THE POINTS OF BIFURCATION ALONG THE MACLAURIN, THE JACOBI, AND THE JEANS SEQUENCES

S. Chandrasekhar
University of Chicago
Received January 23, 1963

ABSTRACT

The role which the second- and the third-order virial equations governing equilibrium can play in isolating points of neutral stability along equilibrium sequences is discussed and clarified. It is shown that a necessary condition for the occurrence of a neutral point is that a non-trivial Lagrangian displacement exists for which the first variations of *all* of the integral relations (five in the second order and fifteen in the third order), provided by the virial equations, vanish. By using this condition, it is possible, for example, to isolate the point of bifurcation along the Jacobian sequence without *any* prior specification of the nature of the sequence which follows bifurcation. As further illustrations of the method, the known points of neutral stability along the Maclaurin and the Jeans sequences are also derived.

I. INTRODUCTION

In an earlier paper (Chandrasekhar 1962; this paper[1] will be referred to hereafter as "Paper I") the occurrence of points of bifurcation along the Maclaurin and the Jacobian sequences was deduced from certain general integral relations provided by the second- and third-order virial equations. The point of bifurcation along the Jacobian sequence was deduced, for example, from the required invariance of an integral property (provided by the third-order virial equations) to an infinitesimal Lagrangian displacement which deforms the Jacobi ellipsoid into a pear-shaped configuration. However, the preoccupation with this matter of the exhibition and the isolation of the point of bifurcation along the Jacobian sequence was allowed to obscure (and to some extent confuse) the entirely general nature of the relations provided by the virial theorem and its extensions and the role they can be called upon to play in distinguishing equilibrium configurations which are neutrally stable along a given sequence. In this paper an attempt will be made to clarify this entire matter. As an instance of the generality achieved in the interpretation, it may be stated here that the point of bifurcation along the Jacobian sequence can be isolated by four (apparently) independent relations which do not presuppose any knowledge of the kind of equilibrium configurations which succeed the Jacobi ellipsoids beyond the point of bifurcation.

II. SOME REMARKS ON THE TERMINOLOGY

In view of the diverse (and sometimes conflicting) terminology which is current in the subject, it may be useful to clarify the different circumstances which must be distinguished and the particular terminology which will be adopted in the present discussion.

First, there is the concept of the *point of bifurcation* itself. There is no ambiguity in what can be meant by this term: a sequence of equilibrium configurations is given which can be arranged "linearly" with respect to some parameter; as we follow the sequence, we come to a point where a parting of the ways occurs; beyond such a point we can distinguish two equilibrium configurations where there was only one before the point was reached. A point of bifurcation defined in this manner, clearly presupposes that the *existence* of equilibrium configurations on both prongs of the fork has been established.

The occurrence of a point of bifurcation implies that a non-trivial infinitesimal Lagrangian displacement ξ exists such that the deformation, by this displacement, of the

[1] The subsequent papers by Chandrasekhar and Lebovitz (1963*a*, *b*, and *c*) will be referred to as "Papers II, III, and IV," respectively.

member of the sequence at the point of bifurcation will leave its equilibrium unaffected. The Lagrangian displacement in question is, in fact, the one which will deform the configuration from the "shape" it has in one of the branches to the shape it has in the branch after bifurcation. From the existence of a displacement ξ which leaves the equilibrium unaffected, we may draw two inferences: *first*, that at the point of bifurcation the equilibrium configuration must have a definite non-trivial neutral mode of oscillation; and *second*, that if J is any integral property (or, more generally, a functional) of the configuration which vanishes as a condition of equilibrium, then its first variation due to the displacement ξ must also vanish at the point of bifurcation. More explicitly, the content of the second statement is the following. The functionals J with which we shall be mostly concerned are of the general form

$$J=\int_V \rho(x)Q_1(x)\,dx+\int_V\int_V \rho(x)\,\rho(x')Q_2(x,\,x')\,dx\,dx'\,, \tag{1}$$

where $Q_1(x)$ and $Q_2(x, x')$ are functions which are defined for all points x and pairs of points (x, x'), respectively, and the integrations over x and x' are effected over the volume V occupied by the fluid. And by the first variation of such a functional due to the displacement ξ we mean

$$\begin{aligned}\delta J=&\int_V \rho(x)\,\xi_l(x)\frac{\partial Q_1}{\partial x_l}\,dx\\&+\int_V\int_V \rho(x)\,\rho(x')\left[\xi_l(x)\frac{\partial}{\partial x_l}+\xi_l(x')\frac{\partial}{\partial x_l'}\right]Q_2(x,\,x')\,dx\,dx'\,.\end{aligned} \tag{2}$$

The statement now is that, if

$$J=0 \text{ as a condition of equilibrium}\,, \tag{3}$$

then

$$\delta J=0 \text{ at the point of bifurcation for some } \xi\,. \tag{4}$$

It seems to be generally assumed that, from the existence of a neutral mode of oscillation along an equilibrium sequence, we may, conversely, infer the occurrence of bifurcation. It is not clear to what extent this converse statement is true. In any event, it would appear that the concepts of a point of bifurcation and of a point of neutral stability are logically distinct. On this account, we shall define a *neutral point* along an equilibrium sequence as one at which the configuration has a definite non-trivial mode belonging to a zero characteristic frequency in a proper analysis of the normal modes. Clearly, a point of bifurcation is also a neutral point; but it would appear that a neutral point need not necessarily be a point of bifurcation.

Neutral points can be of two kinds. The distinction arises in this way. Let ϑ be a parameter which labels the members of a branch of a sequence; and let ϑ_0 be a neutral point along that branch. By definition, we can infer the existence of a characteristic frequency of oscillation σ such that

$$\sigma^2(\vartheta)=0 \quad \text{when} \quad \vartheta=\vartheta_0\,, \tag{5}$$

and σ^2 is real in a finite neighborhood of ϑ_0. The two cases to be distinguished are

$$\begin{aligned}&\text{case (i)}: \sigma^2(\vartheta)>0 \quad \text{for} \quad \vartheta<\vartheta_0 \quad \text{and} \quad \sigma^2(\vartheta)<0 \quad \text{for} \quad \vartheta>\vartheta_0\,,\\&\qquad \textit{or} \quad \sigma^2(\vartheta)<0 \quad \text{for} \quad \vartheta<\vartheta_0 \quad \text{and} \quad \sigma^2(\vartheta)>0 \quad \text{for} \quad \vartheta>\vartheta_0\,,\end{aligned} \tag{6}$$

and

$$\text{case (ii)}: \sigma^2(\vartheta)>0 \quad \text{for} \quad \vartheta\neq\vartheta_0 \quad \text{in a finite neighborhood of } \vartheta_0\,. \tag{7}$$

The physical distinction between the two cases is that, while in case (i) the configurations along the branch considered are definitely unstable on one or the other side of the neutral point, in case (ii) the configurations are stable on both sides of the neutral point. We shall call a neutral point as of the *first kind* in case (i) and as of the *second kind* in case (ii). In the literature on the subject it is customary to distinguish the two cases by referring to a *point of ordinary instability* in case (i) and to a *point of secular instability* in case (ii). The reason for attributing instability in case (ii), as well, is the *belief* that we will, in fact, find instability if any dissipative mechanism is operative and the *presumption* that such mechanisms must be operative in any real physical system. What is implied by the terminology is that if we were to investigate the stability of the system by the standard methods of small oscillations, we will find in case (ii) that instability does, in fact, arise when the limit set by ϑ_0 is transgressed and, further, that the growth rate of the instability depends directly on the magnitude of the dissipative mechanisms which are operative; whereas in case (i), when instability arises, the growth rate of the instability will be comparable to the natural frequencies of oscillation of the system. While all this is implicitly and generally understood, the attributes of stability and instability are often ascribed without a detailed investigation of the stability of the system by the method of small oscillations.[2]

The principal objection to the current terminology is, however, its ambiguity and its inadequacy. For example, systems which are stable (in the absence of any dissipative mechanisms) on both sides of a neutral point often become unstable at a subsequent point ϑ_1 (say) by *overstability* (i.e., by oscillations of increasing amplitude). What generally happens in these cases is that the characteristic frequency which has the behavior (7) at ϑ_0 becomes coincident with another characteristic frequency (belonging to another mode) at ϑ_1, beyond which point the two frequencies become complex conjugates of one another. Under these circumstances we should say that the system becomes "ordinarily unstable" at ϑ_1 (since the instability occurs in the absence of any dissipative mechanisms); but such an "ordinary instability" has no bearing on the occurrence or otherwise of a point of bifurcation.

Returning to the question of isolating neutral points and points of bifurcation, we may recall that at these points we must necessarily have Lagrangian displacements which will leave the equilibrium unaffected. Consequently, if J is any functional of the configuration which vanishes as a condition of equilibrium, then *a necessary condition for the occurrence of a neutral point* (and *a fortiori* for a point of bifurcation) *is that a nontrivial Lagrangian displacement exists for which* $\delta J = 0$. It is necessary to emphasize, however, that the location of a neutral point by considering the vanishing of the first variations of such functionals will not enable us to discriminate the kind of neutral point that is located.

III. THE FIRST POINT OF BIFURCATION ALONG THE MACLAURIN SEQUENCE

The origin of the first point of bifurcation along the Maclaurin sequence has been considered in Paper I (Sec. II) on the basis of certain integral properties provided by the second-order virial theorem; and the arguments given on page 1050 (Paper I) leading to the condition for its occurrence are, of course, valid. But the arguments as given do not depend on the invariance of the same integral properties to first variations at the point of bifurcation. We shall now show how the point of bifurcation can also be deduced from these invariance requirements. The demonstration illustrates in the simplest context the essential elements of the present method.

[2] For example, it has always been stated that the Maclaurin spheroid *is* "secularly unstable" at the point of bifurcation where the Jacobi ellipsoids branch off. But only very recently has the problem of the oscillations of a *viscous* Maclaurin spheroid been treated successfully by Roberts and Stewartson (1963). And they do show that the viscous Maclaurin spheroid becomes unstable at the point of bifurcation.

After the elimination of Π $(=\int p d x)$, the second-order virial equations governing equilibrium are

$$\mathfrak{W}_{13}=\mathfrak{W}_{23}=0\,,\quad I_{13}=I_{23}=0\,,\quad \mathfrak{W}_{12}+\Omega^2 I_{12}=0\,, \tag{8}$$

$$\mathfrak{W}_{11}-\mathfrak{W}_{22}+\Omega^2(I_{11}-I_{22})=0\,, \tag{9}$$

and

$$\mathfrak{W}_{11}+\mathfrak{W}_{22}-2\mathfrak{W}_{33}+\Omega^2(I_{11}+I_{22})=0\,, \tag{10}$$

where

$$\mathfrak{W}_{ij}=-\tfrac{1}{2}\int_V \rho\mathfrak{B}_{ij}d\boldsymbol{x} \quad \text{and} \quad I_{ij}=\int_V \rho x_i x_j d\boldsymbol{x} \tag{11}$$

are the potential energy and the moment of inertia tensors. *At a neutral point (or a point of bifurcation) there must exist a non-trivial Lagrangian displacement $\boldsymbol{\xi}$ for which the first variations of all of the foregoing equations must vanish.*

The question might occur whether one is justified in treating Ω as a constant in carrying out the variations of equations (8)–(10). Actually the question is irrelevant, since $\boldsymbol{\xi}$, as we have seen, is the displacement belonging to a well-defined mode among the normal modes of oscillations of the system; and in the analysis of such modes, the various parameters (such as Ω) specifying the initial state of the system are certainly to be kept constant. For the same reason, the Lagrangian displacement can be restricted by the assumption (cf. Paper I, eqs. [104] and [AII, 2])

$$V_i=\langle\rho\xi_i\rangle=\int_V \rho\xi_i d\boldsymbol{x}=0\,, \tag{12}$$

since this assumption merely corresponds to keeping the center of mass stationary in studying the oscillations of a system (cf. Paper II, Sec. II).

Returning to the first variations of equations (8)–(10), we first observe that

$$\delta I_{ij}=V_{ij}\,, \tag{13}$$

where

$$V_{ij}=\int_V \rho(x_i\xi_j+x_j\xi_i)\,d\boldsymbol{x}=V_{j;i}+V_{i;j} \tag{14}$$

is the symmetrized second-order virial. The corresponding first variations of the potential energy tensor of a homogeneous ellipsoid have been given in an earlier paper (Paper IV, eqs. [47] and [48]); they are

$$\delta\mathfrak{W}_{ij}=-2B_{ij}V_{ij} \qquad (i\neq j) \tag{15}$$

and

$$\delta\mathfrak{W}_{ii}=-(2B_{ii}-a_i^2A_{ii})\,V_{ii}+a_i^2\sum_{l\neq i}A_{il}V_{ll} \tag{16}$$

(no summation over repeated indices in eqs. [15] and [16]) ,

where the A_{ij}'s and the B_{ij}'s are the two index symbols which have been defined in earlier papers (Chandrasekhar and Lebovitz 1962*a;* also Paper II, eq. [114]). A common factor $\pi G\rho a_1a_2a_3$ has been suppressed in writing equations (15) and (16).

For the Maclaurin spheroids, $a_1=a_2$; and this equality implies that the values of A_{ij} and B_{ij} are unaltered if the index 1 (wherever it may occur) is replaced by the index 2 (and conversely).

Now, making use of equations (13), (15), and (16), we readily find that the vanishing of the first variations of equations (8)–(10) requires

$$B_{13}V_{13}=0\,,\quad B_{23}V_{23}=0\,,\quad V_{13}=V_{23}=0\,, \tag{17}$$

$$(\Omega^2-2B_{11})V_{12}=0\,,\quad (\Omega^2-2B_{11})(V_{11}-V_{22})=0\,, \tag{18}$$

and

$$[-2(B_{11} - a_1^2 A_{11} + a_3^2 A_{13}) + \Omega^2](V_{11} + V_{22}) + 2(2B_{33} - a_3^2 A_{33} + a_1^2 A_{13})V_{33} = 0 \,. \tag{19}$$

If the additional assumption that the Lagrangian displacement is divergence-free is made, then the foregoing equations must be supplemented by the condition (Lebovitz 1961, eq. [83])

$$V_{11} + V_{22} = -\frac{a_1^2}{a_3^2} V_{33} \,. \tag{20}$$

From equations (17) and (18) it follows that

$$V_{13} = V_{23} = 0 \,, \tag{21}$$

and

$$V_{12} = 0 \quad \text{and} \quad V_{11} = V_{22} \quad \text{if} \quad \Omega^2 - 2B_{11} \neq 0 \,. \tag{22}$$

And, finally, equations (19) and (20) give

$$\left[2(B_{11} - a_1^2 A_{11} + a_3^2 A_{13}) - \Omega^2 + 2\frac{a_3^2}{a_1^2}(2B_{33} - a_3^2 A_{33} + a_1^2 A_{13})\right](V_{11} + V_{22}) = 0 \,. \tag{23}$$

An examination of the coefficient of $(V_{11} + V_{22})$ in equation (23) shows that it does not vanish along the Maclaurin sequence. Accordingly,

$$V_{11} + V_{22} = 0 \quad \text{and} \quad V_{33} = 0 \,. \tag{24}$$

Hence, if $\Omega^2 \neq 2B_{11}$, all the six symmetrized virials must vanish; and this means that there is no non-trivial $\boldsymbol{\xi}$ belonging to the second harmonics which satisfies equations (8)–(10). Therefore, a *necessary* condition for the occurrence of a neutral point (where a neutral mode belonging to the second harmonics can exist) is

$$\frac{\Omega^2}{\pi G \rho} = 2a_1^2 a_3 B_{11} = 2a_1^2 a_3 (A_1 - a_1^2 A_{11}) \,; \tag{25}$$[3]

and, moreover, the Lagrangian displacement must be such that

$$V_{13} = V_{23} = V_{33} = 0 \,, \quad V_{11} = -V_{22} \,, \quad \text{and} \quad V_{12} \neq 0 \,. \tag{26}$$

The condition is also a *sufficient* one, since there are, in fact, three linearly independent Lagrangian displacements which satisfy the requirements (26); thus

$$\xi_1 = \alpha x_1 \,, \quad \xi_2 = -\alpha x_2 \,, \quad \xi_3 = 0 \,,$$

and

$$\xi_1 = \beta x_2 \,, \quad \xi_2 = \gamma x_1 \,, \quad \text{and} \quad \xi_3 = 0 \,, \tag{27}$$

where α, β, and γ are arbitrary (non-zero) constants.

The condition (25) is exactly the one which determines the point along the Maclaurin sequence where the Jacobi ellipsoids branch off.[4] It should be noted that, in deducing

[3] The factor $\pi G \rho a_1 a_2 a_3$ which had been suppressed in writing equations (15) and (16) has been restored in this equation.

[4] The condition

$$\delta(\mathfrak{W}_{12} + \Omega^2 I_{12}) = (-2B_{12} + \Omega^2)V_{12} = 0$$

is satisfied identically all along the Jacobian sequence, since $\Omega^2 = 2B_{12}$ is the equation which determines Ω^2 along this sequence (cf. Paper I, eq. [AI, 7]). Consequently, the mode which is neutral at the point of bifurcation remains neutral along the entire Jacobian sequence.

that a neutral point occurs when condition (25) is met, we do not need to *demand* that the Lagrangian displacement be solenoidal. On this account, a neutral point will occur at the same place along the sequence of *compressible* homogeneous Maclaurin spheroids (as has, indeed, been found by Chandrasekhar and Lebovitz 1962*b*).

IV. THE POINT OF BIFURCATION ALONG THE JACOBIAN SEQUENCE

The point of bifurcation along the Jacobian sequence, where the pearshaped configurations first become possible as figures of equilibrium, was exhibited and isolated in Paper I by considering the vanishing of the first variation of the functional

$$J = \mathfrak{W}_{13;3} + \mathfrak{W}_{12;2} + \mathfrak{W}_{33;1} - \mathfrak{W}_{22;1}\,, \tag{28}$$

where

$$\mathfrak{W}_{ij;k} = -\tfrac{1}{2}\int_V \rho\,\mathfrak{V}_{ij} x_k d\,x \tag{29}$$

for the particular Lagrangian displacement (defined in Paper I, eqs. [10], [11], and [75]–[77]) which deforms an ellipsoidal into a pear. However, the statement in Paper I (in n. 3 on p. 1057) that (28) is the only functional "available for our present purposes" is an error.[5] It is clear now that the first variations of all of the fifteen equations in Paper I, equations (57)–(65), must, in fact, vanish at the point of bifurcation for the Lagrangian displacement considered.

The proper way of treating the fifteen third-order virial equations (obtained after the elimination of the Π_k's) will now be described.

First, we shall rewrite the fifteen equations in Paper I, equations (57)–(65), in four non-combining groups into which they fall (in what sense they are non-combining will be made clear presently):

A:
$$\mathfrak{W}_{13;2} = \mathfrak{W}_{23;1} = \Omega^2 I_{123} + \mathfrak{W}_{12;3} = 0\,; \tag{30}$$

B:
$$\begin{gathered}\Omega^2 I_{113} + \mathfrak{W}_{11;3} - \mathfrak{W}_{33;3} = \Omega^2 I_{223} + \mathfrak{W}_{22;3} - \mathfrak{W}_{33;3} = 0\,,\\ \mathfrak{W}_{13;1} = \mathfrak{W}_{23;2} = 0\,;\end{gathered} \tag{31}$$

C:
$$\begin{gathered}\Omega^2 I_{122} + 2\mathfrak{W}_{12;2} = \Omega^2 I_{133} + 2\mathfrak{W}_{13;3} = 0\,,\\ \Omega^2 I_{111} - 2(\mathfrak{W}_{22;1} - \mathfrak{W}_{12;2} - \mathfrak{W}_{11;1}) = 0\,,\\ \mathfrak{W}_{13;3} + \mathfrak{W}_{12;2} + \mathfrak{W}_{33;1} - \mathfrak{W}_{22;1} = 0\,;\end{gathered} \tag{32}$$

D:
$$\begin{gathered}\Omega^2 I_{112} + 2\mathfrak{W}_{12;1} = \Omega^2 I_{233} + 2\mathfrak{W}_{23;3} = 0\,,\\ \Omega^2 I_{222} - 2(\mathfrak{W}_{11;2} - \mathfrak{W}_{12;1} - \mathfrak{W}_{22;2}) = 0\,,\\ \mathfrak{W}_{32;3} + \mathfrak{W}_{12;1} + \mathfrak{W}_{33;2} - \mathfrak{W}_{11;2} = 0\,.\end{gathered} \tag{33}$$

And as we have stated, at a neutral point (or a point of bifurcation) the first variations of *all* of the foregoing equations must vanish.

[5] This has already been pointed out in the "Note added in proof" (Paper I, p. 1068). But the qualification in this note that "Ω^2 can be considered as invariable for deformations belonging to the third harmonic" is unnecessary: Ω^2 can be treated as a constant without any *proviso*.

It is found that the first variations of all the quantities which occur in equations (30)–(33) can be expressed as linear combinations of the symmetric third-order virials:[6]

$$V_{ijk}=\int_V \rho(\xi_i x_j x_k+\xi_j x_k x_i+\xi_k x_i x_j)\,d\boldsymbol{x} \\ = V_{i;jk}+V_{j;ki}+V_{k;ij}\,. \tag{34}$$

Thus

$$\delta I_{ijk}=V_{ijk}\,. \tag{35}$$

The expressions for the first variations of the $\mathfrak{W}_{ij;k}$'s have been derived in Paper II (Sec. V); and the coefficients of the virials in the expansions of the $\delta\mathfrak{W}_{ij;k}$'s have been tabulated for all combinations of indices (Paper II, Table 1). It will be seen that the expansion of a particular $\delta\mathfrak{W}_{ij;k}$ involves, at most, only three of the virials; and, more particularly, that the variations of the $\mathfrak{W}_{ij;k}$'s included in any of the groups A, B, C, and D involve only the same virials and are mutually exclusive. It is in this sense that the four groups are non-combining. Designating by δA, δB, δC, and δD the equations which are obtained by taking the first variations of the equations in the respective groups, we observe that the association of the groups and the virials is the following:

$$\begin{aligned} &\delta A:\ V_{123}\,; \quad \delta B:\ V_{113}\,,\quad V_{223}\,,\quad V_{333}\,; \\ &\delta C:\ V_{111}\,,\quad V_{122}\,,\quad V_{133}\,; \quad \delta D:\ V_{222}\,,\quad V_{233}\,,\quad V_{211}\,. \end{aligned} \tag{36}$$

We must now investigate whether a non-trivial Lagrangian displacement exists which will satisfy all the variational equations included in δA, δB, δC, and δD.

The discussion of the group δA is particularly simple: the equations belonging to this system are (cf. Paper II, eq. [119])

$$\delta A:\ (\Omega^2-2B_{12;3})V_{123}=-2B_{13;2}V_{123}=-2B_{23;1}V_{123}=0\,, \tag{37}$$

where

$$B_{ij;k}=B_{ij}+a_k^2 B_{ijk}\,; \tag{38}$$

and these equations clearly demand that

$$V_{123}=0\,. \tag{39}$$

The discussion of the groups δB, δC, and δD is not so simple. Each of these groups provides a system of four linear homogeneous equations for the three virials which they involve. For the existence of a non-trivial solution of the variational equations, it is clearly necessary that at least one of the three 4×3 rectangular matrices, representing the linear systems of equations in the three groups, is of rank at most 2.

For the Jacobi ellipsoids as customarily defined ($a_1>a_2>a_3$ in a right-handed system of co-ordinates), it can be directly verified (by the method to be described presently in connection with the group δC) that the equations of the groups δB and δD do not allow any non-trivial solution. Accordingly (cf. eq. [36]),

$$V_{113}=V_{223}=V_{333}=V_{112}=V_{222}=V_{332}=0\,. \tag{40}$$

It remains to find out whether the group δC allows a non-trivial solution.

[6] A qualification is necessary: when $\delta\mathfrak{W}_{ij;k}$ is evaluated for an arbitrary Lagrangian displacement $\boldsymbol{\xi}$, we find (cf. Paper II, eqs [120]–[122]) that it involves, in addition to the V_{ijk}'s, the first-order virials V_i also; but, as we have seen earlier, all of these can be set equal to zero (cf eq. [12] and the remarks preceding it).

In discussing the equations of the group δC, it is convenient to choose the following functionals instead of those listed under C (eqs. [32]):

$$J_1 = -\Omega^2 I_{122} - 2\mathfrak{W}_{12;2}\,, \qquad J_2 = -\Omega^2 I_{133} - 2\mathfrak{W}_{13;3}\,,$$
$$J_3 = \Omega^2(I_{111} - 3I_{122}) + S_{122}\,, \quad J_4 = \Omega^2(I_{111} - I_{133}) + S_{133}\,, \tag{41}$$

where

$$S_{ijj} = -4\mathfrak{W}_{ij;j} - 2\mathfrak{W}_{jj;i} + 2\mathfrak{W}_{ii;i} \tag{42}$$

(no summation over repeated indices) .

The functionals J_1 and J_2 are, apart from sign, the same as the first two listed under C; and J_3 and J_4 are certain specific linear combinations.[7]

Equilibrium requires that all the J's vanish. Therefore, a necessary condition for the occurrence of a neutral point along the Jacobian sequence is that a Lagrangian displacement exists which will satisfy the requirements (39) and (40) and allow a non-trivial solution for the variational equations

$$\delta J_1 = -\Omega^2 V_{122} - 2\delta\mathfrak{W}_{12;2} = 0\,, \tag{43}$$

$$\delta J_2 = -\Omega^2 V_{133} - 2\delta\mathfrak{W}_{13;3} = 0\,, \tag{44}$$

$$\delta J_3 = \Omega^2(V_{111} - 3V_{122}) + \delta S_{122} = 0\,, \tag{45}$$

and

$$\delta J_4 = \Omega^2(V_{111} - V_{133}) + \delta S_{133} = 0\,. \tag{46}$$

The $\delta\mathfrak{W}_{ij;k}$'s and δS_{ijj}'s which occur in equations (43)–(46) are expressible as linear combinations of V_{111}, V_{122}, and V_{133} with coefficients which are listed in Paper II, Table 2. Accordingly, we may write

$$\delta J_i = \langle i|111\rangle V_{111} + \langle i|122\rangle V_{122} + \langle i|133\rangle V_{133} \qquad (i = 1, 2, 3, 4)\,, \tag{47}$$

where $\langle i|111\rangle$, etc., are certain matrix elements which are known.

If we should now require that the Lagrangian displacement be also solenoidal, then we should supplement equation (47) by the further condition (cf. Paper II, Sec. VII)

$$\frac{V_{111}}{a_1^2} + \frac{V_{122}}{a_2^2} + \frac{V_{133}}{a_3^2} = 0\,. \tag{48}$$

The existence of a non-trivial neutral mode belonging to the third harmonic requires, therefore, that, for some member of the Jacobian sequence, the 4×3 matrix representing equation (47) (or the 5×3 matrix if eq. [48] is also included) is at most of rank 2. Instead of examining the problem in this entirely general way from the outset, we shall find it instructive and useful to investigate Lagrangian displacements of progressively increasing generality.

a) The Lagrangian Displacement Which Deforms an Ellipsoid into a Pear

We may argue as in Paper I (Sec. III) that *if* a point of bifurcation occurs along the Jacobian sequence, then the new figure of equilibrium must be one which can be obtained by deforming an ellipsoid by the Lagrangian displacement

$$\xi_j = \text{Constant}\,\frac{\partial}{\partial x_j}\,x_1\left(\frac{x_1^2}{a_1^2+\lambda} + \frac{x_2^2}{a_2^2+\lambda} + \frac{x_3^2}{a_3^2+\lambda} - 1\right), \tag{49}$$

[7] The functional (28) considered in Paper I is $\frac{1}{2}(J_2 + J_3 - J_4 - 3J_1)$.

where λ is the numerically larger of the two roots of the equation

$$\frac{3}{a_1^2+\lambda}+\frac{1}{a_2^2+\lambda}+\frac{1}{a_3^2+\lambda}=0\,. \tag{50}$$

The components of this displacement can be written in the following forms (Paper I, eqs. [75] and [76]):

$$\xi_1 = (\alpha+\beta)x_1^2 - \alpha x_2^2 - \beta x_3^2 - 1\,,$$
$$\xi_2 = -2\alpha x_1 x_2\,, \quad \text{and} \quad \xi_3 = -2\beta x_1 x_3\,, \tag{51}$$

where

$$\alpha = -(a_2^2+\lambda)^{-1} \quad \text{and} \quad \beta = -(a_3^2+\lambda)^{-1}\,. \tag{52}$$

The Lagrangian displacement defined by the foregoing equations is clearly divergence-free; it satisfies the requirement (Paper I, eq. [104])

$$V_i=\int_V \rho\,\xi_i d\boldsymbol{x} = 0\,; \tag{53}$$

and, moreover, all the virials listed in equations (39) and (40) vanish. It would, in fact, appear that the displacement, as defined, is the only one (apart from a constant factor) which will satisfy all these requirements.

TABLE 1

VALUES OF δJ_i IN THE NEIGHBORHOOD OF THE POINT OF BIFURCATION FOR A LAGRANGIAN DISPLACEMENT WHICH DEFORMS AN ELLIPSOID INTO A PEAR

$\cos^{-1} a_3/a_1$	δJ_1	δJ_2	δJ_3	δJ_4
68° . . .	−0 0763	−0 0343	−1 297	−1 092
69°. . .	− .0277	− .0136	−0 534	−0 464
69°8166. .	+ 0001	+ 0000	+0 002	+0 001
70°. .	+ 0051	+ .0027	+0 112	+0 100
71° . .	+0 0262	+0 0149	+0.658	+0 599

For the Lagrangian displacement considered, it can be shown (by making use of Paper I, eqs. [103] and [104]) that

$$V_{111} = 6a_1^2[a_1^2(\alpha+\beta)-1]\,, \quad V_{122} = -2a_2^2[(2a_1^2+a_2^2)\alpha+1]\,,$$
$$\text{and} \quad V_{133} = -2a_3^2[(2a_1^2+a_3^2)\beta+1]\,. \tag{54}$$

Using these particular values for V_{111}, V_{122}, and V_{133}, we can readily evaluate the δJ_i's for different members of the Jacobian sequence. The results of the calculations are given in Table 1. It will be observed that the δJ_i's do seem to vanish simultaneously at one point along the Jacobian sequence; and the accuracy of Darwin's original determination of the point of bifurcation would appear to be fully confirmed.

b) A Lagrangian Displacement Which Is Divergence-free

If we did not know that it is a sequence of pear-shaped configurations which branches off from the Jacobian sequence, we might still wish to demand that the requisite Lagrangian displacement be divergence-free. In that case, the existence of a non-trivial solution will require that the 5×3 matrix representing equations (47) and (48) is (at most) of rank 2. All six determinants of the form

$$\Delta_{ij} = \begin{Vmatrix} \langle i|111\rangle & \langle i|122\rangle & \langle i|133\rangle \\ \langle j|111\rangle & \langle j|122\rangle & \langle j|133\rangle \\ \frac{1}{a_1^2} & \frac{1}{a_2^2} & \frac{1}{a_3^2} \end{Vmatrix} \quad (i \neq j) \tag{55}$$

must, therefore, vanish simultaneously at the point of bifurcation. Table 2 shows that this does happen.

TABLE 2

VALUES OF Δ_{ij} IN THE NEIGHBORHOOD OF THE POINT OF BIFURCATION

$\cos^{-1} a_3/a_1$	69°	69°.8166	70°
Δ_{12}	+0 000882	−0 000003	−0 000177
Δ_{23}	− .01365	+ 00004	+ .00309
Δ_{34}	+ .2994	− 0010	− 0742
Δ_{41}	+ 02148	− .00007	− .00473
Δ_{13}	+ .00687	− 00002	− 00152
Δ_{24}	−0 00426	+0 00001	+0 00097

TABLE 3

VALUES OF Δ_i IN THE NEIGHBORHOOD OF THE POINT OF BIFURCATION

$\cos^{-1} a_3/a_1$	Δ_1	Δ_2	Δ_3	Δ_4
69°.	+0 00236	+0 00366	−0 000221	−0 000221
69°.8166	− 00001	− 00001	+ 000001	+ 000001
70°	−0 00048	−0 00073	+0 000041	+0 000041

c) *A Lagrangian Displacement Which Is Completely Unspecified*

And, finally, if we do not wish to impose any restrictions whatsoever on the Lagrangian displacement, then the existence of a non-trivial solution will require that the 4×3 matrix representing equation (47) is (at most) of rank 2. All four determinants of order 3 which we can form from the 4×3 matrix must, therefore, vanish simultaneously at the point of bifurcation. Denoting by Δ_i the determinant of order 3 obtained by omitting the ith row in equation (47), we list in Table 3 its values in the neighborhood of the point of bifurcation. We observe that all the Δ_i's do vanish at the point of bifurcation, as required.

We can go even further. Suppose that we did not even wish to restrict the Lagrangian displacement by the requirement that the first-order virials V_i vanish. We should then have found an equation of the form

$$\delta J_i = \langle i|111\rangle V_{111} + \langle i|122\rangle V_{122} + \langle i|133\rangle V_{133} + \langle i|1\rangle\, V_1 \quad (i = 1, 2, 3, 4)\,, \tag{56}$$[8]

[8] From Paper I, eqs. (114)–(117) it is apparent that the groups which we have designated δB, δC, and δD involve (besides the third-order virials enumerated in [36]) the first-order virials V_3, V_1, and V_2, respectively. And the consideration of the groups δB and δD would have led us (at an earlier stage) to infer the vanishing of V_2 and V_3 in addition to the requirements (39) and (40).

in place of equation (47). And the existence of a non-trivial solution will now simply require that the determinant of this system of equations vanish. From the results of Section IV*a*, *b*, and *c*, it is clear that we should now simply find that $V_1 = 0$. It appears, then, that we can isolate the point of bifurcation along the Jacobian sequence almost blindfolded!

It is not easy to see a priori why the different ways of looking at the problem are all consistent with one another; and, in particular, why the solenoidal character of the Lagrangian displacement and the vanishing of the first-order virials are *deducible* as necessary conditions for the occurrence of a neutral point. But a self-consistent way of looking at the problem is the following. Suppose it has been established (as it has been established by Darwin) that a pear-shaped configuration is a possible figure of equilibrium. Since the virial equations (30)–(33) are consequences of the same equations governing equilibrium, it is clearly *necessary* that the first variations of the virial equations vanish at the point of bifurcation for the particular Lagrangian displacement, considered in Section IV*a*, which deforms an ellipsoid into a pear; and this we simply verify.

V. THE SECOND AND THE THIRD NEUTRAL POINTS ALONG THE MACLAURIN SEQUENCE

We return once more to the Maclaurin sequence, but this time to isolate further neutral points, beyond the first, where neutral modes of oscillation belonging to the third harmonics may occur. For the exhibition and the isolation of these further neutral points, we may use the same functionals as those we considered in the discussion of the Jacobian sequence in Section IV.

When we consider the first variations of the equations listed under groups A and B (eqs. [30] and [31]), we find that these equations do not allow any non-trivial solution. Therefore, at a neutral point, we must necessarily have

$$V_{123} = V_{333} = V_{311} = V_{322} = 0 . \tag{57}$$

In view of the equality of a_1 and a_2, the variations of the equations in groups C and D will lead to identical equations governing the respective sets of virials; this is apparent from the fact that the coefficients in the expansions of the relevant quantities in terms of the two sets of virials are the same (see Paper III, Table 1). It will, therefore, suffice to consider equations (43)–(46) derived from group C.

Using the coefficients of the virials in the expansion of δS_{122} given in Paper III, Table 2, we find that, at a neutral point,

$$\delta J_3 = [\Omega^2 - 2(B_{11} + a_1^2 B_{111})](V_{111} - 3V_{122}) = 0 . \tag{58}$$

Hence a Lagrangian displacement for which

$$V_{111} \neq 3V_{122} \neq 0 \tag{59}$$

and all the remaining virials vanish will belong to a neutral mode, where

$$\Omega^2 = 2(B_{11} + a_1^2 B_{111}) . \tag{60}$$

This is the same neutral point as that isolated in Paper III (eq. [8]) from a direct investigation of the problem of small oscillations; and, as we have shown, the condition (60) is met along the Maclaurin sequence where

$$e = 0.89926 . \tag{61}$$

The Lagrangian displacement which belongs to the neutral mode of stability at $e = 0.89926$ is given by

$$\xi_j = \text{Constant}\,\frac{\partial}{\partial x_j}\,x_1(x_1^2 - 3x_2^2)\,, \tag{62}$$

or

$$\xi_1 = a(x_1^2 - x_2^2)\,, \quad \xi_2 = -2ax_1x_2\,, \quad \text{and} \quad \xi_3 = 0\,, \tag{63}$$

where a is a constant. It can be readily verified that the foregoing displacement satisfies all of the conditions (57) and (59).[9]

Turning to the consideration of the functionals J_1, J_2, and J_4, we should set in their variations

$$V_{111} = 3V_{122}\,, \tag{64}$$

TABLE 4

VALUES OF Δ_{ij} IN THE NEIGHBORHOOD OF THE NEUTRAL POINT

e	Δ_{24}	Δ_{12}	Δ_{41}
0 96...	+0 015737	+0 0052456	+0 13906
.969373...	+ 000001	+ .0000003	+ .00001
0 98..... .	−0 009929	−0.0033097	−0 17047

in order that we may not be inconsistent with equation (58). The coefficients of the virials V_{122} and V_{133} which then occur in the expansions of the relevant $\delta\mathfrak{W}_{ij;k}$'s and δS_{ijj}'s are those listed in Paper III, Table 2; and, using those coefficients, we find

$$\begin{vmatrix} \delta J_1 \\ \delta J_2 \\ \delta J_4 \end{vmatrix} = \begin{vmatrix} -\Omega^2 + 2(B_{11} + 3a_1^2B_{111}) & a_1^2B_{113} \\ 4a_3^2B_{113} & 2B_{13} + 3a_2^2B_{133} - \Omega^2 \\ 3\Omega^2 + 4(2a_1^2 + a_3^2)B_{113} - 18a_1^2B_{111} - 6B_{11} & -\Omega^2 + 3[B_{13} + B_{33} + (a_1^2 + 2a_3^2)B_{133} - a_1^2B_{113}] \end{vmatrix} \begin{vmatrix} V_{122} \\ V_{133} \end{vmatrix} = 0\,. \tag{65}$$

For the existence of a non-trivial solution for this system of equations, it is necessary that the rank of the 3×2 matrix on the right-hand side is 1. All three determinants of order 2 which can be formed from the 3×2 matrix must, therefore, vanish simultaneously for some member of the Maclaurin sequence; and this happens when (cf. Paper I, eq. [AII, 10])

$$e = 0.969373\,, \tag{66}$$

as is clear from Table 4, which gives the values of the determinant Δ_{ij} formed out of the ith and the jth rows of the matrix on the right-hand side of equation (65).

[9] The consideration of the functional $\Omega^2(V_{222} - 3V_{211}) + S_{112}$ (in group D) would have led to the same condition (60), but with the associated displacement

$$\xi_j = \text{Constant}\,\frac{\partial}{\partial x_j}\,x_2(x_2^2 - 3x_1^2)\,.$$

In deriving the conditions for the occurrence of neutral points based on equations (58) and (65), no demands were made on the Lagrangian displacement that it be solenoidal. If the solenoidal requirement were made, then equation (65) should be supplemented by the further condition

$$V_{122}=-\frac{a_1^2}{4a_3^2}V_{133}\,, \tag{67}$$

which is the form that the general condition

$$V_{111}+V_{122}=-\frac{a_1^2}{a_3^2}V_{133} \tag{68}$$

takes when relation (64) obtains.

When the equations requiring $\delta J_1 = \delta J_2 = 0$ are supplemented by the condition (67), they lead to the equations

$$\Omega^2 = 2(B_{11}+3a_1^2B_{111})-4a_3^2B_{113} \tag{69}$$

and

$$\Omega^2 = 2B_{13}+3a_3^2B_{133}-a_1^2B_{113}\,. \tag{70}$$

These two conditions are not independent, since it can be shown that the quantities on the right-hand sides of the equations are identically the same[10] and, moreover, determine the same value (66) for *e;* the value (66) was, in fact, determined with the aid of equations (69) and (70).

The vanishing of δJ_4, together with equation (67), leads to a further condition for the occurrence of a neutral point. In writing this last condition, it is more convenient to consider

$$\begin{aligned}\delta J_4+3\delta J_1 = \{-\Omega^2+3[B_{13}+B_{33}+(a_1^2+2a_3^2)B_{133}]\}V_{133}&\\ +4(2a_1^2+a_3^2)B_{113}V_{122}=0\,;&\end{aligned} \tag{71}$$

or, making use of equation (67), we have

$$\Omega^2=3\,[B_{13}+B_{33}+(a_1^2+2a_3^2)B_{133}]-\frac{a_1^2}{a_3^2}(2a_1^2+a_3^2)B_{113}\,. \tag{72}$$

It is found that this condition is also satisfied[11] at the same value of *e;* but the right-hand side of equation (72) is *not* identically the same as those of equations (69) and (70).

Again it is remarkable that the solenoidal character of the Lagrangian displacement should be deducible as a necessary condition for the occurrence of a neutral point.

[10] To prove the identity of the two sides, we first rewrite the equations in terms of the symbols A_{ijk}. We have

$$\Omega^2 = 2A_1-a_1^2A_{11}-4a_3^2A_{13}+5a_1^2a_3^2A_{113}$$

and

$$\Omega^2 = 2A_1-a_1^2A_{11}+a_3^2A_{13}+a_1^2a_3^2A_{113}-3a_3^4A_{133}\,;$$

and the equality of the right-hand sides now requires that

$$3a_3^2A_{133} = -4a_1^2A_{113}+5A_{13}\,.$$

This last equality can be readily established by making use of the relations which express the symbols of a certain order in terms of those of the lower order. In the same way it can be shown that the condition obtained in Paper I, eq (AII, 9) is identical with (69) and (70).

[11] Thus Ω^2 (in the unit $\pi G\rho$) given by the right-hand side of eq. (72), for *e* given by (66) is 0.41419, whereas the value found otherwise is 0.41413.

And, finally, we may note that the Lagrangian displacement which belongs to the third neutral point at $e = 0.96937$ is (cf. Paper I, eqs. [AII, 1]-[AII, 3])

$$\xi_j = \text{Constant}\, \frac{\partial}{\partial x_j}\, x_1 [\, x_1^2 + x_2^2 - 4x_3^2 - \tfrac{4}{5}(\, a_1^2 - a_3^2)\,] \,. \tag{73}$$

VI. THE NEUTRAL POINTS ALONG THE JEANS SEQUENCE

The equilibrium and stability of homogeneous masses under the action of a tidal potential of the form

$$\mathfrak{B}_T = -\tfrac{1}{2}\mu(x_1^2 + x_2^2 + x_3^2) + \tfrac{3}{2}\mu x_1^2 \tag{74}$$

was first studied by Jeans (1917) and more recently by Chandrasekhar and Lebovitz (Paper IV). In equation (74),

$$\mu = \frac{GM'}{R^3}, \tag{75}$$

where M' is the mass of the tidally distorting secondary and R is the distance between the centers of mass of the two objects.

It is known that prolate spheroidal forms are consistent with the equations of hydrostatic equilibrium so long as μ is less than a certain maximum value,

$$\mu_{\max} = 0.125536\, \pi G \rho \,, \tag{76}$$

which it attains when the eccentricity of the spheroid takes the value

$$e = 0.883026\,. \tag{77}$$

In this section we shall isolate the neutral points along the Jeans sequence which belong to the second and the third harmonics.

a) The Neutral Point at $\mu_{\max}$

The second-order virial equations governing equilibrium are (Paper IV, eqs. [8] and [9])

$$\mathfrak{W}_{12} = \mathfrak{W}_{13} = 0\,, \qquad \mathfrak{W}_{23} = \mu I_{23}\,, \tag{78}$$

$$\mathfrak{W}_{22} - \mathfrak{W}_{33} - \mu(I_{22} - I_{33}) = 0\,, \tag{79}$$

and

$$\mathfrak{W}_{11} - \mathfrak{W}_{22} + \mu(2I_{11} + I_{22}) = 0\,. \tag{80}$$

At a neutral point the first variations of all of these equations must vanish for a non-trivial Lagrangian displacement.

Combined with equation (15), the first variations of equations (78) give

$$B_{12}V_{12} = B_{13}V_{13} = 0 \quad \text{and} \quad (2B_{23} + \mu)V_{23} = 0\,. \tag{81}$$

Since the B_{ij}'s are non-zero positive constants, it follows that

$$V_{12} = V_{13} = V_{23} = 0 \tag{82}$$

at a neutral point.

Considering next the first variation of equation (79) and making use of equation (16) (remembering that in the present context $a_2 = a_3$), we find

$$(2B_{22} + \mu)(V_{22} - V_{33}) = 0\,. \tag{83}$$

Hence the Lagrangian displacement at a neutral point must satisfy the further condition

$$V_{22} = V_{33} . \tag{84}$$

Finally, the variation of equation (80), namely,

$$\delta\mathfrak{W}_{11} - \delta\mathfrak{W}_{22} + \mu(2V_{11} + V_{22}) = 0 , \tag{85}$$

gives

$$\begin{aligned} -(2B_{11} - a_1^2A_{11} + a_2^2A_{21})V_{11} + 2(B_{22} - a_2^2A_{22} + a_1^2A_{12})V_{22} \\ + \mu(2V_{11} + V_{22}) = 0 . \end{aligned} \tag{86}$$

By supplementing equation (86) by the further condition

$$V_{11} = -\frac{a_1^2}{a_2^2}(V_{22} + V_{33}) = -2\frac{a_1^2}{a_2^2}V_{22} , \tag{87}$$

which must be satisfied if the displacement is to be solenoidal, we obtain the necessary condition for the occurrence of a neutral point. The condition which we obtain in this manner is the same as that found in Paper IV (eq. [67]) by a direct solution of the problem of small oscillations.

From a comparison of equations (80) and (85) it is apparent that the neutral point must occur where μ attains its maximum value. While the present method cannot discriminate the *kind* of neutral point which occurs at $\mu_{\max}$, we know from the analysis of Paper IV that it is, in fact, of the first kind.

b) The Second Neutral Point along the Jeans Sequence

For the isolation of the second neutral point along the Jeans sequence, we must make use of the third-order virial equations which allows for the action of the tidal field (74). The required equation is readily obtained. We have (cf. Paper I, eq. [24])

$$\begin{aligned} \frac{d}{dt}\int_V \rho u_i x_j x_k d\boldsymbol{x} = 2(\mathfrak{T}_{ij;k} + \mathfrak{T}_{ik;j}) + \mathfrak{W}_{ij;k} + \mathfrak{W}_{ik;j} \\ - \mu I_{ijk} + 3\mu\delta_{i1}I_{1jk} + \delta_{ij}\Pi_k + \delta_{ik}\Pi_j , \end{aligned} \tag{88}$$

where the various symbols have their standard meanings.

When no relative motions are present and hydrostatic equilibrium prevails, equation (88) gives

$$\mathfrak{W}_{ij;k} + \mathfrak{W}_{ik;j} - \mu I_{ijk} + 3\mu\delta_{i1}I_{1jk} = -\delta_{ij}\Pi_k - \delta_{ik}\Pi_j . \tag{89}$$

Writing out explicitly the different components of equation (89), we obtain the following eighteen equations:

$$\mathfrak{W}_{11;1} + \mu I_{111} = -\Pi_1 , \tag{90}$$

$$\mathfrak{W}_{22;1} + \mathfrak{W}_{21;2} - \mu I_{122} = -\Pi_1 , \tag{91}$$

$$\mathfrak{W}_{33;1} + \mathfrak{W}_{31;3} - \mu I_{133} = -\Pi_1 , \tag{92}$$

$$\mathfrak{W}_{12;2} + \mu I_{122} = \mathfrak{W}_{13;3} + \mu I_{133} = 0 , \tag{93}$$

$$\mathfrak{W}_{23;1} + \mathfrak{W}_{21;3} = \mathfrak{W}_{13;2} + \mathfrak{W}_{23;1} = \mu I_{123} , \tag{94}$$

$$\mathfrak{W}_{12;3} + \mathfrak{W}_{13;2} = -2\mu I_{123} , \tag{95}$$

$$\mathfrak{W}_{11;2} + \mathfrak{W}_{12;1} + 2\mu I_{112} = -\Pi_2 \,, \tag{96}$$

$$\mathfrak{W}_{33;2} + \mathfrak{W}_{32;3} - \mu I_{233} = -\Pi_2 \,, \tag{97}$$

$$2\mathfrak{W}_{22;2} - \mu I_{222} = -2\Pi_2 \,, \tag{98}$$

$$2\mathfrak{W}_{12;1} = \mu I_{112} \,, \quad 2\mathfrak{W}_{23;3} = \mu I_{233} \,, \tag{99}$$

$$\mathfrak{W}_{11;3} + \mathfrak{W}_{13;1} + 2\mu I_{113} = -\Pi_3 \,, \tag{100}$$

$$\mathfrak{W}_{22;3} + \mathfrak{W}_{32;2} - \mu I_{223} = -\Pi_3 \,, \tag{101}$$

$$2\mathfrak{W}_{33;3} - \mu I_{333} = -2\Pi_3 \,, \tag{102}$$

$$2\mathfrak{W}_{13;1} = \mu I_{113} \,, \quad 2\mathfrak{W}_{23;2} = \mu I_{223} \,. \tag{103}$$

We next eliminate the Π_k's from equations (90)–(103). After their elimination we shall be left with fifteen equations; and, by suitably combining these remaining equations, we can arrange them in the following four "non-combining" groups:

$$A: \quad \mathfrak{W}_{12;3} = \mathfrak{W}_{13;2} = -\mu I_{123} \,, \quad \mathfrak{W}_{23;1} = 2\mu I_{123} \,; \tag{104}$$

$$B: \quad \begin{aligned} 2\mathfrak{W}_{12;1} &= \mu I_{112} \,, & 2\mathfrak{W}_{23;3} &= \mu I_{233} \,, \\ S_{112} &= \mu(I_{222} + 3I_{112}) \,, & S_{233} &= \mu(I_{222} - 3I_{233}) \,; \end{aligned} \tag{105}$$

$$C: \quad \begin{aligned} 2\mathfrak{W}_{13;1} &= \mu I_{113} \,, & 2\mathfrak{W}_{23;2} &= \mu I_{223} \,, \\ S_{113} &= \mu(I_{333} + 3I_{113}) \,, & S_{223} &= \mu(I_{333} - 3I_{223}) \,; \end{aligned} \tag{106}$$

$$D: \quad \begin{aligned} \mathfrak{W}_{12;2} &= -\mu I_{122} \,, & \mathfrak{W}_{13;3} &= -\mu I_{133} \,, \\ S_{122} &= S_{133} = -2\mu I_{111} \,. \end{aligned} \tag{107}$$

In the foregoing equations, S_{ijj} has the same meaning as in equation (42).

At a neutral point a non-trivial Lagrangian displacement must exist such that the first variations of all of the equations in the groups A, B, C, and D vanish. When the variations are carried out, we find (as in the discussion of the Jacobi ellipsoids in Sec. IV) that the equations derived from the different groups involve different virials and are mutually exclusive. If δA, δB, δC, and δD denote the equations which are obtained by taking the first variations of the equations in the respective groups, then the association of the groups and the virials is the following:

$$\begin{aligned} &\delta A: \; V_{123} \,; \quad \delta B: \; V_{222} \,, \; V_{233} \,, \; V_{211} \,; \\ &\delta C: \; V_{333} \,, \; V_{311} \,, \; V_{322} \,; \quad \delta D: \; V_{111} \,, \; V_{122} \,, \; V_{133} \,. \end{aligned} \tag{108}$$

The coefficients of the virials in the expansions of the $\delta\mathfrak{W}_{ij;k}$'s and δS_{ijj}'s which occur in the varied forms of the equations (104)–(107) can be read off from Table 2 in Paper II. (There are some obvious simplifications arising from the present equality of a_2 and a_3.)

We readily find that the equations in the groups δA, δB, and δC do not allow any non-trivial solution. Therefore, at a neutral point, we must necessarily have

$$V_{123} = V_{222} = V_{233} = V_{211} = V_{333} = V_{311} = V_{322} = 0 \,. \tag{109}$$

The occurrence of a neutral point now depends on whether the remaining group δD allows a non-trivial solution. The equations to be considered can be written in the forms (cf. eq. [107])

$$-\delta\mathfrak{W}_{12;2} + \delta\mathfrak{W}_{13;3} - \mu(V_{122} - V_{133}) = 0\,, \tag{110}$$

$$\delta S_{122} - \delta S_{133} = 0\,, \tag{111}$$

$$-(\delta\mathfrak{W}_{12;2} + \delta\mathfrak{W}_{13;3}) - \mu(V_{122} + V_{133}) = 0\,, \tag{112}$$

and

$$\tfrac{1}{2}(\delta S_{122} + \delta S_{133}) + 2\mu V_{111} = 0\,. \tag{113}$$

Making use of the results of Paper II, Table 2, we find that equations (110) and (111) give

$$(B_{12} + a_2^2 B_{122} - \mu)(V_{122} - V_{133}) = 0 \tag{114}$$

and

$$[3(B_{22} + B_{12}) + (5a_2^2 + a_1^2)B_{122}](V_{122} - V_{133}) = 0\,; \tag{115}$$

and these equations clearly require that

$$V_{122} = V_{133}\,. \tag{116}$$

Using this last result in equations (112) and (113), we find

$$a_2^2 B_{112} V_{111} + 2(B_{12} + 2a_2^2 B_{122} - \mu)V_{122} = 0 \tag{117}$$

and

$$\begin{aligned}&[2\mu + (2a_1^2 + a_2^2)B_{112} - 5a_1^2 B_{111} - 2B_{11}]V_{111}\\ &+ [3(B_{22} + B_{12}) + (5a_1^2 + 7a_2^2)B_{122} - 6a_1^2 B_{112}]V_{122} = 0\,.\end{aligned} \tag{118}$$

For the existence of a non-trivial solution, the determinant of equations (117) and (118) must vanish; and we find that this happens when

$$e = 0.94774 \quad \text{and} \quad \mu = 0.10913\,\pi G\rho\,. \tag{119}$$

These values agree with those derived by Jeans (1917) by a different argument.

In deriving the condition for the neutral point based on equations (117) and (118), no demands were made on the Lagrangian displacement that it be solenoidal. If the solenoidal requirement were made, then equations (117) and (118) should be supplemented by the further condition

$$V_{111} = -\frac{a_1^2}{a_2^2}(V_{122} + V_{133}) = -2\frac{a_1^2}{a_2^2}V_{122}\,. \tag{120}$$

With this additional condition, equations (117) and (118) give

$$\mu = B_{12} + 2a_2^2 B_{122} - a_1^2 B_{112} \tag{121}$$

and

$$\begin{aligned}&[3(B_{22} + B_{12}) + (5a_1^2 + 7a_2^2)B_{122} - 6a_1^2 B_{112}]\\ &- 2\frac{a_1^2}{a_2^2}[2\mu + (2a_1^2 + a_2^2)B_{112} - 5a_1^2 B_{111} - 2B_{11}] = 0\,.\end{aligned} \tag{122}$$

We find that these two conditions are, in fact, satisfied at $e = 0.94774$. Thus the solenoidal character of the Lagrangian displacement is again deducible as a necessary condition for the occurrence of a neutral point.

And, finally, we may note that the Lagrangian displacement which belongs to this second neutral point at $e = 0.94774$ is

$$\xi_j = \text{Constant} \frac{\partial}{\partial x_j} x_1 [\tfrac{2}{3} x_1^2 - (x_2^2 + x_3^2) - \tfrac{2}{5}(a_1^2 - a_2^2)] . \tag{123}$$

The research reported in this paper has in part been supported by the Office of Naval Research under contract Nonr-2121(24) with the University of Chicago.

REFERENCES

Chandrasekhar, S. 1962, *Ap. J.*, **136**, 1048 (referred to as "Paper I").
Chandrasekhar, S., and Lebovitz, N. R. 1962*a*, *Ap. J.*, **136**, 1037.
———. 1962*b*, *ibid.*, p. 1069.
———. 1963*a*, *ibid.*, **137**, 1142 (referred to as "Paper II").
———. 1963*b*, *ibid.*, p. 1162 (referred to as "Paper III").
———. 1963*c*, *ibid.*, p. 1172 (referred to as "Paper IV").
Jeans, J. H. 1917, *Mem. R.A.S*, **62**, 1.
Lebovitz, N. R., 1961, *Ap. J.*, **134**, 500.
Roberts, P. H, and Stewartson, K. 1963, *Ap. J.*, **137**, 777.