NON-RADIAL OSCILLATIONS AND CONVECTIVE INSTABILITY OF GASEOUS MASSES

S. Chandrasekhar and Norman R. Lebovitz
University of Chicago and Massachusetts Institute of Technology

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ABSTRACT

Modes of non-radial oscillation of gaseous masses belonging to spherical harmonics of orders $l = 1$ and 3 are considered on the basis of the first- and the third-order virial equations. For an assumed Lagrangian displacement $\boldsymbol{\xi}$ of the form

$$\xi_i = (L_{i;jk} x_j x_k + L_i) e^{i\sigma t}$$

(where $L_{i;jk}$ and L_i represent a total of twenty-one unspecified constants and σ is the characteristic frequency to be determined), the theory predicts the occurrence of modes of oscillation of two different types: modes (belonging to $l = 3$) which are analogous to the Kelvin modes of an incompressible sphere and modes (belonging to $l = 1$) which are analogous to those discovered by Pekeris for a homogeneous compressible sphere and which exhibit its convective instability. For the latter modes, the virial equations lead to a characteristic equation for σ^2 of degree 2 whose coefficients are integrals over the variables of the unperturbed configuration, including its superpotential. The theory is applied to the polytropic gas spheres, and it is shown that they are convectively unstable (for the modes belonging to $l = 1$) if the ratio of the specific heats γ is less than a certain critical value The critical values of γ predicted by the (approximate) theory differ from $1 + 1/n$ (where n is the polytropic index) by less than 1 per cent over the range of n (≤ 3.5) considered; the extent of this agreement is a measure of the accuracy of the method based on the virial equations and the assumed form of the Lagrangian displacement.

I. INTRODUCTION

As Cowling (1942) and Ledoux (cf. Ledoux and Walraven 1958) have emphasized, a theory of the non-radial oscillations of gaseous masses is relevant in two different connections: in discovering modes of oscillation which may lead to "explosive instability" (particularly under external influences such as tidal action) and in establishing criteria for convective stability on a sound theoretical basis. It is with the latter aspect of the theory of non-radial oscillations that we shall be concerned in this paper.

The criterion for convective stability as commonly applied is that of Karl Schwarzschild, which requires the prevailing temperature gradient, at every point, not to exceed the adiabatic gradient. The manner in which one generally establishes this criterion is to examine how a small isolated element will react to a fluctuation which makes its physical state slightly different from that of its immediate environment. While the arguments as usually presented are reasonable, they are, really, no substitute for a proper treatment of the stability of the system in terms of normal modes and initial conditions: a system reacts to a perturbation *in toto* and is indivisible. From this strict point of view, the qualitative arguments which are used to validate the Schwarzschild criterion are no more than a suggestion that instability via modes of sufficiently high order (i.e., modes belonging to spherical harmonics of high orders l and radial functions with many nodes) will arise if the criterion is violated in any small isolated region. On this account, Cowling and Ledoux have attempted to establish the Schwarzschild criterion by exploring methods of approximation which may be suitable for treating these modes of non-radial oscillation which have many nodes both in the transverse and in the radial directions. The principal approximation underlying these treatments is the neglect of the change in the gravitational potential caused by the perturbation. While this approximation may be a good one[1] for treating modes of oscillations of high orders, it is hardly one which can

[1] See, however, the remarks in the last Sec. VII.

be applied to modes belonging to $l = 1$; and these are the modes that are most relevant for the discernment of convective instability in many important cases. Thus, Pekeris' (1939) first exact treatment of the non-radial adiabatic oscillations revealed the instability of a homogeneous compressible sphere for modes belonging to all l's including $l = 1$. The instability in this instance clearly derives from the fact that *a uniform density is superadiabatic everywhere for any finite ratio of the specific heats* γ. That, under these circumstances, instability should arise already for $l = 1$ is entirely to be expected: it is, for example, the mode by which convective instability first manifests itself in a *viscous* fluid sphere heated within (Chandrasekhar 1961). What is true of the homogeneous sphere must be equally true of the polytropic gas spheres in the sense that when they become convectively unstable, they must also do so for modes of oscillation belonging to $l = 1$. This is apparent when it is noted that a polytropic distribution in which the pressure and the density are related by

$$p = \text{constant } \rho^{1+1/n} \tag{1}$$

is superadiabatic everywhere if

$$\gamma < 1 + \frac{1}{n}. \tag{2}$$

Consequently, if $1 + 1/n$ should exceed γ, the Schwarzschild criterion will be violated simultaneously throughout the entire mass, and the instability can assert itself with the largest permissible pattern of circulation, i.e., by a mode of oscillation belonging to $l = 1$. We may conclude, then, that *a polytrope of index* n *will be convectively unstable for* $\gamma < 1 + 1/n$ *and that the instability will be manifested already by a mode of oscillation belonging to* $l = 1$. But this fundamental result has never been properly established.

In this paper we shall develop a method based on the third-order virial equations (see Chandrasekhar 1962; this paper will be referred to hereafter as "Paper I") which will enable us to treat modes of oscillation belonging to $l = 1$ and 3, taking full account of the variations in the gravitational potential during the oscillations. The linearized form of the virial equations permits exact and explicit solutions of problems associated with homogeneous masses. Thus, the problem of the oscillations and the stability of the Jacobi ellipsoids has recently been solved with their aid (Chandrasekhar and Lebovitz 1963*a*; see also the further paper 1963*b* on the Maclaurin spheroids; these papers will be referred to hereafter as "Papers II" and "III," respectively). And we shall see in Section III below that the Pekeris instability of the homogeneous compressible sphere for $l = 1$ can be derived equally with their aid. However, configurations which are not homogeneous cannot be treated exactly with the virial equations only; but they do provide a basis for an approximative treatment. And since the answer may be considered "known" in the case of the polytropes, the application of the method to them will provide a useful test of the precision of the method.

II. THE VIRIAL EQUATIONS OF THE THIRD ORDER FOR THE TREATMENT OF THE OSCILLATIONS OF A SPHERICAL DISTRIBUTION OF MASS

For departures from equilibrium described by a Lagrangian displacement of the form

$$\xi(x)e^{i\sigma t}, \tag{3}$$

where σ denotes the characteristic frequency to be determined, the virial equations of the third order give (cf. Paper I, eq. [25])

$$-\sigma^2 V_{i;jk} = \delta\mathfrak{W}_{ij;k} + \delta\mathfrak{W}_{ik;j} + \delta_{ij}\delta\Pi_k + \delta_{ik}\delta\Pi_j, \tag{4}$$

where

$$V_{i;jk} = \int_V \rho\xi_i x_j x_k d x \tag{5}$$

defines the third-order virial and $\delta\mathfrak{W}_{ij;k}$ and $\delta\Pi_k$ are the first variations (due to the displacement $\boldsymbol{\xi}$) of the quantities

$$\mathfrak{W}_{ij;k} = -\tfrac{1}{2}\int_V \rho\mathfrak{V}_{ij}x_k dx \qquad \text{and} \qquad \Pi_k = \int_V p x_k dx\,, \tag{6}$$

which have been defined in Paper I.

We have (Paper I, eq. [72])

$$-2\,\delta\mathfrak{W}_{ij;k} = \int_V \rho\mathfrak{V}_{ij}\xi_k dx + \int_V \rho\xi_l \frac{\partial\mathfrak{V}_{ij}}{\partial x_l} x_k dx + \int_V \rho\xi_l \frac{\partial\mathfrak{D}_{ij;k}}{\partial x_l} dx\,, \tag{7}$$

where $\mathfrak{V}_{ij}$ is the tensor potential and $\mathfrak{D}_{ij;k}$ is the same tensor potential for the fictitious density distribution ρx_k.

If the oscillations are assumed to take place adiabatically with a ratio of the specific heats γ, then

$$\delta\left(\frac{p}{\rho}\right) = -(\gamma-1)\frac{p}{\rho}\operatorname{div}\boldsymbol{\xi}\,, \tag{8}$$

and

$$\delta\Pi_k = \delta\int_V \frac{p}{\rho}x_k\rho dx = -(\gamma-1)\int_V p x_k \operatorname{div}\boldsymbol{\xi}\; dx + \int_V p\xi_k dx\,. \tag{9}$$

An alternative form of $\delta\Pi_k$, which we obtain after an integration by parts, is

$$\delta\Pi_k = (\gamma-1)\int_V x_k\boldsymbol{\xi}\cdot \operatorname{grad} p\; dx + \gamma\int_V p\xi_k dx\,. \tag{10}$$

In treating the eighteen equations represented by equation (4), we shall find it convenient to introduce the symmetrized virial,

$$V_{ijk} = V_{i;jk} + V_{j;ki} + V_{k;ij}\,, \tag{11}$$

and combine and group the equations into different non-combining sets. Such a grouping was accomplished in Paper II for the more general system of equations governing a uniformly rotating configuration with no special symmetry (except that implied by the rotation itself). In the present instance when there is no rotation and the unperturbed configuration is spherically symmetric, the desired reduction is, of course, much simpler. However, in writing the different equations, we shall refer to the equations in Papers II and III and specialize them appropriately; but they can be derived quite readily, *ab initio*, from equation (4).

(*a*) *First*, we have *six* equations of the form (cf. Paper II, eqs. [62], [76], [77], [96], and [97])

$$\sigma^2(V_{122} - \tfrac{1}{3}V_{111}) = \delta S_{122}\,, \tag{12}$$

where

$$\delta S_{122} = -4\delta\mathfrak{W}_{12;2} - 2\delta\mathfrak{W}_{22;1} + 2\delta\mathfrak{W}_{11;1}\,. \tag{13}$$

The other five equations of this set can be written down by selecting other pairs of indices ($i, j, i \neq j$) besides (1, 2) to which equations (12) and (13) belong.

(*b*) *Second*, we have a set of *three* equations of the form (Paper II, eqs. [30] and [34])

$$\sigma^2(V_{1;22} - V_{1;33}) = -2\delta\mathfrak{W}_{12;2} + 2\delta\mathfrak{W}_{13;3}\,. \tag{14}$$

(*c*) *Third*, we have the *three* equations (Paper II, eqs. [38], [42], [43], and [93])

$$\sigma^2 V_{123} = \delta S_{123}\,, \tag{15}$$

$$\sigma^2(V_{1;23} - V_{2;31}) = 0\,, \quad \text{and} \quad \sigma^2(V_{2;13} - V_{3;12}) = 0\,, \tag{16}$$

where

$$\delta S_{123} = -2\delta\mathfrak{W}_{12;3} - 2\delta\mathfrak{W}_{23;1} - 2\delta\mathfrak{W}_{31;2} = -6\delta\mathfrak{W}_{12;3} . \tag{17}$$

In writing equations (16) and (17) we have made use of the fact (which we shall verify later; see eq. [51] below) that, under circumstances of spherical symmetry,

$$\delta\mathfrak{W}_{12;3} = \delta\mathfrak{W}_{23;1} = \delta\mathfrak{W}_{31;2} . \tag{18}$$

(*d*) *Fourth*, we have *three pairs* of equations of the form (Paper I, eqs. [30], [34], [48], [51], and [87])

$$\tfrac{1}{3}\sigma^2 V_{111} + 2\delta\mathfrak{W}_{11;1} = -2\delta\Pi_1 \tag{19}$$

and

$$\sigma^2(V_{1;22} + V_{1;33}) = -2\delta\mathfrak{W}_{12;2} - 2\delta\mathfrak{W}_{13;3} . \tag{20}$$

It will be noticed that, among the four groups of equations, only those in the fourth group (*d*) involve the $\delta\Pi_k$'s.

III. THE MODES OF OSCILLATION OF A COMPRESSIBLE HOMOGENEOUS SPHERE BELONGING TO THE HARMONICS OF ORDERS ONE AND THREE

For a homogeneous sphere, the equations of the preceding section can be solved exactly and explicitly: for, in this case, the various $\delta\mathfrak{W}_{ij;k}$'s, δS_{ijk}'s and $\delta\Pi_k$'s can all be expressed linearly in terms of the virials themselves with simple numerical coefficients. The required coefficients for homogeneous ellipsoids and spheroids have been tabulated in Paper II (Tables 1 and 2) and Paper III (Tables 1, 2, and 3). For a sphere, the coefficients are much simplified, since the symbols $A_{ijk\ldots}$ and $B_{ijk\ldots}$, in terms of which they are expressed, depend only on the *number* of the indices and not on what they are. Thus, suppressing a common factor $1/a^3$ and measuring length in units of the radius (a) of the sphere, we have

$$A_i = A_1 = \tfrac{2}{3}\,, \quad A_{ij} = A_{11} = \tfrac{2}{5}\,, \quad A_{ijk} = A_{111} = \tfrac{2}{7} \tag{21}$$

and

$$B_i = B_1 = \tfrac{4}{3}\,, \quad B_{ij} = B_{11} = \tfrac{4}{15}\,, \quad B_{ijk} = B_{111} = \tfrac{4}{35} . \tag{22}$$

The subsidiary symbols $A_{ij;k}$ and $B_{ij;k}$ (Paper II, eq. [123]) have the values

$$A_{ij;k} = A_{11} + A_{111} = \tfrac{24}{35} \quad \text{and} \quad B_{ij;k} = B_{11} + B_{111} = \tfrac{8}{21} . \tag{23}$$

With the coefficients listed in Paper III (Tables 1 and 3), we now find

$$\delta S_{122} = 6(B_{11} + B_{111})(V_{122} - \tfrac{1}{3}V_{111}) = \tfrac{16}{7}(V_{122} - \tfrac{1}{3}V_{111}) \tag{24}$$

and

$$\delta S_{123} = 6(B_{11} + B_{111}) = \tfrac{16}{7}V_{123}\,, \tag{25}$$

where a common factor $\pi G\rho$ has been suppressed.

In view of equations (24) and (25), the six equations of group (*a*) and equation (15) of group (*c*) all lead to the same root,

$$\sigma^2 = \tfrac{16}{7}\pi G\rho\,, \tag{26}$$

where the factor $\pi G\rho$ has been restored. The root (26) is of multiplicity 7.

The characteristic root given by equation (26) coincides with the Kelvin frequency for $l = 3$; its multiplicity 7 is in agreement with this identification.

In the equations of the remaining groups (*b*) and (*d*) we must set

$$V_{122} = V_{133} = \tfrac{1}{3}V_{111}\,, \qquad \text{etc.,} \tag{27}$$

to exclude the root (26) and not to be inconsistent with the equations of group (*a*).

In view of the equalities (27), we now have (cf. Paper III, Table 2)

$$-2\delta\mathfrak{W}_{12;2} = -2\delta\mathfrak{W}_{13;3} = \tfrac{1}{3}(2B_{11} + 7B_{111})V_{111} = \tfrac{4}{9}V_{111}\,. \tag{28}$$

The equations of group (*b*) and the two equations (16) of group (*c*) lead to the root

$$\sigma^2 = 0\,, \tag{29}$$

with multiplicity 5.

Finally, considering the equations of the group (*d*), we first note (cf. Paper II, Table 1)

$$\begin{aligned} -2\delta\mathfrak{W}_{11;1} &= [2(B_{11} + 2B_{111}) - A_{11;1}]V_{111} + (2B_{111} - A_{11;1})(V_{122} + V_{133}) \\ &= [2(\tfrac{4}{15} + \tfrac{8}{35}) - \tfrac{24}{35} + \tfrac{2}{3}(\tfrac{8}{35} - \tfrac{24}{35})]V_{111} = 0\,. \end{aligned} \tag{30}$$

The equations to be considered are, therefore,

$$\tfrac{1}{3}\sigma^2 V_{111} = -2\delta\Pi_1 \tag{31}$$

and

$$\sigma^2(V_{1;22} + V_{1;33}) = \tfrac{8}{9}V_{111}\,, \tag{32}$$ [2]

where we have made use of equation (28).

It remains to determine $\delta\Pi_1$. Now the pressure distribution in a homogeneous sphere is given by

$$p = \tfrac{2}{3}\rho(1 - r^2)\,, \tag{33}$$

where the same factor $\pi G\rho$ has been suppressed and r is measured in the unit a. For this pressure distribution, equation (10) gives

$$\delta\Pi_1 = -\tfrac{4}{3}(\gamma - 1)\int_V \rho\xi_j x_j x_1 d\boldsymbol{x} - \tfrac{2}{3}\gamma\int_V \rho\xi_1 x_j x_j d\boldsymbol{x} + \tfrac{2}{3}\int_V \rho\xi_1 d\boldsymbol{x}\,. \tag{34}$$

The last term vanishes by the condition requiring the stationariness of the center of mass (cf. Paper II, Sec. II); and the remaining terms can be combined to give

$$\begin{aligned} \delta\Pi_1 &= -\tfrac{4}{3}(\gamma - 1)V_{j;j1} - \tfrac{2}{3}\gamma V_{1;jj} \\ &= -\tfrac{2}{3}(\gamma - 1)V_{1jj} - \tfrac{2}{3}(\tfrac{1}{3}V_{111} + V_{1;22} + V_{1;33}) \\ &= -\tfrac{2}{9}(5\gamma - 4)V_{111} - \tfrac{2}{3}(V_{1;22} + V_{1;33})\,. \end{aligned} \tag{35}$$

Inserting this value of $\delta\Pi_1$ in equation (31), we obtain

$$\tfrac{1}{3}\sigma^2 V_{111} = \tfrac{4}{9}(5\gamma - 4)V_{111} + \tfrac{4}{3}(V_{1;22} + V_{1;33})\,. \tag{36}$$

Equations (32) and (36) now lead to the characteristic equation

$$\sigma^4 - \tfrac{4}{3}(5\gamma - 4)\sigma^2 - \tfrac{32}{9} = 0\,. \tag{37}$$

The roots of this equation are

$$\sigma^2 = \tfrac{2}{3}(5\gamma - 4) \pm \tfrac{2}{3}[(5\gamma - 4)^2 + 8]^{1/2}\,; \tag{38}$$

and each of these roots is of multiplicity 3. The roots given by equation (38) agree with those found by Pekeris for $l = 1$.

[2] Actually, we can set $V_{1;22} = V_{1;33}$ in this and in the subsequent equations for consistency with the equations of group (*b*).

One of the two roots given by equation (38) is clearly negative; and this implies instability. We have thus derived the convective instability of the homogeneous sphere for modes of oscillation belonging to $l = 1$.

IV. THE SUPERPOTENTIALS AND THE TENSOR POTENTIALS OF A SPHERICAL DISTRIBUTION OF MASS

The quantities $\delta\mathfrak{W}_{ij;k}$ which occur in the virial equations of Section II are, as we have seen, expressible as integrals over the tensor potentials $\mathfrak{B}_{ij}$ and $\mathfrak{D}_{ij;k}$. In this section we shall assemble the necessary formulae.

The superpotentials χ and Φ are, in case of spherical symmetry, governed by the differential equations

$$\chi'' + \frac{2}{r}\chi' = -2\mathfrak{B} \tag{39}$$

and

$$\Phi'' + \frac{2}{r}\Phi' = -4\chi, \tag{40}$$

where the primes denote differentiations with respect to r. Once χ and Φ have been determined as solutions of these differential equations (satisfying the appropriate boundary conditions), the required tensor potentials follow from the equations (Chandrasekhar and Lebovitz 1962*b*, eqs. [6] and [33])

$$\mathfrak{B}_{ij} = \mathfrak{B}\,\delta_{ij} + \frac{\partial^2\chi}{\partial x_i \partial x_j} \tag{41}$$

and

$$\mathfrak{D}_{ij;k} = \frac{\partial^3\Phi}{\partial x_i \partial x_j \partial x_k} + \frac{\chi'}{r}(x_i\delta_{jk} + x_j\delta_{ki} + x_k\delta_{ij}) + x_k\mathfrak{B}_{ij}. \tag{42}$$

Inserting the foregoing expressions in equation (7), we obtain

$$\begin{aligned} -2\,\delta\mathfrak{W}_{ij;k} = {} & 2\int_V \rho\mathfrak{B}_{ij}\xi_k dx + 2\int_V \rho\xi_l \frac{\partial\mathfrak{B}_{ij}}{\partial x_l} x_k dx + \int_V \rho\xi_l \frac{\partial^4\Phi}{\partial x_l \partial x_i \partial x_j \partial x_k} dx \\ & - \int_V \rho\xi_l x_l (x_i\delta_{jk} + x_j\delta_{ki} + x_k\delta_{ij})\left(2\frac{\mathfrak{B}}{r^2} + 3\frac{\chi'}{r^3}\right) dx \\ & + (\delta_{li}\delta_{jk} + \delta_{lj}\delta_{ki} + \delta_{lk}\delta_{ij})\int_V \rho\xi_l \frac{\chi'}{r} dx. \end{aligned} \tag{43}$$

The derivatives of χ and Φ which occur in equations (41)–(43) can all be expressed, in view of the spherical symmetry of these functions, in terms of the derivatives with respect to r; and in the resulting expressions all derivatives of χ and Φ of orders second and higher can be reduced to the first by successive use of the defining equations (39) and (40). We thus find

$$\frac{\partial^2\chi}{\partial x_i \partial x_j} = \frac{\chi'}{r}\delta_{ij} + \frac{B}{r^2}x_i x_j, \tag{44}$$

$$\frac{\partial^3\chi}{\partial x_i \partial x_j \partial x_k} = \frac{B}{r^2}(x_i\delta_{jk} + x_j\delta_{ki} + x_k\delta_{ij}) + \frac{F}{r^2}x_i x_j x_k, \tag{45}$$

$$\frac{\partial^3\Phi}{\partial x_i \partial x_j \partial x_k} = C(x_i\delta_{jk} + x_j\delta_{ki} + x_k\delta_{ij}) + \frac{D}{r^2}x_i x_j x_k, \tag{46}$$

$$\frac{\partial^4 \Phi}{\partial x_i \partial x_j \partial x_k \partial x_l} = C(\delta_{li}\delta_{jk} + \delta_{lj}\delta_{ki} + \delta_{lk}\delta_{ij}) + \frac{E}{r^4} x_i x_j x_k x_l$$
$$+ \frac{D}{r^2}(x_l x_i \delta_{jk} + x_l x_j \delta_{ki} + x_l x_k \delta_{ij} + x_j x_k \delta_{il} + x_k x_i \delta_{jl} + x_i x_j \delta_{kl}), \tag{47}$$

where we have introduced the abbreviations

$$A = \mathfrak{V} + \frac{\chi'}{r}, \qquad D = -\left(5C + 4\frac{\chi'}{r}\right),$$
$$B = -\left(2\mathfrak{V} + 3\frac{\chi'}{r}\right), \qquad E = 8\mathfrak{V} + 35C + 40\frac{\chi'}{r}, \tag{48}$$
$$C = -\frac{1}{r^2}\left(4\chi + 3\frac{\Phi'}{r}\right), \qquad F = -\frac{1}{r^2}(5B + 2\mathfrak{V}'r).$$

In terms of these abbreviations, we can now write

$$\mathfrak{V}_{ij} = A\,\delta_{ij} + \frac{B}{r^2} x_i x_j, \tag{49}$$

and

$$\frac{\partial \mathfrak{V}_{ij}}{\partial x_l} = \frac{\mathfrak{V}'}{r} x_l \delta_{ij} + \frac{B}{r^2}(x_i \delta_{jl} + x_j \delta_{li} + x_l \delta_{ij}) + \frac{F}{r^2} x_i x_j x_l. \tag{50}$$

Returning to equation (43) and inserting for the various quantities in accordance with equations (47), (49), and (50), we obtain, after some further regrouping of the terms,

$$-2\delta\mathfrak{W}_{ij;k} = 2\delta_{ij}\int_V \rho A\,\xi_k dx + 2\delta_{ij}\int_V \rho\left(\frac{\mathfrak{V}'}{r} + \frac{B}{r^2}\right)\xi_l x_l x_k dx$$
$$+ \int_V \rho\left(\frac{2B}{r^2} + \frac{D}{r^2}\right)(\xi_i x_j x_k + \xi_j x_k x_i + \xi_k x_i x_j)\,dx$$
$$+ \int_V \rho\left(\frac{B}{r^2} + \frac{D}{r^2}\right)\xi_l x_l(x_i \delta_{jk} + x_j \delta_{ki} + x_k \delta_{ij})\,dx \tag{51}$$
$$+ \int_V \rho\left(C + \frac{\chi'}{r}\right)(\xi_k \delta_{ij} + \xi_i \delta_{jk} + \xi_j \delta_{ik})\,dx$$
$$+ \int_V \rho\left(2\frac{F}{r^2} + \frac{E}{r^4}\right)\xi_l x_l x_i x_j x_k dx.$$

V. THE VIRIAL EQUATIONS FOR AN ASSUMED LAGRANGIAN DISPLACEMENT OF THE FORM $\xi_i = L_{i;jk}x_j x_k + L_i$

In using the virial equations of Section II in conjunction with the formulae of Section IV, we shall suppose that it will suffice to consider a Lagrangian displacement of the form

$$\xi_i = L_{i;jk} x_j x_k + L_i, \tag{52}$$

where $L_{i;jk}$ and L_i represent a total of twenty-one unspecified constants. The form (52) for $\boldsymbol{\xi}$ is, in the present context of the third-order virial equations, the proper generalization of the form

$$\xi_i = L_{i;j} x_j \tag{53}$$

assumed in an earlier paper (Chandrasekhar and Lebovitz 1962*a*) in the context of the second-order virial equations.[3] And the justification for assuming the form (52) is that it conforms to the *exact* solution for a homogeneous sphere and that we may, therefore, expect it to be adequate as a first "trial function" for configurations with not excessive degrees of central concentrations. We shall verify that this expectation is borne out by applications to the polytropes.

As has been pointed out in Paper II (Sec. II), the virial equations of the third order must be considered together with the equations of the first order which require

$$V_i = \int_V \rho \xi_i d\boldsymbol{x} = 0 \tag{54}$$

as a condition for the stationariness of the center of mass. For the form of the Lagrangian displacement assumed, condition (54) gives

$$\tfrac{1}{3} L_{i;jj} \int_V \rho r^2 d\boldsymbol{x} + L_i \int_V \rho d\boldsymbol{x} = 0 \,, \tag{55}$$

or

$$L_i = - \frac{\int_V \rho r^2 d\boldsymbol{x}}{3 \int_V \rho d\boldsymbol{x}} L_{i;jj} \,. \tag{56}$$

Relations (56) reduce the number of unspecified constants in (52) to eighteen: the same as the number of the virial equations of the third order.

a) *The Kelvin Modes*

We shall first consider equations (12) and (15) in groups (*a*) and (*c*) (in Sec. II). We find, by direct calculation,

$$V_{122} - \tfrac{1}{3} V_{111} = \tfrac{2}{15} (L_{122} - \tfrac{1}{3} L_{111}) \int_V \rho r^4 d\boldsymbol{x} \tag{57}$$

and

$$V_{123} = \tfrac{2}{15} L_{123} \int_V \rho r^4 d\boldsymbol{x} \,, \tag{58}$$

where L_{ijk} without the semicolon (like the symmetrized virial V_{ijk}) is defined in the manner

$$L_{ijk} = L_{i;jk} + L_{j;ki} + L_{k;ij} \,. \tag{59}$$

[3] It should be noted that greater generality is not achieved by an assumption of the form

$$\xi_i = L_{i;jk} x_j x_k + L_{i;j} x_j + L_i \,,$$

since the terms in $L_{i;j}$ will vanish identically in all the equations provided by the third-order virial equations. On the other hand, if one went to the virial equations of the fourth order, the form for the Lagrangian displacement one would assume is

$$\xi_i = L_{i;jkl} x_j x_k x_l + L_{i;j} x_j \,;$$

and the second- and the fourth-order virial equations will together provide a number of equations equal to the number of the constants $L_{i;jkl}$ and $L_{i;j}$ even as the first- and the third-order virial equations provide a number of equations equal to the number of the constants $L_{i;jk}$ and L_i. And, in general, the consideration of the virial equations of increasing (even or odd) orders corresponds to solving the exact problem with "trial functions" for ξ_i which are polynomials in the co-ordinates of increasing (odd or even) orders.

The reduction of the expressions for the required $\delta\mathfrak{W}_{ij;k}$'s and δS_{ijk}'s, in accordance with equation (51) for the assumed form of $\boldsymbol{\xi}$, to integrals over functions only of r is long but straightforward. We shall omit all the details of the reduction and quote only the final results.

We find

$$\delta S_{122} = -\tfrac{8}{35}(L_{122} - \tfrac{1}{3}L_{111}) \int_V \rho \mathfrak{V}' r^3 dx \tag{60}$$

and

$$\delta S_{123} = -\tfrac{8}{35} L_{123} \int_V \rho \mathfrak{V}' r^3 dx \,. \tag{61}$$

From equations (57), (58), (60), and (61) it is apparent that the equations of group (a) and equation (15) all lead to the same root:

$$\sigma^2 = -\frac{12}{7} \frac{\int_V \rho \mathfrak{V}' r^3 dx}{\int_V \rho r^4 dx} \,. \tag{62}$$

This root is of multiplicity 7 and belongs to $l = 3$; it represents the analogue of the corresponding Kelvin mode for the homogeneous sphere.

The integral in the numerator of the expression for σ^2 can be transformed in this manner:

$$\int_V \rho \frac{d\mathfrak{V}}{dr} r^3 dx = 4\pi \int \frac{dp}{dr} r^5 dr = -20\pi \int p r^4 dr = -5 \int_V p r^2 dx \,. \tag{63}$$

We may accordingly write

$$\sigma^2 = \frac{60}{7} \frac{\int_V p r^2 dx}{\int_V \rho r^4 dx} \,. \tag{64}$$

For polytropes, we find, on expressing r, ρ, and p in terms of the usual Emden variables (cf. Chandrasekhar and Lebovitz 1962c, eq. [8]), that

$$\frac{\sigma^2}{4\pi G \rho_c} = \frac{60}{7(n+1)} \frac{\int_0^{\xi_1} \theta^{n+1} \xi^4 d\xi}{\int_0^{\xi_1} \theta^n \xi^6 d\xi} \,. \tag{65}$$

The integrals occurring in this expression for σ^2 are among those reduced and tabulated in the appendix (Table 1). And using the results given in the appendix, we obtain the values listed in the accompanying tabulation (eq. [66]).

n	$\sigma^2/(4\pi G\rho_c)$	n	$\sigma^2/(4\pi G\rho_c)$
0	0 57143	2 0	0 16609
1 0	0 29866	3 0	0 084302
1 5	0 22328	3 5	0 054834

(66)

b) The Characteristic Equation for the Modes of Oscillation Belonging to $l = 1$

We turn now to a consideration of equations (19) and (20). In considering them in the framework of the assumption (52), we must set

$$L_{122} = L_{133} = \tfrac{1}{3}L_{111}\,, \text{ etc.}\,, \tag{67}$$

to be consistent with the equations of group (a) and exclude the root (62).

We find, by direct calculation, that

$$V_{1;11} = \tfrac{1}{5}(L_{1;11} + \tfrac{1}{3}M_1)\int_V \rho r^4 dx + \tfrac{1}{3}L_1\int_V \rho r^2 dx \tag{68}$$

and

$$V_{1;22} + V_{1;33} = \tfrac{2}{15}(L_{1;11} + 2M_1)\int_V \rho r^4 dx + \tfrac{2}{3}L_1\int_V \rho r^2 dx\,, \tag{69}$$

where we have introduced the abbreviation

$$M_1 = L_{1;22} + L_{1;33}\,. \tag{70}$$ [4]

From equation (51) we similarly find (after some lengthy reductions) that, for the assumed form for ξ (with the additional restrictions [67]),

$$\begin{aligned} -2\delta\mathfrak{W}_{12;2} - 2\delta\mathfrak{W}_{13;3} = &-\tfrac{4}{15}\int_V \rho(L_{1;11}r^2 + L_1)(\mathfrak{B} + 2\mathfrak{B}'r)dx \\ &+ \tfrac{2}{15}(2L_{1;11} - M_1)\int_V \rho\left(4\mathfrak{B} + 5\frac{\chi'}{r}\right)r^2 dx \end{aligned} \tag{71}$$

and

$$\begin{aligned} -2\delta\mathfrak{W}_{11;1} = &\tfrac{2}{15}\int_V \rho(L_{1;11}r^2 + L_1)(2\mathfrak{B} - \mathfrak{B}'r)dx \\ &+ \tfrac{1}{15}(2L_{1;11} - M_1)\int_V \rho\left(2\mathfrak{B} + 5\frac{\chi'}{r}\right)r^2 dx\,. \end{aligned} \tag{72}$$

The coefficients of L_1 in equations (71) and (72) can be simplified by making use of the relations

$$\mathfrak{W} = -\tfrac{1}{2}\int_V \rho\mathfrak{B}dx = -3\int_V p\,dx \tag{73}$$

and

$$\int_V \rho\frac{d\mathfrak{B}}{dr}r\,dx = 4\pi\int_V \frac{dp}{dr}r^3 dr = -12\pi\int_V p r^2 dr = \mathfrak{W}\,. \tag{74}$$

Thus,

$$\int_V \rho(\mathfrak{B} + 2\mathfrak{B}'r)dx = 0\,, \quad \text{and} \quad \tfrac{2}{15}\int_V \rho(2\mathfrak{B} - \mathfrak{B}'r)dx = 2\int_V p\,dx\,; \tag{75}$$

and we can write

$$\begin{aligned} -2\delta\mathfrak{W}_{12;2} - 2\delta\mathfrak{W}_{13;3} = &\tfrac{4}{15}L_{1;11}\int_V \rho\left(3\mathfrak{B} - 2\mathfrak{B}'r + 5\frac{\chi'}{r}\right)r^2 dx \\ &-\tfrac{2}{15}M_1\int_V \rho\left(4\mathfrak{B} + 5\frac{\chi'}{r}\right)r^2 dx \end{aligned} \tag{76}$$

[4] In this equation we can set $L_{1;22} = L_{1;33}$ to be consistent with eq. (14) and exclude the zero root.

and

$$-2\delta\mathfrak{W}_{11;1}=\tfrac{2}{15}L_{1;11}\int_V\rho\left(4\mathfrak{V}-\mathfrak{V}'r+5\frac{\chi'}{r}\right)r^2dx$$
$$-\tfrac{1}{15}M_1\int_V\rho\left(2\mathfrak{V}+5\frac{\chi'}{r}\right)r^2dx+2L_1\int_V p\,dx\,. \tag{77}$$

It remains to determine $\delta\Pi_1$ in accordance with equation (9). For the assumed form of ξ (with the additional restrictions [67]), we find

$$\begin{aligned}\delta\Pi_1&=-\tfrac{2}{3}(\gamma-1)L_{j;j1}\int_V p\,r^2dx+\tfrac{1}{3}L_{1;jj}\int_V p\,r^2dx+L_1\int_V p\,dx\\&=[-\tfrac{1}{3}(4\gamma-5)L_{1;11}+\tfrac{1}{3}\gamma M_1]\int_V p\,r^2dx+L_1\int_V p\,dx\,.\end{aligned}\tag{78}$$

Now substituting from equations (68), (69), (76), (77), and (78) in equations (19) and (20), we obtain

$$\begin{aligned}&-\sigma^2\left[\tfrac{1}{5}(L_{1;11}+\tfrac{1}{3}M_1)\int_V\rho r^4dx+\tfrac{1}{3}L_1\int_V\rho r^2dx\right]\\&+\tfrac{2}{15}L_{1;11}\int_V\rho\left(4\mathfrak{V}-\mathfrak{V}'r+5\frac{\chi'}{r}\right)r^2dx-\tfrac{1}{15}M_1\int_V\rho\left(2\mathfrak{V}+5\frac{\chi'}{r}\right)r^2dx\\&=\tfrac{2}{3}[-(4\gamma-5)L_{1;11}+\gamma M_1]\int_V p\,r^2dx\end{aligned}\tag{79}$$

and

$$\begin{aligned}&-\sigma^2\left[\tfrac{2}{15}(L_{1;11}+2M_1)\int_V\rho r^4dx+\tfrac{2}{3}L_1\int_V\rho r^2dx\right]\\&+\tfrac{4}{15}L_{1;11}\int_V\rho\left(3\mathfrak{V}-2\mathfrak{V}'r+5\frac{\chi'}{r}\right)r^2dx-\tfrac{2}{15}M_1\int_V\rho\left(4\mathfrak{V}+5\frac{\chi'}{r}\right)r^2dx=0\,.\end{aligned}\tag{80}$$

We notice that in these final equations the "super-superpotential" Φ does not appear: the terms in it canceled at an earlier stage.

Equations (79) and (80) must be supplemented by the further equation (cf. eq. [56])

$$L_1=-\frac{\int_V\rho r^2dx}{3\int_V\rho\,dx}(L_{1;11}+M_1)\,. \tag{81}$$

With the aid of this last equation, L_1 can be eliminated from equations (79) and (80). After the elimination of L_1, we shall be left with a system of two linear homogeneous equations for $L_{1;11}$ and M_1; and the requirement that the determinant of the system vanish will lead to the desired characteristic equation of σ^2. It is to be particularly noted that the coefficients of the characteristic equation for σ^2 are integrals which involve, in addition to the usual quantities, the superpotential χ explicitly.

VI. THE CONVECTIVE INSTABILITY OF THE POLYTROPES FOR $\gamma<1+1/n$

The characteristic equation for σ^2 which follows from equations (79)–(81) has been obtained in an explicit form for a few polytropes. Before we write down the characteristic equations which were obtained, we may note that when r, ρ, p, $\mathfrak{V}$, and χ are expressed in terms of the Emden variables ξ and θ, equations (79)–(81) take the forms

$$-\sigma^2\left[(3L_{1;11}+M_1)\int_0^{\xi_1}\theta^n\xi^6 d\xi+5L_1\int_0^{\xi_1}\theta^n\xi^4 d\xi\right]$$
$$+2L_{1;11}\int_0^{\xi_1}\theta^n\left[4(\theta+c_0)-\theta'\xi+5\frac{\chi'}{\xi}\right]\xi^4 d\xi-M_1\int_0^{\xi_1}\theta^n\left[2(\theta+c_0)+5\frac{\chi'}{\xi}\right]\xi^4 d\xi \tag{82}$$
$$=\frac{10}{n+1}[-(4\gamma-5)L_{1;11}+\gamma M_1]\int_0^{\xi_1}\theta^{n+1}\xi^4 d\xi,$$

$$-\sigma^2\left[(L_{1;11}+2M_1)\int_0^{\xi_1}\theta^n\xi^6 d\xi+5L_1\int_0^{\xi_1}\theta^n\xi^4 d\xi\right] \tag{83}$$
$$+2L_{1;11}\int_0^{\xi_1}\theta^n\left[3(\theta+c_0)-2\theta'\xi+5\frac{\chi'}{\xi}\right]\xi^4 d\xi$$
$$-M_1\int_0^{\xi_1}\theta^n\left[4(\theta+c_0)+5\frac{\chi'}{\xi}\right]\xi^4 d\xi=0,$$

and

$$L_1=-\frac{\int_0^{\xi_1}\theta^n\xi^4 d\xi}{3\xi_1^2|\theta'(\xi_1)|}(L_{1;11}+M_1), \tag{84}$$

where a prime denotes differentiation with respect to ξ, ξ_1 is the first zero of the Lane-Emden function, and σ^2 is measured in the unit $4\pi G\rho_c$; also in the chosen units (cf. Chandrasekhar and Lebovitz 1962*c*)

$$\mathfrak{V}=\theta+c_0, \quad \text{where} \quad c_0=-\xi_1\left(\frac{d\theta}{d\xi}\right)_{\xi_1}, \tag{85}$$

and

$$\chi''+\frac{2}{\xi}\chi'=-2(\theta+c_0). \tag{86}$$

The various integrals occurring in equations (82)–(84) are reduced and tabulated in the appendix (Table 1). And we finally obtain

$$\begin{aligned}
&\sigma^4-0.76513\,(\gamma-0.91079)\,\sigma^2+0.029036\,\,(\gamma-1.9955)=0\quad(n=1.0),\\
&\sigma^4-0.55037\,(\gamma-0.94660)\,\sigma^2+0.018768\,\,(\gamma-1.6609)=0\quad(n=1.5),\\
&\sigma^4-0.39698\,(\gamma-0.97624)\,\sigma^2+0.011071\,\,(\gamma-1.4931)=0\quad(n=2.0),\\
&\sigma^4-0.19192\,(\gamma-1.02493)\,\sigma^2+0.0029138\,(\gamma-1.3258)=0\quad(n=3.0),\\
&\sigma^4-0.12241\,(\gamma-1.04520)\,\sigma^2+0.0012184\,(\gamma-1.2784)=0\quad(n=3.5).
\end{aligned} \tag{87}$$

From these equations it is apparent that the polytropes are unstable when

$$\begin{aligned}
&\gamma<1.995 \text{ for } n=1.0, \text{ whereas } 1+1/n=2.000,\\
&\gamma<1.661 \text{ for } n=1.5, \text{ whereas } 1+1/n=1.667,\\
&\gamma<1.493 \text{ for } n=2.0, \text{ whereas } 1+1/n=1.500,\\
&\gamma<1.326 \text{ for } n=3.0, \text{ whereas } 1+1/n=1.333,\\
&\gamma<1.278 \text{ for } n=3.5, \text{ whereas } 1+1/n=1.286.
\end{aligned} \tag{88}$$

The accuracy with which the critical values of γ are predicted is remarkable: even for $n = 3.5$ (when $\rho_c = 153\bar{\rho}$), the error does not exceed 0.7 per cent; and this is indeed gratifying.

VII. CONCLUDING REMARKS

As we have remarked in the introductory section, the fact that the convective instability of polytropes asserts itself already for a mode of oscillation belonging to $l = 1$ should not cause any surprise: it is very much to be expected. However, in cases where, as in Cowling's point-source model for a star, the effective polytropic index decreases from a relatively high value in the exterior to a subcritical value in the interior, the circumstances are, of course, different. But even in these cases it would seem that the most relevant modes are again those which belong to spherical harmonics of low orders (including $l = 1$) and radial functions having one or more nodes. If this presumption is valid, then the modes of oscillation belonging to really high values of l are not as basic to the theory of convective instability as one might be inclined to suppose.

A further aspect of the analysis of the preceding sections that is noteworthy is the following. Apart from the fact that the virial equations of the third order appear to provide an adequate basis for treating non-radial oscillations belonging to $l = 1$ and 3, an important feature of the equations is the role of the superpotential χ. By virtue of its relationship to the tensor potential $\mathfrak{B}_{ij}$, the superpotential χ expresses, even more than the Newtonian potential $\mathfrak{B}$ does, the co-operative aspects of the gravitational interaction between the different parts of a body. The dependence of the characteristic frequencies of non-radial oscillations on the superpotential χ may be considered, then, as an indication of their dependence in some significant way on a delicate balance in the co-operative aspects of the gravitational interactions. It is indeed clear from the point of view of the virial equations that the higher modes of non-radial oscillations will depend on the superpotentials of still higher orders such as Φ, etc. It would not, therefore, seem that, when dealing with compressible masses, the usual assumption (derived from a knowledge of the reactions of incompressible masses) that the variations in the gravitational potential can be neglected in the treatment of *all* manner of modes of oscillation (belonging to a given l) is necessarily a valid one. An examination of the modes of oscillation belonging to $l = 4$ from the point of view of the virial equations of the fourth order may contribute to a clarification of these issues; and this examination is now under consideration.

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APPENDIX

The Reduction of Certain Integrals in the Theory of Polytropes

Equations (82)–(84) involve the following integrals:

$$\text{I.}\ \int_0^{\xi_1} \theta^n \xi^4 d\xi\,; \qquad \text{II.}\ \int_0^{\xi_1} \theta^n \xi^6 d\xi\,; \qquad \text{III.}\ \int_0^{\xi_1} \theta^{n+1} \xi^4 d\xi\,;$$
$$\text{IV.}\ \int_0^{\xi_1} \theta^n \theta' \xi^5 d\xi\,; \quad \text{and} \quad \text{V.}\ \int_0^{\xi_1} \theta^n \chi' \xi^3 d\xi\,. \tag{A.1}$$

We shall reduce these integrals to simpler forms:

I. $$\int_0^{\xi_1} \theta^n \xi^4 d\xi = -\xi_1^4 \theta_1' - 6\int_0^{\xi_1} \theta \xi^2 d\xi , \tag{A.2}$$

where a subscript 1 to a quantity indicates that its value at ξ_1 (the first zero of θ) is meant. The reduction (A. 2) was effected in an earlier paper (Chandrasekhar and Lebovitz 1962*c*, Appendix I). In an exactly similar way we find

II. $$\int_0^{\xi_1} \theta^n \xi^6 d\xi = -\xi_1^6 \theta_1' + 4\int_0^{\xi_1} \theta' \xi^5 d\xi ; \tag{A.3}$$

III. $$\int_0^{\xi_1} \theta^{n+1} \xi^4 d\xi = \frac{n+1}{n+11}\left[\xi_1^5(\theta_1')^2 - 3\int_0^{\xi_1} \theta^2 \xi^2 d\xi\right]. \tag{A.4}$$

Equation (A.4) is a special case of the following general formula due to Milne (1929, eq. [53]):

$$\begin{aligned}[n(s-3)+3s-1]\int_0^{\xi_1} \theta^{n+1}\xi^s d\xi = (n+1)\Big[\xi_1^{s+1}(\theta_1')^2 \\ -\tfrac{1}{2}(s-3)(s-2)(s-1)\int_0^{\xi_1}\xi^{s-2}\theta^2 d\xi\Big]\ (s>1) ;\end{aligned} \tag{A.5}$$

IV. $$\int_0^{\xi_1} \theta^n \theta' \xi^5 d\xi = -\frac{5}{n+1}\int_0^{\xi_1} \theta^{n+1}\xi^4 d\xi . \tag{A.6}$$

This is equivalent to the general relation (63) established in the text.

V. $$\begin{aligned}\int_0^{\xi_1} \theta^n \frac{d\chi}{d\xi}\xi^3 d\xi &= -\int_0^{\xi_1} \xi \frac{d\chi}{d\xi}\frac{d}{d\xi}\left(\xi^2 \frac{d\theta}{d\xi}\right) d\xi \\ &= -\xi_1^3 \chi_1' \theta_1' + \int_0^{\xi_1} \xi^3 \frac{d\theta}{d\xi}\left(\frac{d^2\chi}{d\xi^2} + \frac{1}{\xi}\frac{d\chi}{d\xi}\right) d\xi \\ &= -\xi_1^3 \chi_1' \theta_1' - \int_0^{\xi_1} \xi^3 \frac{d\theta}{d\xi}\left[2(\theta+c_0) + \frac{1}{\xi}\frac{d\chi}{d\xi}\right] d\xi \\ &= -\xi_1^3 \chi_1' \theta_1' - \int_0^{\xi_1} \xi^3 \frac{d}{d\xi}(\theta+c_0)^2 d\xi - \int_0^{\xi_1} \frac{d\theta}{d\xi}\left(\xi^2 \frac{d\chi}{d\xi}\right) d\xi \\ &= -\xi_1^3 \chi_1' \theta_1' - \xi_1^3 c_0^2 + 3\int_0^{\xi_1} \xi^2 (\theta+c_0)^2 d\xi - 2\int_0^{\xi_1} \theta(\theta+c_0)\xi^2 d\xi \\ &= -\xi_1^3 \chi_1' \theta_1' + 4c_0\int_0^{\xi_1} \theta\xi^2 d\xi + \int_0^{\xi_1} \theta^2 \xi^2 d\xi ,\end{aligned} \tag{A.7}$$

where, in the reductions, use has been made (at two different times) of the equation (cf. eq [86])

$$\chi'' + \frac{2}{\xi}\chi' = -2(\theta + c_0) \qquad (c_0 = -\xi_1 \theta_1') \tag{A.8}$$

governing χ.

Thus the five integrals listed in (A. 1) can be expressed in terms of the following three:

$$\int_0^{\xi_1} \theta \xi^2 d\xi , \quad \int_0^{\xi_1} \theta' \xi^5 d\xi , \quad \text{and} \quad \int_0^{\xi_1} \theta^2 \xi^2 d\xi . \tag{A.9}$$

Of these three integrals, the first has already been evaluated in a different connection (Chandrasekhar and Lebovitz 1962*c*, Appendix I and Table 5). The remaining two integrals have now been evaluated; and their values, along with the others are listed in Table 1.

TABLE 1

A TABLE OF INTEGRALS

	$n=1\ 0$*	$n=1\ 5$	$n=2\ 0$	$n=3\ 0$	$n=3\ 5$
$\int\theta\xi^2 d\xi$	3 14159	4 18545	5 84540	14 1915	26 6943
$-\int\theta'\xi^5 d\xi$	60 7836	104 6806	198 4622	1114 165	3868.37
$\int\theta^2\xi^2 d\xi$	1 57080	1 91883	2 41105	4 32705	6 45430
$\int\theta^n\xi^4 d\xi$	12 15672	11 1197	10 6110	10 8516	11 7455
$\int\theta^n\xi^6 d\xi$	62 8853	64 9770	71 7372	109 7484	158 689
$\int\theta^{n+1}\xi^4 d\xi$	4 38231	4 23148	4 17018	4 31761	4 56828
$-\int\theta^n\chi'\xi^3 d\xi$	12 8169	9 8058	7 9827	6 0957	5 6829

* The values of the integrals down this column are π, $5\pi^3 - 30\pi$, $\pi/2$, $\pi^3 - 6\pi$, $\pi^5 - 20\pi^3 + 120\pi$, $(\pi^3 - \frac{3}{2}\pi)/6$, and $\frac{2}{3}\pi^3 - 2\ 5\pi$, respectively

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