1}{a_3^2} & \dfrac{1}{a_1^2}+\dfrac{1}{a_2^2}+
\end{matrix}
$$

approp e verified that, in the limits $\Omega^2 = 0$ and $\mu = 0$, equation (77) provides characteristic roots which are
d). oids and the Jacobi ellipsoids[8] (or the Maclaurin spheroids if the indices 1 and 2 are not distinguished

= 0) alon $^2 = 2B_{12}\pi G\rho a_1 a_2 a_3$ (after restoring the suppressed factor) for these ellipsoids, the occurrence of a neutral mode ($\lambda^2$ =
om equation (77).

## VIII. THE CHARACTERISTIC FREQUENCIES OF OSCILLATION BELONGING TO THE SECOND HARMONICS: THE POINT AT WHICH INSTABILITY SETS IN ALONG THE ROCHE SEQUENCE

The characteristic equations (66) and (77) have been solved for the different Roche sequences for which the equilibrium properties have been tabulated in Section III; and the results are given in Table 3.

An examination of the roots listed in Table 3 shows that the Roche ellipsoids do become unstable by a mode of oscillation belonging to the second harmonics. Their instability arises, in fact, by the same mode by which the Jeans spheroid becomes unstable at $\mu_{max}$ and for which the Jacobi ellipsoids are neutral along their entire sequence. Figure 3 exhibits the behavior of the corresponding characteristic root ($\sigma_3^2$) as we pass

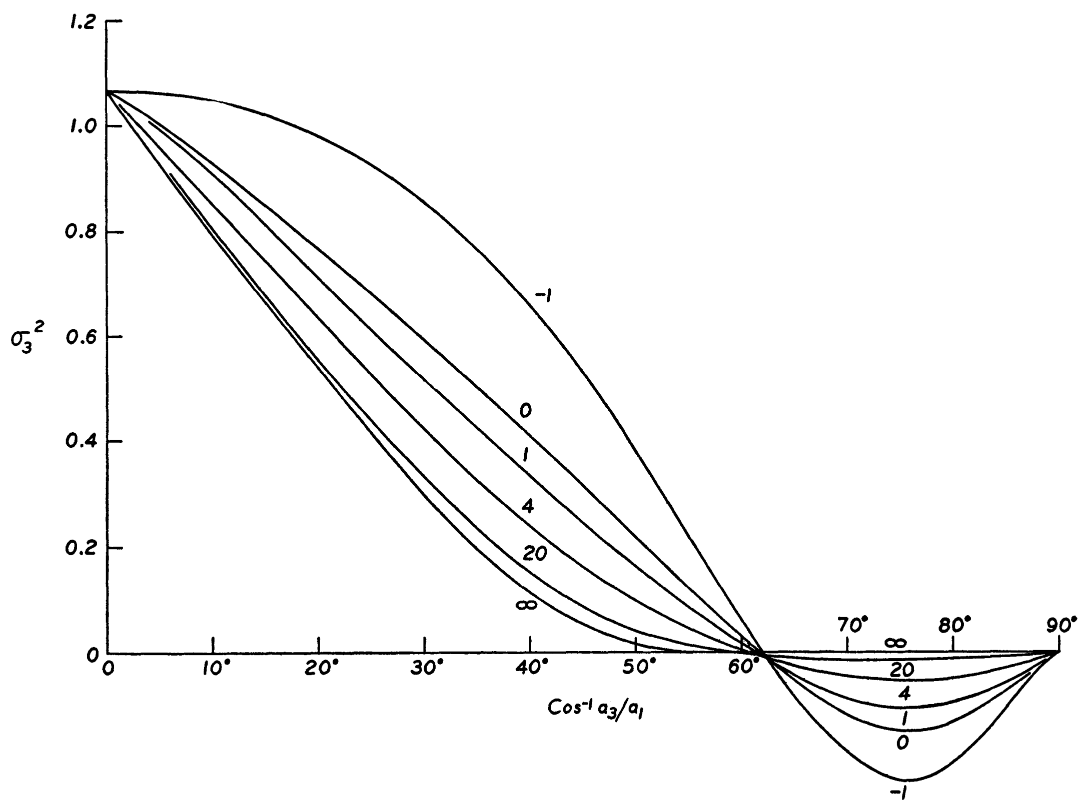



FIG. 3.—The squares of the characteristic frequency $\sigma_3^2$ (measured in the unit $\pi G\rho$) belonging to the mode by which the Roche ellipsoids and the Jeans spheroids become unstable. The curves for the different Roche sequences are labeled by the values of $p$ to which they belong; the curve labeled by $\infty$ belongs to the combined Maclaurin-Jacobi sequence and the curve labeled by $-1$ belongs to the Jeans sequence.

from the Jeans sequence to the limiting Maclaurin-Jacobi sequence via the Roche sequences of increasing $p$. The initial question which was raised in Section VI has thus been answered. And an important result which has emerged from answering the question is that *the Jacobi ellipsoids are unstable under the tidal action of the least object.* In Section XII, we shall briefly consider the possible bearing of this result for cosmogony.

The points, beyond which the Roche ellipsoids are unstable, were determined by interpolation among the roots $\sigma_3^2$ listed in Table 3. Their positions, together with some additional information, are given in Table 4. In Figure 1 the locus of these points, at

TABLE 3

THE SQUARES OF THE CHARACTERISTIC FREQUENCIES BELONGING TO THE SECOND HARMONICS

($\sigma^2$ Is Listed in the Unit $\pi G\rho$)

| $\phi$ | EVEN MODES | | | ODD MODES | | |
|---|---|---|---|---|---|---|
| | $\sigma_1^2$ | $\sigma_2^2$ | $\sigma_3^2$ | $\sigma_4^2$ | $\sigma_5^2$ | $\sigma_6^2$ |
| | | | $p=0$ | | | |
| 0° | 1 067 | 1.0667 | +1 0667 | 1.067 | 1 0667 | 0 |
| 24 | 1 323 | 1.1239 | +0 7007 | 1.272 | 0.9370 | 0 0241 |
| 36 | 1 429 | 1.1393 | +0 4866 | 1.412 | 0.8718 | .0551 |
| 48 | 1 569 | 1 0445 | +0 2574 | 1.577 | 0.7757 | .0954 |
| 57 | 1 697 | 0 8788 | +0 0836 | 1 708 | 0.6640 | .1234 |
| 60 | 1 740 | 0 8070 | +0 0285 | 1.752 | 0 6168 | .1301 |
| 61 | 1 754 | 0 7814 | +0 0109 | 1 767 | 0 5997 | .1318 |
| 62 | 1 767 | 0 7552 | −0 0063 | 1.781 | 0 5822 | .1333 |
| 63 | 1 783 | 0 7284 | −0 0232 | 1.793 | 0 5642 | .1344 |
| 66 | 1 824 | 0 6442 | −0 0695 | 1 833 | 0.5063 | .1356 |
| 71 | 1 886 | 0 4948 | −0 1278 | 1.893 | 0 3987 | .1284 |
| 72 | 1 898 | 0 4640 | −0 1357 | 1.903 | 0.3758 | .1253 |
| 75 | 1 928 | 0.3711 | −0 1495 | 1.933 | 0.3049 | .1125 |
| 79 | 1 962 | 0.2482 | −0 1416 | 1.963 | 0.2076 | 0 0865 |
| | | | $p=1$ | | | |
| 0° | 1 067 | 1 0667 | +1.0667 | 1.067 | 1 0667 | 0 |
| 12 | 1.280 | 1 0333 | +0 8740 | 1.181 | 0 9799 | 0 0094 |
| 24 | 1 392 | 1.1398 | +0 6294 | 1.313 | 0.9078 | .0380 |
| 36 | 1 494 | 1.1901 | +0 4045 | 1 464 | 0 8440 | .0854 |
| 48 | 1 602 | 1 1410 | +0 1952 | 1.625 | 0.7610 | .1440 |
| 54 | 1 673 | 1.0468 | +0.1011 | 1.705 | 0.6994 | .1713 |
| 59 | 1 738 | 0 9309 | +0 0303 | 1.770 | 0 6321 | .1884 |
| 60 | 1 752 | 0 9040 | +0 0171 | 1.782 | 0.6166 | .1909 |
| 61 | 1 765 | 0 8760 | +0 0044 | 1 795 | 0.6004 | .1930 |
| 66 | 1 830 | 0.7210 | −0.0521 | 1.853 | 0 5088 | .1965 |
| 72 | 1.901 | 0.5127 | −0.0976 | 1 914 | 0.3771 | .1803 |
| 78 | 1 955 | 0.3000 | −0.1054 | 1 961 | 0 2298 | 1354 |
| 81 | 1 975 | 0.1999 | −0 0905 | 1 978 | 0 1559 | 0 1017 |
| | | | $p=4$ | | | |
| 0° | 1.067 | 1.0667 | +1 0667 | 1 067 | 1 0667 | 0 |
| 12 | 1.299 | 1.0886 | +0 8070 | 1.206 | 0.9565 | 0 0146 |
| 24 | 1 478 | 1.1560 | +0.5453 | 1 368 | 0 8712 | .0582 |
| 36 | 1 583 | 1.2442 | +0 3083 | 1.543 | 0.8050 | .1275 |
| 48 | 1.643 | 1.2650 | +0.1241 | 1.708 | 0.7373 | .2081 |
| 58 | 1.736 | 1.0995 | +0 0213 | 1.822 | 0.6467 | .2576 |
| 60 | 1.760 | 1.0430 | +0 0059 | 1.841 | 0.6210 | .2625 |
| 61 | 1.772 | 1.0121 | −0.0012 | 1.851 | 0.6069 | .2641 |
| 72 | 1.903 | 0.5882 | −0 0532 | 1.936 | 0.3969 | .2318 |
| 75 | 1.932 | 0.4600 | −0 0576 | 1.954 | 0.3241 | .2042 |
| 80 | 1.970 | 0.2560 | −0.0520 | 1.977 | 0.1961 | 0.1388 |

TABLE 3—*Continued*

| $\phi$ | EVEN MODES | | | ODD MODES | | |
|---|---|---|---|---|---|---|
| | $\sigma_1^2$ | $\sigma_2^2$ | $\sigma_3^2$ | $\sigma_4^2$ | $\sigma_5^2$ | $\sigma_6^2$ |
| | | | $p=20$ | | | |
| 0°. | 1.067 | 1 0667 | +1 0667 | 1 067 | 1 0667 | 0 |
| 36 | 1 664 | 1 2959 | +0 2204 | 1 637 | 0.7620 | 0 1714 |
| 48. . | 1 656 | 1 4017 | +0 0588 | 1 826 | 0 7038 | .2738 |
| 58 | 1.712 | 1.2684 | +0 0042 | 1 921 | 0 6452 | 3242 |
| 60 . . | 1.739 | 1.2052 | −0 0008 | 1.931 | 0 6275 | .3253 |
| 66 . | 1 827 | 0 9641 | −0 0106 | 1.953 | 0.5528 | .3063 |
| 72 . | 1 903 | 0 6739 | −0 0157 | 1 967 | 0 4403 | 2533 |
| 75. | 1 933 | 0.5230 | −0 0166 | 1.974 | 0.3699 | 0.2149 |
| | | | $p=100$ | | | |
| 0° . . | 1.067 | 1 0667 | +1 0667 | 1 067 | 1 0667 | 0 |
| 48 . . . . . . | 1 639 | 1 4624 | +0 0332 | 1 886 | 0 6840 | 0 3027 |
| 52 . . . . | 1 605 | 1.4785 | +0 0106 | 1 937 | 0 6666 | 3349 |
| 56. . . | 1 631 | 1 4132 | +0 0018 | 1 967 | 0 6500 | .3532 |
| 58 . . . . . | 1 665 | 1 3531 | +0 0001 | 1 974 | 0 6401 | .3549 |
| 60 . . . | 1 709 | 1 2795 | −0 0010 | 1 978 | 0.6274 | .3519 |
| 70 . . . | 1.876 | 0 8132 | −0 0033 | 1.981 | 0 4996 | .2804 |
| 75 . | 1.933 | 0 5475 | −0 0036 | 1.982 | 0 3892 | .2146 |
| 83 | 1 986 | 0 1668 | −0 0025 | 1 991 | 0 1592 | 0 0844 |

TABLE 4

THE POINT AT WHICH INSTABILITY SETS IN ALONG THE ROCHE SEQUENCES

($\Omega^2$ and $\mu$ Are Listed in the Unit $\pi G\rho$)

| $p$ | $\theta$ | $\Omega^2$ | $\mu$ | $\bar{a}_1$ | $\bar{a}_2$ | $\bar{a}_3$ |
|---|---|---|---|---|---|---|
| − 1 | 62°009 | 0 | 0 12554 | 1.656 | 0.7771 | 0 7771 |
| 0 . . | 61 63 | 0 090034 | 090034 | 1.611 | 0 8108 | .7655 |
| 1 . . | 61 35 | .141229 | .070614 | 1.581 | 0 8350 | .7577 |
| 4 . | 60 83 | .216581 | 043316 | 1.526 | 0.8808 | .7439 |
| 20 . | 59.66 | .305937 | .014568 | 1.430 | 0.9683 | .7222 |
| 100 . . | 58 18 | .350303 | 0 003468 | 1.343 | 1 0523 | .7078 |
| ∞ | 54 358 | 0 374230 | 0 | 1.197 | 1 1972 | 0.6977 |

which instability sets in along the Roche sequences, is the full-line curve joining $T_2$ and $M_2$. It is at once apparent that the Roche ellipsoids do not become unstable at the Roche limit but at a subsequent point.

That the two points, the point where $\Omega^2$ and $\mu$ attain their maxima and the point where instability sets in, are distinct, follows from a comparison of equation (47), which determines the former point, and the equation

$$(4B_{12}-2\Omega^2-\mu)\begin{Vmatrix} -\Omega^2-2\mu+3B_{11}-B_{12} & -(B_{13}-B_{23}) & -3\mu+3(B_{11}-B_{22})+B_{13}+B_{23} \\ \Omega^2+B_{12}-B_{13} & \mu+3B_{33}-B_{23} & \Omega^2+3(B_{22}-B_{33})+B_{12}-B_{13} \\ \frac{1}{a_1^2} & -\frac{1}{a_3^2} & \frac{1}{a_1^2}+\frac{1}{a_2^2}+\frac{1}{a_3^2} \end{Vmatrix}$$

$$-3\mu\Omega^2\begin{Vmatrix} B_{13}-B_{23} & -3\mu+3(B_{11}-B_{22})+B_{13}-B_{23} \\ \frac{1}{a_3^2} & \frac{1}{a_1^2}+\frac{1}{a_2^2}+\frac{1}{a_3^2} \end{Vmatrix}=0, \quad (78)$$

which determines the latter point. That the two equations are necessarily distinct can be seen more directly from a comparison of equations (34) and (35) and the limiting forms of equations (70) and (71) for $\lambda = 0$. Thus while equation (71), in the limit $\lambda = 0$, is the same as equation (34), this is not the case with respect to equations (70) and (35). The difference in the latter case arises from the circumstance that, by virtue of equations (67) and (68), the additional term in equation (70), namely,

$$2\lambda\Omega(V_{1;2}-V_{2;1}) = 2\Omega^2(V_{11}+V_{22})+\frac{6\Omega\mu}{\lambda}V_{12}$$

$$= 2\Omega^2(V_{11}+V_{22})-6\Omega^2\mu\frac{V_{11}-V_{22}}{\lambda^2+4B_{12}-2\Omega^2-\mu} \quad (79)$$

does *not* tend to zero as $\lambda \to 0$; it tends, instead, to the finite limit:

$$\lim_{\lambda\to 0} 2\lambda\Omega(V_{1;2}-V_{2;1}) = 2\Omega^2\left(V_{11}+V_{22}-3\mu\frac{V_{11}-V_{22}}{4B_{12}-2\Omega^2-\mu}\right)\neq 0. \quad (80)[9]$$

And this circumstance cannot have been foreseen by any simple consideration based on equilibrium only. The basic reason for the occurrence of the neutral point beyond the Roche limit is the presence of the Coriolis term in the equations of motion. The matter is further clarified by Lebovitz (1963) in the paper following this one.

And finally it should be noted that the fact that the neutral point does not coincide with the Roche limit is in agreement with a theorem due to Karl Schwarzschild (1898) that *along a Roche sequence there is no point of bifurcation where a second ellipsoidal sequence branches off*.

In Section XII we shall consider the bearing of the foregoing results for a proper interpretation of the Roche limit.

[9] However, the limit *is* zero for the Jeans sequence (for which $\Omega^2 = 0$); and this is in agreement with the fact that along this sequence the point where $\mu$ attains its maximum coincides with the point where instability sets in. And along the Maclaurin sequence (for which $\mu = 0$) the limit is also zero *if* $V_{11} + V_{22} = 0$, a condition which is necessary to *exclude* the point where $\Omega^2$ attains its maximum (at $M_{max}$) and *determine* the point of bifurcation (at $M^2$) (cf. Chandrasekhar 1963*b* and n. 7 on p. 1190).

## IX. THE NEUTRAL POINT ALONG THE ROCHE SEQUENCE BELONGING TO THE THIRD HARMONICS

It is known that along the Jacobian sequence a point of bifurcation occurs where the Jacobi ellipsoid becomes unstable by a mode of oscillation belonging to the third harmonics; this is the point $J_3$ in Figure 1. It is also known that along the Jeans sequence a second neutral point belonging also to the third harmonics, occurs; this is the point $T_3$ in Figure 1. The occurrence of these two neutral points, $J_3$ and $T_3$, suggests that a similar neutral point occurs along each Roche sequence. Such a neutral point, if one such exists, can be exhibited and isolated by the method which has been described in the contexts of the Jacobian and the Jeans sequences (Paper I, Secs. IV and VI). The method is based on the integral properties which follow, *as identities*, from the relevant third-order virial equation. Accordingly, we shall first derive the analogous properties appropriate for configurations which are both rotationally and tidally distorted.

### *a) The Third-Order Virial Equations and the Integral Properties Governing Equilibrium*

By multiplying equation (9) by $x_j x_k$ and integrating over the volume $V$ occupied by the fluid, we obtain in the usual manner the required third-order virial equation (cf. Chandrasekhar 1962, Sec. IV, and Paper I, eq. [88]):

$$\frac{d}{dt}\int_V \rho u_i x_j x_k d\boldsymbol{x} = 2(\mathfrak{T}_{ij;k} + \mathfrak{T}_{ik\cdot j}) + \mathfrak{W}_{ij;k} + \mathfrak{W}_{ik\cdot j} + (\Omega^2 - \mu) I_{ijk} - \Omega^2 \delta_{i3} I_{3jk} + 3\mu\delta_{i1} I_{1jk} + \delta_{ij}\Pi_k + \delta_{ik}\Pi_j + 2\Omega \int_V \rho \epsilon_{il3} u_l x_j x_k d\boldsymbol{x}\,, \tag{81}$$

where

$$\mathfrak{T}_{ij;k} = \tfrac{1}{2}\int_V \rho u_i u_j x_k d\boldsymbol{x}\,, \qquad \mathfrak{W}_{ij\cdot k} = -\tfrac{1}{2}\int_V \rho \mathfrak{V}_{ij} x_k d\boldsymbol{x}\,,$$
$$I_{ijk} = \int_V \rho x_i x_j x_k d\boldsymbol{x}\,, \qquad \text{and} \qquad \Pi_k = \int_V p x_k d\boldsymbol{x}\,. \tag{82}$$

When no relative motions are present in the frame of reference considered, and hydrostatic equilibrium prevails, equation (81) becomes

$$\mathfrak{W}_{ij;k} + \mathfrak{W}_{ik;j} + (\Omega^2 - \mu) I_{ijk} - \Omega^2\delta_{i3} I_{3jk} + 3\mu\delta_{i1} I_{1jk} = -\delta_{ij}\Pi_k - \delta_{ik}\Pi_j\,. \tag{83}$$

Equation (83) represents a total of eighteen equations. These eighteen equations fall into four groups distinguished by their parity with respect to the indices 1, 2, and 3: a group of three equations which are odd in all three indices, and three groups of five equations each which are odd with respect to one of the three indices and even with respect to the remaining two. The four groups of equations are

$$\mathfrak{W}_{12;3} + \mathfrak{W}_{31;2} + (\Omega^2 + 2\mu) I_{123} = 0\,,$$
$$\mathfrak{W}_{23;1} + \mathfrak{W}_{12;3} + (\Omega^2 - \mu) I_{123} = 0\,, \tag{84}$$
$$\mathfrak{W}_{31;2} + \mathfrak{W}_{23;1} - \mu I_{123} = 0\,;$$

$$\begin{aligned}
&2\mathfrak{W}_{11;1}+(\Omega^2+2\mu)I_{111}=-2\Pi_1\,,\\
&\mathfrak{W}_{22;1}+\mathfrak{W}_{21;2}+(\Omega^2-\mu)I_{221}=-\Pi_1\,,\\
&\mathfrak{W}_{33;1}+\mathfrak{W}_{31;3}-\mu I_{331}=-\Pi_1\,,\\
&2\mathfrak{W}_{12;2}+(\Omega^2+2\mu)I_{122}=2\mathfrak{W}_{13;3}+(\Omega^2+2\mu)I_{133}=0\,;
\end{aligned} \tag{85}$$

$$\begin{aligned}
&2\mathfrak{W}_{22;2}+(\Omega^2-\mu)I_{222}=-2\Pi_2\,,\\
&\mathfrak{W}_{11;2}+\mathfrak{W}_{12;1}+(\Omega^2+2\mu)I_{112}=-\Pi_2\,,\\
&\mathfrak{W}_{33;2}+\mathfrak{W}_{32;3}-\mu I_{332}=-\Pi_2\,,\\
&2\mathfrak{W}_{21;1}+(\Omega^2-\mu)I_{211}=2\mathfrak{W}_{23;3}+(\Omega^2-\mu)I_{233}=0\,;
\end{aligned} \tag{86}$$

and

$$\begin{aligned}
&2\mathfrak{W}_{33;3}-\mu I_{333}=-2\Pi_3\,,\\
&\mathfrak{W}_{22;3}+\mathfrak{W}_{23;2}+(\Omega^2-\mu)I_{223}=-\Pi_3\,,\\
&\mathfrak{W}_{11;3}+\mathfrak{W}_{13;1}+(\Omega^2+2\mu)I_{113}=-\Pi_3\,,\\
&2\mathfrak{W}_{31;1}-\mu I_{311}=2\mathfrak{W}_{32;2}-\mu I_{322}=0\,.
\end{aligned} \tag{87}$$

After the elimination of the $\Pi_k$'s from the foregoing equations we shall be left with a total of fifteen equations; and by suitably combining them, we obtain the following four groups of equations:

A:
$$\mathfrak{W}_{23;1}=-2\mathfrak{W}_{31;2}=2\mu I_{123},\ \mathfrak{W}_{12;3}=-(\Omega^2+\mu)I_{123}\,; \tag{88}$$

B:
$$\begin{aligned}
&2\mathfrak{W}_{12;2}+(\Omega^2+2\mu)I_{122}=2\mathfrak{W}_{13;3}+(\Omega^2+2\mu)I_{133}=0\,,\\
&S_{122}+(\Omega^2+2\mu)I_{111}-3\Omega^2 I_{122}=0\,,\\
&S_{133}+(\Omega^2+2\mu)I_{111}-\Omega^2 I_{133}=0\,;
\end{aligned} \tag{89}$$

C:
$$\begin{aligned}
&2\mathfrak{W}_{21;1}+(\Omega^2-\mu)I_{211}=2\mathfrak{W}_{23;3}+(\Omega^2-\mu)I_{233}=0\,,\\
&S_{211}+(\Omega^2-\mu)I_{222}-3(\Omega^2+\mu)I_{211}=0\,,\\
&S_{233}+(\Omega^2-\mu)I_{222}-(\Omega^2-3\mu)I_{233}=0\,;
\end{aligned} \tag{90}$$

D:
$$\begin{aligned}
&2\mathfrak{W}_{31;1}-\mu I_{311}=2\mathfrak{W}_{32;2}-\mu I_{322}=0\,,\\
&S_{322}-\mu I_{333}-(2\Omega^2-3\mu)I_{322}=0\,,\\
&S_{311}-\mu I_{333}-(2\Omega^2+3\mu)I_{311}=0\,,
\end{aligned} \tag{91}$$

where

$$S_{ijj}=-4\mathfrak{W}_{ij;j}-2\mathfrak{W}_{jj;i}+2\mathfrak{W}_{ii;i}\,. \tag{92}$$

(no summation over repeated indices)

### *b) The Neutral Point along the Roche Sequence Belonging to the Third Harmonics*

Now a necessary condition for the occurrence of a neutral point (belonging to the third harmonics) is that a non-trivial Lagrangian displacement exists such that the first variations of all of the equations in the four groups A, B, C, and D vanish at that point.

It has been shown in an earlier paper (Chandrasekhar and Lebovitz 1963*a*, Sec. V;

this paper will be referred to hereafter as "Paper II") that the first variations of all the quantities which occur in equations (88)–(91) are expressible, linearly, in terms of the symmetrized third-order virials:

$$V_{ijk}=\int_V \rho\,(\xi_i x'_j x'_k+\xi_j x'_k x'_i+\xi_k x'_i x'_j)\,dx=\delta I_{ijk}. \tag{93}$$

Moreover, the equations derived (by first variation) from the different groups (A, B, C, and D) involve different virials and are mutually exclusive. Thus, if $\delta$A, $\delta$B, $\delta$C, and $\delta$D denote the equations which are obtained by taking the first variations of the equations in the respective groups, then the association of the groups and the virials is the following:

$$\begin{aligned}\delta\text{A}&: V_{123}\,; &\quad \delta\text{B}&: V_{111},\ V_{122},\ V_{133}\,;\\ \delta\text{C}&: V_{222},\ V_{233},\ V_{211}\,; &\quad \delta\text{D}&: V_{333},\ V_{311},\ V_{322}\,.\end{aligned} \tag{94}$$

The coefficients of the virials in the expansions of the $\delta\mathfrak{W}_{ij;k}$'s and $\delta S_{ijj}$'s which occur in the varied form of equations (88)–(91) are tabulated in Table 2, Paper II.

We readily verify that the equations in the groups $\delta$A, $\delta$C, and $\delta$D do not allow any non-trivial solutions. Therefore, at a neutral point, we must necessarily have

$$V_{123}=V_{222}=V_{233}=V_{211}=V_{333}=V_{311}=V_{322}=0\,. \tag{95}$$

In this respect the situation is the same as in the cases of the Jacobi ellipsoids and the Jeans spheroids (cf. Paper I, eqs. [39], [40], and [109]).

The occurrence of a second neutral point now depends on whether the remaining group $\delta$B allows a non-trivial solution. The equations to be considered are

$$\delta J_1=-2\delta\mathfrak{W}_{12;2}-(\Omega^2+2\mu)V_{122}=0\,, \tag{96}$$

$$\delta J_2=-2\delta\mathfrak{W}_{13;3}-(\Omega^2+2\mu)V_{133}=0\,, \tag{97}$$

$$\delta J_3=\delta S_{122}+(\Omega^2+2\mu)V_{111}-3\Omega^2 V_{122}=0\,, \tag{98}$$

and

$$\delta J_4=\delta S_{133}+(\Omega^2+2\mu)V_{111}-\Omega^2 V_{133}=0\,. \tag{99}$$

It will be observed that these equations represent simple generalizations of the ones considered in the contexts of the Jacobi ellipsoids and the Jeans spheroids (Paper I, eqs. [43]–[46] and [110]–[113]).

Since the $\delta\mathfrak{W}_{ij;k}$'s and $\delta S_{ijj}$'s which occur in equations (96)–(99) are expressible as linear combinations of $V_{111}$, $V_{122}$, and $V_{133}$, we may write

$$\delta J_i=\langle i|111\rangle V_{111}+\langle i|122\rangle V_{122}+\langle i|133\rangle V_{133}=0\quad (i=1,2,3,4)\,, \tag{100}$$

where $\langle i|111\rangle$, etc., are certain matrix elements which are known.

If we should now require that the Lagrangian displacement be also solenoidal, then we should supplement equation (100) by the condition (see Paper II, Sec. VII)

$$\frac{V_{111}}{a_1^2}+\frac{V_{122}}{a_2^2}+\frac{V_{133}}{a_3^2}=0\,. \tag{101}$$

er of thnon-trivial neutral mode belonging to the third harmonics requires, therefore, that for some memb
itrix representing equations (100) and (101) is, at most, of rank 2.
e have expressions for the matrix elements $\langle i|111\rangle$, etc., we find that the $5 \times 3$ matrix whose rank w

$$^2B_{123} \quad {}_2^2B_{112} \qquad 2B_{12}+3a_2^2B_{122}-(\Omega^2+2\mu) \qquad a_2$$

$$B_{133}-({}_3^2B_{113} \qquad a_3^2B_{123} \qquad 2B_{13}+3a_3^2B$$

$$)B_{123}-5a_1^2B_{111}-2B_{11}+\Omega^2+2\mu \quad 3[B_{22}+B_{12}+(2a_2^2+a_1^2)B_{122}-a_1^2B_{112}-\Omega^2] \qquad (a_2^2+2a_1^2$$

$$+a_1^2)B\,5a_1^2B_{111}-2B_{11}+\Omega^2+2\mu \qquad (a_3^2+2a_1^2)B_{123}-3a_1^2B_{112} \qquad 3[B_{33}+B_{13}+(2a_3^2-$$

$$\frac{1}{a_3^2} \quad \frac{1}{a_1^2} \qquad \frac{1}{a_2^2}$$

oy consing a Roche sequence) the determinants of the six different sets of equations which we can form b
This vequations (100), we find that all six determinants vanish, simultaneously, at a determinate point.
fore, ex become two for a particular member of the sequence. A neutral point of the kind sought, there
y, to th195), in the context of analogous considerations relative to the Jacobian sequence, apply, equall
the ranlple matter to isolate the neutral point along a Roche sequence with the aid of the requirement on t
point isin Table 5 were obtained with that aid. And in Figure 1, the corresponding locus of the neutral

## X. SUMMARY OF THE PRINCIPAL RESULTS

Returning to Figure 1, we shall now recapitulate the principal results pertaining to the equilibrium and the stability of the configurations along the Maclaurin, the Jacobi, the Jeans, and the Roche sequences. All these configurations are ellipsoids and their structures are uniquely determined by the virial equations

$$\mathfrak{W}_{11} + (\Omega^2 + 2\mu)I_{11} = \mathfrak{W}_{22} + (\Omega^2 - \mu)I_{22} = \mathfrak{W}_{33} - \mu I_{33} ; \qquad (103)$$

and their characteristic frequencies of oscillation, belonging to the second and the third harmonics, can be ascertained with the aid of the linearized forms of the second- and the third-order virial equations.

In the plane of Figure 1, an ellipsoid (or a spheroid) of equilibrium is represented by the normalized values,

$$\bar{a}_1 = \frac{a_1}{(a_1 a_2 a_3)^{1/3}} \quad \text{and} \quad \bar{a}_2 = \frac{a_2}{(a_1 a_2 a_3)^{1/3}}, \qquad (104)$$

of two of its principal axes.

TABLE 5

THE NEUTRAL POINT ALONG THE ROCHE SEQUENCES BELONGING TO THE THIRD HARMONICS

($\Omega^2$ and $\mu$ Are Listed in the Unit $\pi G\rho$)

| $p$ | $\theta$ | $\Omega^2$ | $\mu$ | $\bar{a}_1$ | $\bar{a}_2$ | $\bar{a}_3$ |
|---|---|---|---|---|---|---|
| − 1 | 71°395 | 0 | 0 1091 | 2.142 | 0.6833 | 0 6833 |
| 0 | 70°98 | 0.07754 | .07754 | 2.080 | .7091 | 6780 |
| 1 | 70 74 | .12073 | .06036 | 2.044 | .7258 | 6741 |
| 4 | 70 40 | .1825 | .03649 | 1 989 | .7535 | .6671 |
| 20 | 70 02 | .2496 | 0 01188 | 1 923 | .7912 | 6572 |
| 100 | 69 86 | . . . . . . | . . . . . . | . . . . . . | . . . . . . | . . . . . . |
| ∞ . . | 69 817 | 0 2840 | 0 | 1 886 | 0 8150 | 0.6507 |

The undistorted sphere is represented by the point

$$S(\bar{a}_1 = \bar{a}_2 = \bar{a}_3 = 1) ; \qquad (105)$$

and the Maclaurin spheroids by the line

$$SM_{3''}: \bar{a}_1 = \bar{a}_2 (\geq 1) . \qquad (106)$$

Along the Maclaurin sequence, the first point of bifurcation (where the Jacobian sequence branches off) occurs at

$$M_2(e = 0.81267, \bar{a}_1 = \bar{a}_2 = 1.1972, \bar{a}_3 = 0.69766) . \qquad (107)$$

At this point, the Maclaurin spheroid becomes unstable if any dissipative mechanism is operative (cf. Roberts and Stewartson 1963). In the absence of such mechanisms, $M_2$ is a neutral point of the second kind (i.e., characterized by stability on either side of the point). In all events, the Maclaurin spheroid becomes unstable at

$$O_2(e = 0.95289, \bar{a}_1 = \bar{a}_2 = 1.4883, \bar{a}_3 = 0.45145) , \qquad (108)$$

by a mode[10] of overstable oscillations belonging to the second harmonics. At

$$M_{\max}(e = 0.929955, \bar{a}_1 = \bar{a}_2 = 1.3959, \bar{a}_3 = 0.51322) , \qquad (109)$$

[10] This is the same mode which is neutral at $M_2$.

intermediate between $M_2$ and $O_2$, $\Omega^2$ attains its maximum value ($0.449332\ \pi G\rho$) along this sequence. At

$$M_{3'}(e = 0.89926,\ \bar{a}_1 = \bar{a}_2 = 1.3174,\ \bar{a}_3 = 0.57623) \tag{110}$$

and

$$M_{3''}(e = 0.96937,\ \bar{a}_1 = \bar{a}_2 = 1.5968,\ \bar{a}_3 = 0.39217) \tag{111}$$

occur two further neutral points of the second kind (belonging to the third harmonics). At each of these points, a (different) sequence of pear-shaped configurations (presumably) branches off. And finally at

$$O_3(e = 0.96696,\ \bar{a}_1 = \bar{a}_2 = 1.5771,\ \bar{a}_3 = 0.40205) \tag{112}$$

the mode, which is neutral at $M_{3'}$, becomes unstable by overstable oscillations.

The Jacobian sequence $M_2J$, which branches off from the Maclaurin sequence at $M_2$, becomes unstable at the point

$$J_3(\bar{a}_1 = 1.8858,\ \bar{a}_2 = 0.81498,\ \bar{a}_3 = 0.65066)\,, \tag{113}$$

where the sequence of the pear-shaped configurations branches off; the analogous point on the Maclaurin sequence is $M_{3''}$. (The point $M_{3'}$ has no analogue on the Jacobian sequence.) While the Jacobi ellipsoid becomes strictly unstable only at $J_3$, it is, nevertheless, characterized by a non-trivial neutral mode of oscillation along its entire sequence; the mode in question is, in fact, the one which becomes neutral at $M_2$.

It should also be noted that along the combined Maclaurin-Jacobi sequence, $SM_2J$, $\Omega^2$ has its maximum ($=0.374230\ \pi G\rho$) at $M_2$.

The Jeans sequence of the pure tidally distorted prolate spheroids is represented by the "hyperbola"

$$ST\colon\ \bar{a}_1\bar{a}_2^2 = 1\,. \tag{114}$$

Along this sequence we have the two neutral points,

$$T_2(e = 0.88303,\ \bar{a}_1 = 1.6558,\ \bar{a}_2 = \bar{a}_3 = 0.77712) \tag{115}$$

and

$$T_3(e = 0.94774,\ \bar{a}_1 = 2.1417,\ \bar{a}_2 = \bar{a}_3 = 0.68331)\,, \tag{116}$$

where the Jeans spheroid becomes unstable by modes of oscillation belonging to the second and the third harmonics, respectively. Moreover, $\mu$ attains its maximum value ($0.125536\ \pi G\rho$) along this sequence at $T_2$.

The Roche sequences (belonging to the different $p$'s) fill the domain bounded by $SM_2$, $M_2J$, and $SR_0$, the Roche sequence for $p = 0$. They are represented by continuous curves which start at $S$ and eventually become asymptotic to $ST$. Along each Roche sequence, $\Omega^2$ and, simultaneously, $\mu$ attain maxima. The locus of these maxima is the dotted curve joining $M_2$ and $T_2$. This locus defines the Roche limit. But the Roche limit does not limit the stability of the Roche ellipsoids. The Roche ellipsoid actually becomes unstable at a somewhat later point by a mode of oscillation belonging to the second harmonics; and the locus which limits the stability of the Roche ellipsoids is the other heavy curve joining $M_2$ and $T_2$. And finally, along each Roche sequence a neutral point occurs where instability, by a mode of oscillation belonging to the third harmonics, sets in; the locus of this neutral point is the curve joining $T_3$ and $J_3$.

In the limit $p \to \infty$, the Roche sequence tends to the combined Maclaurin-Jacobi sequence represented by the broken curve $SM_2J$. In this limit, we must regard the *entire* Jacobian part of the combined Maclaurin-Jacobi sequence as unstable. Stated less abstractly, the conclusion to be drawn is that the Jacobi ellipsoids are unstable under the least tidal action: they are unstable in the presence of a fly!

## XI. THE EFFECT OF COMPRESSIBILITY ON THE STABILITY OF THE ROCHE ELLIPSOIDS

The analysis in Section VII can be readily extended to determine the effect of compressibility on the stability of the Roche ellipsoids. Specifically, the problem to be considered is that of the adiabatic oscillations of rotationally and tidally distorted homogeneous configurations. The assumption of *homogeneity* insures that in the equilibrium state the configurations will be indistinguishable from the incompressible Roche ellipsoids considered in Section III. But the present assumption, that the configurations are *gaseous*, has the consequence that the Lagrangian displacement describing the deformation can no longer be restricted to be solenoidal; instead, we must apply the laws appropriate to a gas subject to adiabatic changes. If the gas is assumed to have a ratio of specific heats $\gamma$, then the condition div $\xi = 0$ must be replaced by the condition

$$\frac{\delta p}{p} = \gamma \frac{\delta \rho}{\rho} = -\gamma \operatorname{div} \xi . \tag{117}$$

These relations will enable us to express the first variation $\delta\Pi$ $(=\delta\int p dx)$ in terms of $\xi$; we have

$$\delta\Pi = (\gamma - 1) \int_V \xi \cdot \operatorname{grad} p dx , \tag{118}$$

where $p$ denotes the pressure in the equilibrium state.

For the case under consideration

$$\operatorname{grad} p = \rho \operatorname{grad} \left[ I - \sum_{i=1}^{3} A_i x_i^2 + \tfrac{1}{2}\Omega^2 (x_1^2 + x_2^2) + \mu (x_1^2 - \tfrac{1}{2} x_2^2 - \tfrac{1}{2} x_3^2) \right] ; \tag{119}$$

and equation (118) gives

$$\delta\Pi = -q[(A_1 - \mu - \tfrac{1}{2}\Omega^2)V_{11} + (A_2 + \tfrac{1}{2}\mu - \tfrac{1}{2}\Omega^2)V_{22} + (A_3 + \tfrac{1}{2}\mu)V_{33}] , \tag{120}$$

where, for the sake of brevity, we have written

$$q = \gamma - 1 . \tag{121}$$

In equation (120) a factor $\pi G \rho a_1 a_2 a_3$ (by which *every* quantity must be divided out) has been suppressed.

### *a) The Characteristic Equation for the Even Modes Which Are Affected by Compressibility*

Turning to the problem of determining the characteristic frequencies of oscillation under the present more general circumstances, we first observe that the modes which we have designated as odd are unaffected: for, in deriving the corresponding characteristic equation (66), we made no use of the solenoidal condition (75). However, the modes which we have designated as even are affected; and the place where the analysis needs to be changed is clear: instead of supplementing equations (67), (73), and (74) by the solenoidal condition (as we did in Section VII*b*), we must now use one of the original equations (57)–(59) and assign to $\delta\Pi$ its present value (120). Thus, choosing equation (57) as the fourth equation, we have

$$\begin{aligned}(\tfrac{1}{2}\lambda^2 + \mu)V_{33} - \delta\mathfrak{W}_{33}& \\ + q[(A_1 - \mu - \tfrac{1}{2}\Omega^2)V_{11} + (A_2 + \tfrac{1}{2}\mu - \tfrac{1}{2}\Omega^2)V_{22} &+ (A_3 + \tfrac{1}{2}\mu)V_{33}] = 0 ;\end{aligned} \tag{122}$$

or substituting for $\delta\mathfrak{W}_{33}$ in accordance with equation (39) and regrouping the terms, we have

$$\begin{aligned}[q(A_1 - \mu - \tfrac{1}{2}\Omega^2) - a_3^2 A_{31}]V_{11} + [q(A_2 + \tfrac{1}{2}\mu - \tfrac{1}{2}\Omega^2) - a_3^2 A_{32}]V_{22}& \\ +[\tfrac{1}{2}\lambda^2 + \mu + 2B_{33} - a_3^2 A_{33} + q(A_3 + \tfrac{1}{2}\mu)]V_{33} = 0 .&\end{aligned} \tag{123}$$

Equations (67), (73), (74), and (123) now lead to the characteristic equation[11] (cf. eq. [76])

[11] We may, once again, draw attention to the fact that *every* quantity in these and similar equations must be considered as having been divided out by $\pi G \rho a_1 a_2 a_3$.

$3B_{11}-B_{12}$ $\quad -\tfrac{1}{2}\lambda^2+\Omega^2-\mu-3B_{22}+B_{12}$ $\quad B_{13}-B_{23}$

$_{11}+B_{12}-2B_{13}$ $\quad +\tfrac{1}{2}\lambda^2+\Omega^2+\mu+3B_{22}+B_{12}-2B_{23}$ $\quad -\lambda^2-2\mu-6B_{33}+B_{13}+B_{23}$

$-\lambda\Omega$ $\quad 0$

$\mu)$ $\quad )-a_3^2A_{31}$ $\quad q(A_2+\tfrac{1}{2}\mu-\tfrac{1}{2}\Omega^2)-a_3^2A_{32}$ $\quad \tfrac{1}{2}\lambda^2+\mu+2B_{33}-a_3^2A_{33}+q(A_3+\tfrac{1}{2}$

can be simplified in the same manner as equation (76); and we find (cf. eq. [77])

$B_{11}-B$ $\Bigg\|$ $\tfrac{1}{2}\lambda^2-\Omega^2-2\mu+3B_{11}-B_{12}$ $\quad -(B_{13}-B_{23})$ $\quad -3\mu+3($

$_2-B_{33})$ $\quad \Omega^2+B_{12}-B_{13}$ $\quad \tfrac{1}{2}\lambda^2+\mu+3B_{33}-B_{23}$ $\quad \Omega^2+3(B_2$

$A_3-2$ $\quad q(A_1-\mu-\tfrac{1}{2}\Omega^2)-a_3^2A_{31}$ $\quad -\tfrac{1}{2}\lambda^2-\mu-2B_{33}+a_3^2A_{33}-q(A_3+\tfrac{1}{2}\mu)$ $\quad \tfrac{1}{2}\lambda^2+\mu+2$

$-B_{23}\Bigg\|$ $\quad 2\lambda^2$ $\quad -(B_{13}-B_{23})$ $\quad -3\mu+3(B_{11}-B_{22})+B_{13}-$

$B_{13}$ $=$ $-(\lambda^2+3\mu)$ $\quad \tfrac{1}{2}\lambda^2+\mu+3B_{33}-B_{23}$ $\quad \Omega^2+3(B_{22}-B_{33})+B_{12}-$

$\Omega^2)\Bigg\|$ $\quad 0$ $\quad -\tfrac{1}{2}\lambda^2-\mu-2B_{33}+a_3^2A_{33}-q(A_3+\tfrac{1}{2}\mu)$ $\quad \tfrac{1}{2}\lambda^2+\mu+2A_3-2+q(2-$

### *b) The Effect of Compressibility on the Onset of Instability*

nd to a attached to the effect of compressibility on the onset of instability along a Roche sequence; ar
ns mar ation (125) for all its roots, since by putting $\lambda^2=0$, we obtain, at once, the equation which gover
er

$-B_{22})$ - $\Bigg\|$ $-\Omega^2-2\mu+3B_{11}-B_{12}$ $\quad -(B_{13}-B_{23})$ $\quad -3\mu+3(B_{11}-$

$B_{33})+ -\mu)$ $\quad \Omega^2+B_{12}-B_{13}$ $\quad \mu+3B_{33}-B_{23}$ $\quad \Omega^2+3(B_{22}-$

$+q(2$ $\quad q(A_1-\mu-\tfrac{1}{2}\Omega^2)-a_3^2A_{13}$ $\quad -\mu-2B_{33}+a_3^2A_{33}-q(A_3+\tfrac{1}{2}\mu)$ $\quad \mu+2A_3-2$

$$+3\mu\Omega^2\left\|\begin{matrix}-(B_{13}-B_{23}) & -3\mu+3(B_{11}-B_{22})+B_{13}-B_{23}\\ -\mu-2B_{33}+a_3^2A_{33}-q(A_3+\tfrac{1}{2}\mu) & \mu+2A_3-2+q(2-\Omega^2)\end{matrix}\right\|=0$$

which sidered as an equation for $\gamma\ (=1+q)$, will determine, for every equilibrium ellipsoid, a value for $\gamma$
ar that ver the results, which are valid for the incompressible case, by letting $\gamma$ tend to infinity, it is clea
ncompre the equilibrium ellipsoid considered is stable or unstable by the criterion which obtains in the in
to infinity when the limit to stability, set by incompressibility, is approached.

It follows from the foregoing remarks that, in considering equation (126), we can restrict ourselves to ellipsoids which are stable according to the considerations of Sections VII and VIII. The deduced values of $\gamma_c$ will then be positive; and the meaning to be attached to $\gamma_c$, in these cases, is that *the ellipsoid considered is stable if* $\gamma > \gamma_c$ *and unstable if* $\gamma < \gamma_c$.

We shall first consider the two special cases, $\Omega^2 = 0$ and $\mu = 0$.

By setting $\Omega^2 = 0$ in equation (126), and remembering that in this case the ellipsoid degenerates to a prolate spheroid and the indices 2 and 3 are indistinguishable, we find that the equation becomes

$$\begin{Vmatrix} -2\mu + 3B_{11} - B_{12} & -3\mu + 3B_{11} - 4B_{22} + B_{12} \\ q(A_1 - \mu) - a_2^2 A_{12} & 2q + \mu - A_1 \end{Vmatrix} = 0 . \tag{127}$$

In agreement with the results of Paper III (Sec. VI, Table 2 and Fig. 2) we find from equation (127) that

$$\gamma_c \to \tfrac{4}{3} \text{ as } \mu \to 0 \qquad \text{and} \qquad \gamma_c \to \infty \text{ as } \mu \to \mu_{\max} . \tag{128}$$

The equations determining the neutral modes along the Maclaurin and the Jacobian sequences must be considered separately. Setting $\mu = 0$ in equation (125) and remembering that $\Omega^2 = 2B_{12}$ along the Jacobian sequence, we observe that $\lambda^2 = 0$ is a characteristic root of the equation. Accordingly, *the neutral mode which obtains along the entire Jacobian sequence, in the incompressible case, is unaffected by compressibility*.[12] And along the Maclaurin sequence, the equation determining the neutral mode can be obtained from equation (126) by setting $\mu = 0$ and remembering that, in this case, the indices 1 and 2 are indistinguishable; we thus find

$$\begin{Vmatrix} 3B_{33} - B_{23} & \Omega^2 + 4B_{11} - 3B_{33} - B_{13} \\ -2B_{33} + a_3^2 A_{33} - qA_3 & 2A_3 - 2 + q(2 - \Omega^2) \end{Vmatrix} = 0 . \tag{129}$$

And this equation determines the same $\gamma_c$'s as have been tabulated in an earlier paper (Chandrasekhar and Lebovitz 1962*b*, Table 1). We may note here in particular, that

$$\gamma_c = 1.22515 \text{ at the point of bifurcation, } M_2 . \tag{130}$$

The values of $\gamma_c$ determined in accordance with equation (126), for the different Roche sequences, are given in Table 6; the table also includes the results for the Jeans sequence derived from equation (127).

[12] And the neutral mode *depending* on compressibility is determined by the equation

$$\begin{Vmatrix} -\Omega^2 + 3B_{11} - B_{12} & -(B_{13} - B_{23}) & 3(B_{11} - B_{22}) + B_{13} - B_{23} \\ +\Omega^2 + B_{12} - B_{13} & 3B_{33} - B_{23} & \Omega^2 + 3(B_{22} - B_{33}) + B_{12} - B_{13} \\ q(A_1 - \tfrac{1}{2}\Omega^2) - a_3^2 A_{13} & -2B_{33} + a_3^2 A_{33} - qA_3 & 2A_3 - 2 + q(2 - \Omega^2) \end{Vmatrix} = 0 .$$

For this equation we find:

| $\cos^{-1} a_3/a_1$ | $\gamma_c$ | $\cos^{-1} a_3/a_1$ | $\gamma_c$ |
|---|---|---|---|
| 54°.358 | 1 22515 | 65° | 1 2194 |
| 55 | 1 22513 | 70 | 1 2118 |
| 56 | 1 2250 | 75 | 1 2000 |
| 60 | 1 2236 | 80 | 1 1820 |
| 63 | 1 2214 | 83 | 1 1664 |

From Figure 4, which exhibits the results, it is apparent that the principal effect of compressibility on the stability of the Roche ellipsoids is to bring closer toward the Maclaurin branch ($SM_2$ in Fig. 1) the locus of the points of marginal stability and reduce, still further, the domain of stability which prevails in the incompressible case. More precisely, the situation is the following: for any $\gamma < \frac{4}{3}$, the locus of the points of marginal stability, in the $(\bar{a}_1, \bar{a}_2)$-plane of Figure 1, is a curve which joins $M_2$ to a point on $ST_2$;

TABLE 6

THE VALUES OF $\gamma_c$ WHICH LIMIT THE STABILITY OF THE HOMOGENEOUS COMPRESSIBLE ROCHE ELLIPSOIDS

| $e$ | $\cos^{-1} a_3/a_i$ | $\gamma_c$ |
|---|---|---|
| | $p=-1$ | |
| 0 | 0 | 1 3333 |
| 0 1 | 5°739 | 1 3334 |
| 2 | 11 537 | 1 3337 |
| 3 | 17 468 | 1 3353 |
| 4 | 23 578 | 1 3403 |
| 5 | 30 | 1 3530 |
| 6 | 36 870 | 1 3837 |
| 7. | 44 427 | 1 4646 |
| 75 | 48 590 | 1 5594 |
| 80 | 53 130 | 1 7768 |
| 82 | 55 085 | 1 9628 |
| 84 | 57 140 | 2 3230 |
| .86 | 59 316 | 3 3098 |
| 88. | 61 642 | 17 329 |
| 0 88303 | 62 009 | ∞ |

| $\cos^{-1} a_3/a_1$ | $\gamma_c$ |
|---|---|
| $p=0$ | |
| 24° | 1 3340 |
| 36 | 1 3541 |
| 48 | 1 4735 |
| 57 | 2 1339 |
| 60 | 4 0719 |
| 61 | 8 795 |
| 61 63 | ∞ |
| $p=1$ | |
| 12° | 1 3319 |
| 24 | 1 3301 |
| 36 | 1 3403 |
| 48 | 1 4329 |
| 54 | 1 6599 |
| 59 | 2 8570 |
| 60 | 4 1806 |
| 61 | 13 023 |
| 61 35 | ∞ |

| $\cos^{-1} a_3/a_1$ | $\gamma_c$ |
|---|---|
| $p=4$ | |
| 12° | 1 3310 |
| 24 | 1 3246 |
| 36 | 1 3209 |
| 48 | 1 3707 |
| 58 | 2 1899 |
| 60 | 4 9399 |
| 60 83 | ∞ |
| $p=20$ | |
| 36° | 1 3017 |
| 48 | 1 2952 |
| 58 | 2 1503 |
| 59 66 | ∞ |
| $p=100$ | |
| 48° | 1 2634 |
| 52 | 1 2646 |
| 56 | 1 4692 |
| 58 | 5 9359 |
| 58 18 | ∞ |

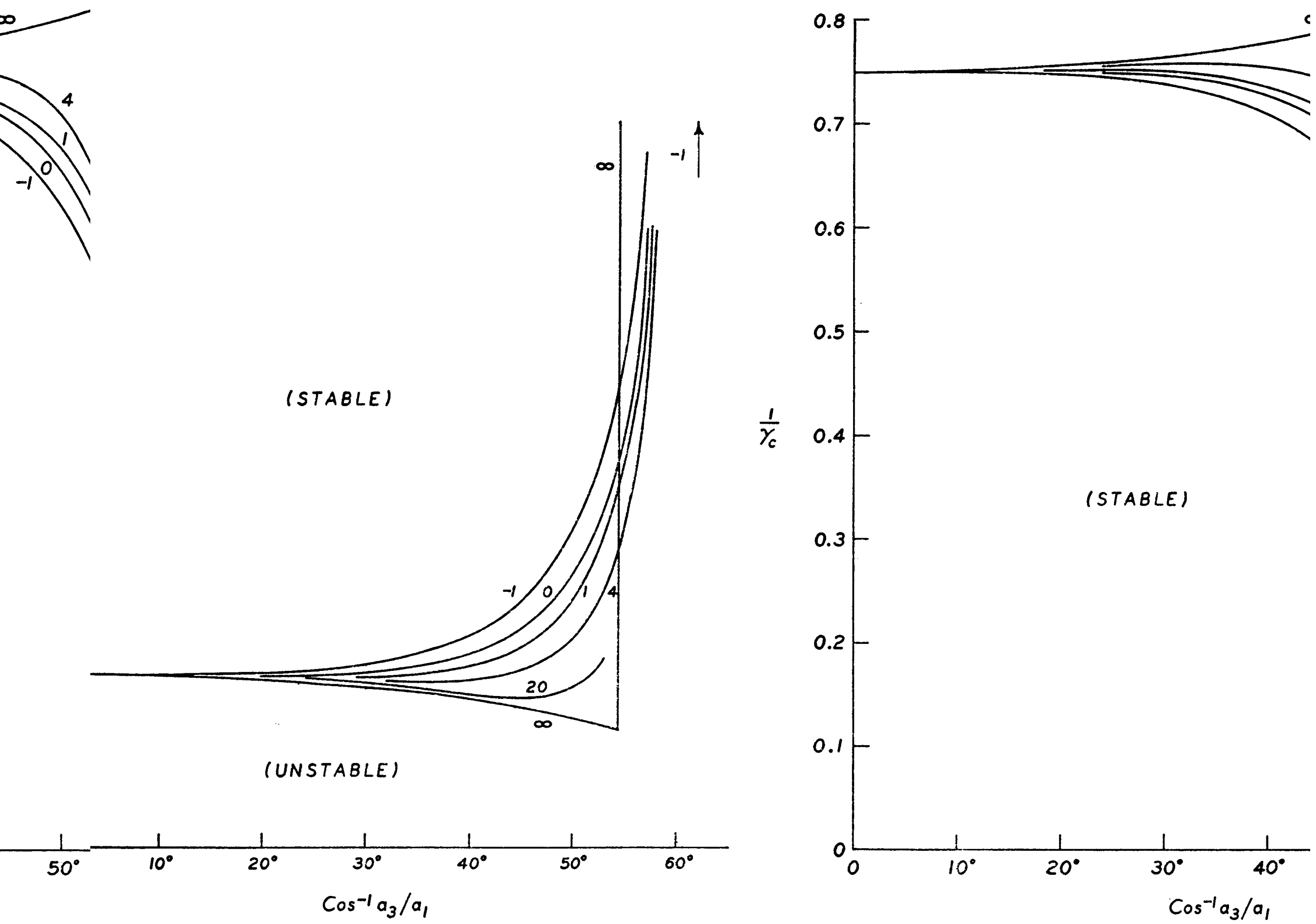
for $\gamma = \frac{4}{3}$, the locus is an arc which joins $M_2$ and $S$; for $1.22515 < \gamma < \frac{4}{3}$, the locus similarly joins $M_2$ to a point on $SM_2$; and for $\gamma \leq 1.22515$ the entire domain is unstable.

And finally we may note explicitly that compressibility does not affect the instability of the Jacobi ellipsoids under the least tidal action.

## XII. CONCLUDING REMARKS

The principal results bearing on the stability of the equilibrium configurations along the Maclaurin, the Jacobi, the Jeans, and the Roche sequences have been summarized in Section X. In this concluding section we shall consider only briefly two questions which occur.

*First*, since the Roche ellipsoids become unstable only after surpassing the Roche limit, what will be their behavior if they should so surpass the limit? And *second*, has the in-




e ellipsoı—The critical values of $\gamma_c$ and $1/\gamma_c$ at which marginal stability occurs for the homogeneous but compressible Roch the values of $p$ to which they belong.

stability of the Jacobi ellipsoid, under the least tidal action, any consequence for the "wider aspects of cosmogony?"

With respect to the first question, we may argue as Jeans (1919, pp. 118–128) has in the context of what may happen to one of his tidally distorted prolate spheroids if $R'$ should decrease below the limit set by $\mu_{max}$. On the assumption that the spheroid (secularly) retains its prolate form, Jeans derived an equation of motion for its eccentricity if $\mu$ varies in some prescribed manner. And he showed that if $\mu$ (during the course of its variation) should transgress $\mu_{max}$, then, the character of the problem will change into "a truly dynamical one" and, in the (very) first instance, the spheroid will continue to evolve down the sequence toward higher eccentricities.[13] If similar considerations can be applied to the Roche ellipsoids and should the Roche limit be surpassed, then the character of the problem will change into "a truly dynamical one" and, in the (very) first instance, the ellipsoid will continue to evolve down the sequence toward higher elongations. Since the point at which instability sets in is only a very little beyond the Roche limit (cf. Fig. 1), the dynamical evolution has not far to proceed before it will be arrested by the instability of one of its normal modes. These considerations are qualitative; but they suggest that it might be useful to formulate them quantitatively in the manner of Jeans.

Turning to the second question, we observe that as $p \to \infty$,

$$\lambda^2 = O\left(\frac{1}{p}\right) \tag{131}$$

along the near Jacobian part of the Roche sequence. Consequently, the tidal action of a body, one hundredth as massive as the primary, will induce an instability with an $e$-folding time of the order of ten times the natural periods of oscillation. If we should now idealize the bar of a barred spiral galaxy as a uniform prolate spheroid (which is the limiting form of the Jacobi ellipsoid), then its natural period of oscillation will be of the order of $10^8$ years (cf. Burbidge, Burbidge, and Prendergast 1960); and the tidal action of an external mass, which is 1 per cent of the mass of the bar, will induce an instability which will become manifest in $10^9$ years. It would appear, then, that the present considerations may have some relevance for cosmogony.

It should be apparent that the magnitude of the numerical work involved in the preparation of this paper is a very large one; and further, that it is particularly true of the present investigation that one cannot, in Kelvin's well-known phrase, "obtain satisfaction from formulas without their numerical magnitude." For that "satisfaction," I am greatly indebted to Miss Donna Elbert: she carried out all the necessary numerical work. I am also indebted to Dr. Norman R. Lebovitz for his careful scrutiny of the present manuscript and for helpful discussions.

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[13] Actually, Jeans's considerations, in the context of his particular problem, are, in a strict sense, unnecessary: for his spheroids (as we have seen) become unstable at $\mu_{max}$, a fact of which Jeans does not seem to have been fully aware (cf. comments in Paper III, Sec. VII).

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