THE ELLIPTICITY OF A SLOWLY ROTATING CONFIGURATION

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ABSTRACT

The second-order virial theorem is used to set upper and lower bounds for m/ϵ_R for a slowly rotating configuration, where m is the ratio of the centrifugal acceleration at the equator to the (average) gravitational acceleration on its surface and ϵ_R is the ellipticity of its slightly oblate figure of equilibrium. The bounds obtained are explicitly evaluated for the polytropes, for a model consisting of a core and a mantle of constant densities, and for a particular model for the earth.

I. INTRODUCTION

For a slowly rotating configuration, the ratio m, of the centrifugal acceleration at the equator to the average gravitational acceleration on its surface, and the ellipticity ϵ_R, of its slightly oblate figure of equilibrium, are in a relationship of cause and effect. It is, in fact, one of Newton's well-known theorems (cf. Todhunter 1873, Sec. 27) that, for a homogeneous configuration,

$$\frac{m}{\epsilon_R}=\frac{4}{5}. \tag{1}$$

Quite generally, m/ϵ_R is functionally dependent on the distribution of the density in the configuration; and the establishment of this dependence by Clairaut is one of the great achievements that followed in Newton's wake.

If one restricts one's self to configurations in which (in the non-rotating state) the density $\rho_0(r)$ at a point, does not exceed the mean density, $\bar{\rho}_0(r)$, interior to that point, then it is known that (cf. Chandrasekhar 1933, eq. [100])

$$1 \leq \frac{5}{4}\frac{m}{\epsilon_R}<2.5 . \tag{2}$$

And if the configuration is one in which the departures from homogeneity are not very great, then an approximate formula, derived by Darwin (1899) on the basis of a transformation of Clairaut's equation due to Radau, relates m/ϵ_R to the moment of inertia,

$$I=\int_V \rho_0(r)\, r^2 dx = 4\pi\int_0 \rho_0(r)\, r^4 dr , \tag{3}$$

of the configuration by

$$\frac{I}{MR^2}=1-\frac{2}{5}\left(\frac{5}{2}\frac{m}{\epsilon_R}-1\right)^{1/2}, \tag{4}$$

where M is its mass and R is its (mean) radius. (In eq. [3], $\rho_0(r)$ is the zero-order spherically symmetric part of the density distribution; see eq. [10] below.)

In Table 1, the values of I/MR^2 for the polytropes are compared with those deduced with the aid of equation (4) and the known values of $5m/2\epsilon_R$. It would appear from this table that Darwin's formula (4) cannot be applied if the central density of the configuration exceeds, say, six times its mean density.

In this paper, we shall set certain upper and lower bounds to $5m/4\epsilon_R$ from an application to this problem of the second-order virial equations which have already proved their utility in other connections (see the various papers of Chandrasekhar, Lebovitz, and Roberts in the recent issues of the *Astrophysical Journal*).

II. AN INTEGRAL RELATION

Let the origin of the co-ordinate system be at the center of mass of the configuration; and let the z-axis coincide with the axis of rotation; and, finally, let Ω denote the uniform angular velocity of the rotation. We shall assume that the configuration has symmetry about the z-axis; this assumption is justified for slowly rotating configurations in which we are presently interested.

A fundamental relation provided by the virial theorem is

$$\Omega^2 I_{11} = \mathfrak{W}_{33} - \mathfrak{W}_{11}\,, \tag{5}$$

where

$$\mathfrak{W}_{ij} = \int_V \rho x_i \frac{\partial \mathfrak{V}}{\partial x_j}\, dx \qquad \text{and} \qquad I_{ij} = \int_V \rho x_i x_j dx \tag{6}$$

are the potential energy and the moment of inertia tensors. In equations (6), $\mathfrak{V}$ denotes the gravitational potential and ρ the density.

TABLE 1

APPLICATION OF DARWIN'S FORMULA TO POLYTROPES*

n	$\rho_c/\bar{\rho}$	$5m/2\epsilon_R$	I/MR^2 Exact	I/MR^2 By Darwin's Formula
0.	1 0	2 0000	0 60000	0 6000
1 0. . .	3 29	3 2899	.39207	.3947
1 5. . .	5.99	3 8865	.30690	.3204
2 0 . . .	11 40	4.3559	.23227	.2673
3 0. . . .	54 18	4 8596	11304	2142
3.5. . .	152 9	4 9513	0 06832	0 2049

* The values of $5m/2\epsilon_R$ given in Chandrasekhar (1933, Table 3) have been revised in accordance with the results of the more precise integrations tabulated in Chandrasekhar and Lebovitz (1962, Table 7).

In the case of axisymmetry, $\mathfrak{W}_{33}$ and $\mathfrak{W}_{11}$ are the two distinct components of the potential energy tensor; and in spherical polar co-ordinates (r, ϑ, φ), the expressions for them become

$$\mathfrak{W}_{33} = \int_V \rho r \left[\mu^2 \frac{\partial \mathfrak{V}}{\partial r} + \frac{\mu(1-\mu^2)}{r} \frac{\partial \mathfrak{V}}{\partial \mu} \right] dx \tag{7}$$

and

$$\mathfrak{W}_{11} = \tfrac{1}{2} \int_V \rho r \left[(1-\mu^2) \frac{\partial \mathfrak{V}}{\partial r} - \frac{\mu(1-\mu^2)}{r} \frac{\partial \mathfrak{V}}{\partial \mu} \right] dx\,, \tag{8}$$

where $\mu = \cos \vartheta$. With the foregoing expressions for $\mathfrak{W}_{33}$ and $\mathfrak{W}_{11}$, equation (5) gives

$$\Omega^2 I_{11} = \int_V \rho r \left[P_2(\mu) \frac{\partial \mathfrak{V}}{\partial r} + \tfrac{3}{2} \frac{\mu(1-\mu^2)}{r} \frac{\partial \mathfrak{V}}{\partial \mu} \right] dx\,. \tag{9}$$

Equation (9) is exact. We shall now apply it to a slowly rotating configuration in which the departures from spherical symmetry, considered small, are governed by the Legendre function, $P_2(\mu)$. We shall write, then,

$$\rho = \rho_0(r) + \rho_2(r) P_2(\mu) \tag{10}$$

and

$$\mathfrak{V} = \mathfrak{V}_0(r) + \mathfrak{V}_2(r) P_2(\mu)\,, \tag{11}$$

where

$$|\rho_2(r)| \ll |\rho_0(r)| \quad \text{and} \quad |\mathfrak{B}_2(r)| \ll |\mathfrak{B}_0(r)| . \tag{12}$$

Under these same circumstances, we may also suppose that the boundary of the configuration is given by

$$R(\mu) = R[1 - \tfrac{2}{3}\epsilon_R P_2(\mu)] , \tag{13}$$

where ϵ_R defines the ellipticity of the figure of equilibrium.

In the framework of the foregoing assumptions, we may clearly replace I_{11} in equation (9) by its zero-order value,

$$\tfrac{1}{3}I = \tfrac{1}{3}\int_V \rho_0(r)\, r^2 dx = \tfrac{4}{3}\pi\int_0^R \rho_0(r)\, r^4 dr . \tag{14}$$

Now substituting for ρ and $\mathfrak{B}$ in accordance with equations (10) and (11) in equation (9), we find (on ignoring all quantities of the second order and after performing the integrations over the angles)

$$\tfrac{1}{3}\Omega^2 I = \tfrac{4}{5}\pi\int_0^R \rho_0(r)\, r^3 \left(\frac{d\mathfrak{B}_2}{dr} + \frac{3}{r}\,\mathfrak{B}_2\right) dr + \tfrac{4}{5}\pi\int_0^R \rho_2(r)\, r^3 \frac{d\mathfrak{B}_0}{dr}\, dr . \tag{15}$$

Since the gravitational potential is derived in a linear fashion from the distribution of density, it is clear that $\mathfrak{B}_0$ and $\mathfrak{B}_2$ should be expressible in terms of ρ_0 and ρ_2, respectively. Thus, by expanding $|x - x'|^{-1}$ in spherical harmonics in the usual manner, we readily find, from Poisson's integral,

$$\mathfrak{B}_0(r) + \mathfrak{B}_2(r)P_2(\mu) = G\int_V \frac{\rho_0(r') + \rho_2(r')P_2(\mu')}{|x - x'|}\, dx' , \tag{16}$$

that

$$\mathfrak{B}_0(r) = 4\pi G\left[\frac{1}{r}\int_0^r \rho_0(s)\, s^2 ds + \int_r^R \rho_0(s)\, s\, ds\right] \tag{17}$$

and

$$\mathfrak{B}_2(r) = \tfrac{4}{5}\pi G\left[\frac{1}{r^3}\int_0^r \rho_2(s)\, s^4 ds + r^2\int_r^R \frac{\rho_2(s)}{s}\, ds\right] . \tag{18}$$

Relations which follow from equations (17) and (18) are[1]

$$\frac{d\mathfrak{B}_0}{dr} = -\frac{4\pi G}{r^2}\int_0^r \rho_0(s)\, s^2 ds = -\frac{GM(r)}{r^2} \tag{19}$$

and

$$\frac{1}{r^3}\frac{d}{dr}(r^3\mathfrak{B}_2) = \frac{d\mathfrak{B}_2}{dr} + \frac{3}{r}\,\mathfrak{B}_2 = 4\pi G r\int_r \frac{\rho_2(s)}{s}\, ds , \tag{20}$$

where, in equation (19), $M(r)$ is the mass interior to r in the zeroth approximation.

Inserting relations (19) and (20) in equation (15), we obtain

$$\Omega^2 I = \frac{48\pi^2 G}{5}\left[\int_0^R dr\, \rho_0(r)\, r^4 \int_r^R ds\, \frac{\rho_2(s)}{s} - \int_0^R dr\, \rho_2(r)\, r\int_0^r ds\, \rho_0(s)\, s^2\right] . \tag{21}$$

[1] These relations are readily seen to be the first integrals of the equations,

$$\frac{d^2\mathfrak{B}_0}{dr^2} + \frac{2}{r}\frac{d\mathfrak{B}_0}{dr} = -4\pi G\rho_0$$

and

$$\frac{d^2\mathfrak{B}_2}{dr^2} + \frac{2}{r}\frac{d\mathfrak{B}_2}{dr} - \frac{6}{r^2}\,\mathfrak{B}_2 = -4\pi G\rho_2 ,$$

governing $\mathfrak{B}_0$ and $\mathfrak{B}_2$.

Inverting the order of the integrations in the second term on the right-hand side, we can write

$$\Omega^2 I = \frac{48\pi^2 G}{5}\int_0^R d\,r\,\rho_0(\,r\,)\,r^2\Big[\,r^2\int_r^R d\,s\,\frac{\rho_2(\,s\,)}{s} - \int_r^R d\,s\,\rho_2(\,s\,)\,s\,\Big], \tag{22}$$

or

$$\Omega^2 I = \frac{12\pi G}{5}\int_0^R d\,r\,\frac{dM(\,r\,)}{d\,r}\Big[\,r^2\int_r^R d\,s\,\frac{\rho_2(\,s\,)}{s} - \int_r^R d\,s\,\rho_2(\,s\,)\,s\,\Big]. \tag{23}$$

An integration by parts now leads to the result

$$\Omega^2 I = -\frac{24\pi G}{5}\int_0^R d\,rM(\,r\,)\,r\int_r^R d\,s\,\frac{\rho_2(\,s\,)}{s}. \tag{24}$$

In virtue of equation (20), we can also write

$$\Omega^2 I = -\tfrac{6}{5}\int_0^R \frac{M(\,r\,)}{r^3}\,\frac{d}{d\,r}\,(\,r^3\mathfrak{B}_2)\,d\,r\,; \tag{25}$$

and this is the desired integral relation.

III. AN INTEGRAL EQUATION GOVERNING THE ELLIPTICITY OF THE SURFACES OF EQUAL GEOPOTENTIAL

The definition of the geopotential is

$$\Psi = \mathfrak{B} + \tfrac{1}{2}\Omega^2 r^2(1-\mu^2)\,, \tag{26}$$

so that

$$\operatorname{grad} p = \rho \operatorname{grad} \Psi\,. \tag{27}$$

Let the surfaces of equal geopotential be

$$r[1-\tfrac{2}{3}\epsilon(r)P_2(\mu)]\,; \tag{28}$$

$\epsilon(r)$ defines, then, the varying ellipticity of these surfaces. At the boundary of the configuration, $\epsilon(r)$ takes the value ϵ_R.

For the assumed form of $\mathfrak{B}$,

$$\Psi = \mathfrak{B}_0(r) + \mathfrak{B}_2(r)P_2(\mu) + \tfrac{1}{3}\Omega^2 r^2[1-P_2(\mu)]\,. \tag{29}$$

The condition that Ψ so defined is constant over the surfaces (28) requires (in our present approximation) that

$$\mathfrak{B}_2(\,r\,) = \tfrac{2}{3}\,r\epsilon(\,r\,)\,\frac{d\mathfrak{B}_0}{d\,r} + \tfrac{1}{3}\Omega^2\,r^2\,; \tag{30}$$

or (cf. eq. [19])

$$\mathfrak{B}_2(\,r\,) = -\tfrac{2}{3}G\epsilon(\,r\,)\,\frac{M(\,r\,)}{r} + \tfrac{1}{3}\Omega^2\,r^2. \tag{31}$$

From this last equation it follows that

$$\frac{d}{d\,r}\,(\,r^3\mathfrak{B}_2) = -\tfrac{2}{3}G\frac{d}{d\,r}[\epsilon(\,r\,)M(\,r\,)\,r^2] + \tfrac{5}{3}\Omega^2\,r^4. \tag{32}$$

Returning to equation (25), we may, in view of equation (32), write it in the form

$$\Omega^2 I = -2\Omega^2\int_0^R M(\,r\,)\,r\,d\,r + \tfrac{4}{5}G\int_0^R \frac{M(\,r\,)}{r^3}\,\frac{d}{d\,r}[\epsilon(\,r\,)M(\,r\,)\,r^2]\,d\,r. \tag{33}$$

On the other hand,

$$2\int_0^R M(r)\,r\,dr = MR^2 - 4\pi\int_0^R \rho_0(r)\,r^4 dr = MR^2 - I\,. \tag{34}$$

Equation (33), therefore, becomes

$$\Omega^2 MR^2 = \tfrac{4}{5}G\int_0^R \frac{M(r)}{r^3}\frac{d}{dr}[\epsilon(r)M(r)\,r^2]\,dr\,. \tag{35}$$

Now, by definition,

$$m = \frac{\Omega^2 R}{GM/R^2} = \frac{\Omega^2 R^3}{GM}\,. \tag{36}$$

In terms of m, we can rewrite equation (35) in the form

$$\tfrac{5}{4}m = \frac{R}{M^2}\int_0^R \frac{M(r)}{r^3}\frac{d}{dr}[\epsilon(r)M(r)\,r^2]\,dr\,; \tag{37}$$

and an integration by parts gives

$$\tfrac{5}{4}m = \epsilon_R + \frac{R}{M^2}\int_0^R \epsilon(r)M(r)\,r^2\frac{d}{dr}\left[-\frac{M(r)}{r^3}\right]dr\,. \tag{38}$$

Making use of the relation

$$-\frac{d}{dr}\frac{M(r)}{r^3} = -\tfrac{4}{3}\pi\frac{d\bar{\rho}_0(r)}{dr} = \frac{4\pi}{r}[\bar{\rho}_0(r) - \rho_0(r)]\,, \tag{39}$$

we can also write

$$\tfrac{5}{4}m = \epsilon_R + 4\pi\frac{R}{M^2}\int_0^R \epsilon(r)M(r)\,r\,[\bar{\rho}_0(r) - \rho_0(r)]\,dr\,. \tag{40}$$

The integrals which occur in equations (37), (38), and (40) are, therefore, all positive definite, so long as

$$\bar{\rho}_0(r) \geq \rho_0(r)\,. \tag{41}$$

IV. UPPER AND LOWER BOUNDS FOR m/ϵ_R

It is known (cf. Jeffreys 1959, p. 149) that, for configurations in which condition (41) is met, $\epsilon(r)$ is a monotonic increasing function of r and that

$$\epsilon_R \geq \epsilon(r) \geq \epsilon_R\left(\frac{r}{R}\right)^3. \tag{42}$$

Therefore, by inserting these bounds for $\epsilon(r)$ in equation (37), (38), or (40), we shall obtain corresponding bounds for m/ϵ_R. Thus, by replacing $\epsilon(r)$ by ϵ_R in equation (38), we obtain

$$\tfrac{5}{4}\frac{m}{\epsilon_R} \leq 1 - \frac{R}{M^2}\int_0^R M(r)\,r^2\frac{d}{dr}\frac{M(r)}{r^3}\,dr\,. \tag{43}$$

Now, by successive integrations by parts, we find

$$\begin{aligned}\int_0^R M(r)\,r^2\frac{d}{dr}\frac{M(r)}{r^3}\,dr &= \int_0^R \frac{M(r)\,dM(r)}{r} + 3\int_0^R M^2(r)\frac{d}{dr}\left(\frac{1}{r}\right)dr\\ &= 3\frac{M^2}{R} - 5\int_0^R \frac{M(r)\,dM(r)}{r} = 3\frac{M^2}{R} - 5\frac{|\mathfrak{W}|}{G}\,,\end{aligned} \tag{44}$$

where $\mathfrak{W}$ $(=\mathfrak{W}_{ii})$ is the gravitational potential energy of the configuration. We thus obtain the inequality,

$$\tfrac{5}{4}\frac{m}{\epsilon_R}\leq 5\,\frac{|\mathfrak{W}|}{GM^2/R}-2\,, \tag{45}$$

which gives the desired upper bound.

Similarly, by replacing $\epsilon(r)$ by $\epsilon_R(r/R)^3$ in equation (37), we obtain

$$\tfrac{5}{4}\frac{m}{\epsilon_R}\geq\frac{1}{M^2R^2}\int_0^R\frac{M(r)}{r^3}\frac{d}{dr}[M(r)\,r^5]\,dr\,. \tag{46}$$

The integral on the right-hand side of this inequality is

$$\tfrac{5}{2}\int_0^R M^2(r)\,d(r^2)+\int_0^R r^2M(r)\,dM(r)=\tfrac{5}{2}M^2R^2-4\int_0^R r^2M(r)\,dM(r)\,. \tag{47}$$

We thus obtain the inequality,

$$\tfrac{5}{4}\frac{m}{\epsilon_R}\geq 2.5-\frac{4}{M^2R^2}\int_0^R r^2M(r)\,dM(r)\,, \tag{48}$$

which gives the desired lower bound.

V. SOME ILLUSTRATIONS

We shall now consider some applications of the inequalities found in the preceding section.

a) The Polytropes

For the gravitational potential energy of a polytrope, we have Emden's well-known formula,

$$\mathfrak{W}=-\frac{3}{5-n}\frac{GM^2}{R}\,; \tag{49}$$

and the inequality (45) giving the upper bound of m/ϵ_R becomes, in this case,

$$\tfrac{5}{4}\frac{m}{\epsilon_R}\leq\frac{5+2n}{5-n}\,. \tag{50}$$

It should be noted that inequality (50) does not provide a meaningful bound for $n>\frac{5}{3}$: for we know that, in all cases, $5\,m/4\epsilon_R<2.5$ (cf. eq. [2]); and only for $n>\frac{5}{3}$ is the right-hand side of (50) less than 2.5.

Turning to inequality (48) giving the lower bound of m/ϵ_R and expressing the various quantities in terms of Emden's variables and units (cf. Chandrasekhar and Lebovitz 1962, eq. [8]), we have

$$\tfrac{5}{4}\frac{m}{\epsilon_R}\geq 2.5+\frac{4}{\xi_1^6(\theta_1')^2}\int_0^{\xi_1}\xi^6\theta^n\frac{d\theta}{d\xi}\,d\xi\,. \tag{51}$$

After an integration by parts, the inequality (51) becomes

$$\tfrac{5}{4}\frac{m}{\epsilon_R}\geq 2.5-\frac{24}{(n+1)\,\xi_1^6(\theta_1')^2}\int_0^{\xi_1}\theta^{n+1}\xi^5 d\xi\,, \tag{52}$$

where it may be noted that, according to a reduction formula due to Milne (1929),

$$\int_0^{\xi_1}\theta^{n+1}\xi^5 d\xi=\frac{n+1}{2(n+7)}\left[\xi_1^6(\theta_1')^2-12\int_0^{\xi_1}\xi^3\theta^2 d\xi\right]. \tag{53}$$

In Table 2, the upper and the lower bounds of $5m/4\epsilon_R$ given by the inequalities (50) and (52) are listed, together with their known values and also the values deduced from Darwin's formula (4).

b) *A Model Consisting of a Core and a Mantle of Different Densities*

A model often used as an idealization of the density distribution which occurs inside the earth consists of a core of a certain constant density and a mantle of a different constant density. We shall consider the implications of the inequalities (45) and (48) for this model.

TABLE 2

UPPER AND LOWER BOUNDS OF $5m/4\epsilon_R$ FOR POLYTROPES

n	Lower Bound	Upper Bound	Exact Value	By Darwin's Formula
0	1.000	1 000	1.0000	1 000
1.0	1 456	1.750	1.6449	1 655
1 5	1 661	2 286	1.9432	2 001
2 0	1.850	...	2 1779	2 312
3.0	2.117	...	2 4298	...
3.5	2.293	...	2 4757	...
5.0	2 500	...	2 5000	...

Let the core occupy a fraction q of the radius R and let α be the ratio of the densities of the core and the mantle. The integrals which occur in the inequalities (45) and (48) are readily evaluated for this model, and we find

$$\frac{5}{4}\frac{m}{\epsilon_R} \leq 3\,\frac{\alpha^2 q^5 + (1-q^5) - 2.5(1-\alpha)q^3(1-q^2)}{(\alpha q^3 + 1 - q^3)^2} - 2 \tag{54}$$

and

$$\frac{5}{4}\frac{m}{\epsilon_R} \geq 2.5 - 1.5\,\frac{\alpha^2 q^8 + (1-q^8) - 1.6(1-\alpha)q^3(1-q^5)}{(\alpha q^3 + 1 - q^3)^2}. \tag{55}$$

For the earth

$$m = 0.003499 \quad \text{and} \quad \epsilon_R = 1/297; \tag{56}$$

and it is further estimated that

$$q = 0.545\,. \tag{57}$$

These values for m, ϵ_R, and q, when inserted in formulae (54) and (55), lead to two quadratic inequalities for α, which, when solved, give

$$4.68 \geq \alpha \geq 2.62\,. \tag{58}$$

The value of α generally used, when the present model for the earth is assumed, is 2.9. The lower bound for α determined by formula (54) is, thus, very close to its "true value."

c) *A Particular Model for the Earth*

As a last illustration, we shall consider a particular model for the earth which Bullard (1946) has used for a direct integration of Clairaut's equation. Using the values of $\rho(r)$ and $\bar{\rho}(r)$ tabulated in Bullard's paper, we find (by numerical integration) that

$$\frac{|\mathfrak{W}|}{GM^2/R} = 0.6669 \quad \text{and} \quad \frac{1}{M^2R^2}\int_0^R r^2 M(r)\,dM(r) = 0.3326\,; \tag{59}$$

and these values, when inserted into inequalities (45) and (48), give

$$1.170 \leq \tfrac{5}{4}\frac{m}{\epsilon_R} \leq 1.335. \tag{60}$$

The actual value of $5m/4\epsilon_R$ for the earth is 1.299; it thus differs from the upper bound given by formula (60) by less than 3 per cent.

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APPENDIX

Clairaut's Equation

It will be noticed that nowhere in the text was there an occasion to define the ellipticity of the surfaces of constant density. However, if the equation of state is barotropic, then it follows from equation (27) that the surfaces of equal geopotential are also surfaces of equal pressure and density. The requirement (in these circumstances) that the density specified in equation (10) is constant over the same surfaces (28) the geopotential is constant, yields the additional relation,

$$\rho_2(r) = \tfrac{2}{3}\epsilon(r)\, r \frac{d\rho_0}{dr}. \tag{A, 1}$$

This relation when inserted in equation (20) gives

$$\frac{d}{dr}(r^3\mathfrak{V}_2) = \tfrac{8}{3}\pi G r^4 \int_r^R \epsilon(s)\frac{d\rho_0(s)}{ds}\, ds; \tag{A, 2}$$

and this equation in combination with equation (32) gives

$$4\pi G \int_r^R \epsilon(s)\frac{d\rho_0(s)}{ds}\, ds = -\frac{G}{r^4}\frac{d}{dr}[\epsilon(r)M(r)r^2] + \tfrac{5}{2}\Omega^2. \tag{A, 3}$$

Differentiating this equation with respect to r, we obtain

$$4\pi\epsilon(r)\frac{d\rho_0}{dr} = \frac{d}{dr}\left\{\frac{1}{r^4}\frac{d}{dr}[\epsilon(r)M(r)r^2]\right\}. \tag{A, 4}$$

Equation (A, 4), as can be verified by expansion, is the same as Clairaut's equation; its present derivation emphasizes (what is sometimes obscured) that its validity strictly depends on the equation of state being barotropic.

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