ON THE STABILITY OF THE JACOBI ELLIPSOIDS

S. Chandrasekhar and Norman R. Lebovitz
University of Chicago and Massachusetts Institute of Technology
Received January 23, 1963

ABSTRACT

In this paper the problem of the small oscillations of the Jacobi ellipsoids is solved, and all the characteristic frequencies belonging to the second and the third harmonics are found. In particular, the variation, along the Jacobian sequence, of the characteristic frequency with respect to which the Jacobi ellipsoid becomes unstable is exhibited.

I. INTRODUCTION

The question of the stability of the Jacobi ellipsoids, particularly at the point of bifurcation where the sequence of the pear-shaped configurations branches off, is central to the theory of the equilibrium and the stability of rotating incompressible masses; it is one to which Poincaré, Darwin, Liapounoff, and Jeans addressed themselves. And yet, it was only in 1924 that it was finally established by Cartan (see Cartan 1928; a detailed account of Cartan's work will be found in Lyttleton 1953) that the Jacobi ellipsoids *must* become unstable at the point of bifurcation with respect to some mode of oscillation belonging to the third harmonics.

While Cartan's investigation settled a basic question of the theory, it did not solve (neither was it an attempt to solve) the problem of the small oscillations; and the problem of determining the characteristic frequencies of the fundamental modes of oscillation of the Jacobi ellipsoids has remained an open one. In this paper we solve this problem and exhibit, in particular, the variation, along the Jacobian sequence, of the characteristic frequency belonging to the mode with respect to which the Jacobi ellipsoid becomes unstable at the point of bifurcation (see Figs. 1 and 2 in Sec. VII).

II. THE VIRIAL EQUATIONS OF THE FIRST ORDER AND THE CONDITIONS FOR THE STATIONARY MAINTENANCE OF THE CENTER OF MASS

Since the instability of the Jacobi ellipsoids occurs via a mode belonging to the third harmonics, greatest interest naturally attaches to these modes. We shall presently see how the characteristic frequencies belonging to these modes can be determined by a straightforward application of the virial equations of the third order which have recently been derived (Chandrasekhar 1962; this paper will be referred to hereafter as "Paper I"). However, it will be found that it is necessary to supplement these equations by the virial equations of the *first order* which one obtains by a direct integration of the equations of motion. Thus, by considering the equations of motion,

$$\rho \frac{du_i}{dt} = -\frac{\partial p}{\partial x_i} + \rho \frac{\partial \mathfrak{B}}{\partial x_i}, \tag{1}$$

written in an inertial frame of reference, and integrating over the entire volume V occupied by the fluid, we obtain

$$\int_V \rho \frac{du_i}{dt} dx = -\int_V \frac{\partial p}{\partial x_i} dx + \int_V \rho \frac{\partial \mathfrak{B}}{\partial x_i} dx . \tag{2}$$

The first of the two integrals on the right-hand side of equation (2) clearly vanishes; and the second also vanishes:

$$\int_V \rho \frac{\partial \mathfrak{B}}{\partial x_i} dx = G \int_V dx\, \rho(x) \frac{\partial}{\partial x_i} \int_V dx' \frac{\rho(x')}{|x - x'|}$$
$$= -G \int_V \int_V \rho(x)\, \rho(x') \frac{x_i - x'_i}{|x - x'|^3} dx dx' = 0 . \tag{3}$$

The result of the integration is therefore[1]

$$\frac{d}{dt} \int_V \rho u_i dx = 0 ; \tag{4}$$

in other words,

$$\int_V \rho u_i dx = \text{Constant} . \tag{5}$$

Equation (5) is no more than the expression of the dynamical requirement that the center of mass of a self-gravitating mass moves only with a constant velocity. For the discussion of the stability of such a system there is clearly no loss of generality in supposing that the center of mass is at rest:

$$\int_V \rho u_i dx \equiv 0 . \tag{6}$$

When the equations of motion are written in a frame of reference rotating with a uniform angular velocity $\mathbf{\Omega}$, the foregoing remarks require some amplification. For, by integrating in the same way the equation of motion,

$$\rho \frac{du_i}{dt} = -\frac{\partial p}{\partial x_i} + \rho \frac{\partial}{\partial x_i} (\mathfrak{B} + \tfrac{1}{2} |\mathbf{\Omega} \times x|^2) + 2 \rho \epsilon_{ilm} u_l \Omega_m , \tag{7}$$

written in a rotating frame of reference, we shall obtain (in place of eq. [4])

$$\frac{d}{dt} \int_V \rho u_i dx = \Omega^2 I_i - \Omega_i \Omega_l I_l + 2 \epsilon_{ilm} \int_V \rho u_l \Omega_m dx , \tag{8}$$

where

$$I_i = \int_V \rho x_i dx . \tag{9}$$

If the direction of $\mathbf{\Omega}$ is chosen to lie along the x_3-axis, equation (8) takes the form

$$\frac{d}{dt} \int_V \rho u_i dx = \Omega^2 (I_i - \delta_{i3} I_3) + 2 \Omega \epsilon_{il3} \int_V \rho u_l dx . \tag{10}$$

If no relative motions (in the rotating frame of reference) are present and hydrostatic equilibrium prevails, it follows from equation (10) that

$$I_1 = I_2 = 0 . \tag{11}$$

These conditions express the requirement that the axis of rotation pass through the center of mass; and we must suppose that this has been so chosen.

[1] The operation d/dt which occurs inside the integral sign in eq. (2) can be taken outside, since the conservation of mass insures the constancy of ρdx as we follow the element during its motion.

Now let an initial state in which hydrostatic equilibrium prevails be slightly disturbed. And let the ensuing motions be described in terms of a Lagrangian displacement, ξ, such that

$$u_i = \frac{d\xi_i}{dt}. \tag{12}$$

The corresponding linearized form of equation (10) is

$$\frac{d^2 V_i}{dt^2} = \Omega^2 (V_i - \delta_{i3} V_3) + 2\Omega \epsilon_{il3} \frac{dV_l}{dt}, \tag{13}$$

where

$$V_i = \int_V \rho \xi_i d\boldsymbol{x}. \tag{14}$$

Explicitly, the equations governing the different components of $\boldsymbol{V}$ are

$$\frac{d^2 V_1}{dt^2} - 2\Omega \frac{dV_2}{dt} = \Omega^2 V_1, \qquad \frac{d^2 V_2}{dt^2} + 2\Omega \frac{dV_1}{dt} = \Omega^2 V_2, \tag{15}$$

and

$$\frac{d^2 V_3}{dt^2} = 0. \tag{16}$$

The general solutions of equations (15) and (16) can be readily written down. But they are not needed[2] (neither are they relevant) for a discussion of the stability of a rotating system: equations (15) and (16) are in no way dependent on the construction or the constitution of the particular system which may be under consideration. We can therefore assume, without any loss of generality, that

$$V_i \equiv 0. \tag{17}$$

The meaning of this assumption is that we are considering the system in a frame of reference whose origin is permanently located at the center of mass of the system. Since the only motion which the center of mass of a self-gravitating system is capable of is a uniform one, no generality is lost by the assumption (17); and this assumption will be made in our subsequent discussion (see Sec. VI; also Appendix II, where the matter is considered from a different point of view).

III. THE THIRD-ORDER VIRIAL EQUATIONS GOVERNING SMALL OSCILLATIONS ABOUT EQUILIBRIUM OF A UNIFORMLY ROTATING SYSTEM

The general third-order virial equations governing the equilibrium of a rotating fluid mass have been written down in Paper I, Section V. We shall now suppose that the system is slightly disturbed; and that the ensuing motions are described by a Lagrangian displacement of the form

$$\xi(x) e^{\lambda t}, \tag{18}$$

where λ is a parameter whose characteristic values are to be determined. To the first order in ξ, the virial equation in Paper I, equation (25), gives

$$\begin{aligned} \lambda^2 V_{i;jk} - 2\lambda \epsilon_{ilm} V_{l;jk} \Omega_m &= \delta \mathfrak{W}_{ij;k} + \delta \mathfrak{W}_{ik;j} \\ &+ \Omega^2 \delta I_{ijk} - \Omega_i \Omega_l \delta I_{ljk} + \delta_{ij} \delta \Pi_k + \delta_{ik} \delta \Pi_j , \end{aligned} \tag{19}$$

[2] It may, however, be noted that eqs. (15) allow solutions for V_1 and V_2 which are periodic with a frequency Ω.

where

$$V_{i;jk} = \int_V \rho \xi_i x_j x_k d\boldsymbol{x} , \tag{20}$$

and $\delta \mathfrak{W}_{ij;k}$, δI_{ijk}, and $\delta \Pi_k$ are the first variations of the quantities

$$\mathfrak{W}_{ij;k} = -\tfrac{1}{2}\int_V \rho \mathfrak{B}_{ij} x_k d\boldsymbol{x} , \qquad I_{ijk} = \int_V \rho x_i x_j x_k d\boldsymbol{x} , \tag{21}$$

and

$$\Pi_k = \int_V p x_k d\boldsymbol{x} , \tag{22}$$

which have been defined in Paper I. (Note that an index after the semicolon indicates that a moment with respect to the associated space co-ordinate is involved.)

With the direction of $\boldsymbol{\Omega}$ chosen along the x_3-axis, equation (19) takes the form

$$\begin{aligned} \lambda^2 V_{i;jk} - 2\lambda\Omega \epsilon_{il3} V_{l;jk} = \delta \mathfrak{W}_{ij;k} + \delta \mathfrak{W}_{ik;j} + \Omega^2 \delta I_{ijk} - \Omega^2 \delta_{i3} \delta I_{3jk} \\ + \delta_{ij} \delta \Pi_k + \delta_{ik} \delta \Pi_j . \end{aligned} \tag{23}$$

The cases $i = 3$ and $i \neq 3$ are clearly distinguished. Thus, when $i = 3$, equation (23) gives

$$\lambda^2 V_{3;jk} = \delta \mathfrak{W}_{3j;k} + \delta \mathfrak{W}_{3k;j} + \delta_{3j} \delta \Pi_k + \delta_{3k} \delta \Pi_j , \tag{24}$$

while, when $i = \varpi = 1$ or 2,

$$\lambda^2 V_{\varpi;jk} - 2[\varpi, \varpi^*, 3]\lambda\Omega V_{\varpi^*;jk} = \delta \mathfrak{W}_{\varpi j;k} + \delta \mathfrak{W}_{\varpi k;j} + \Omega^2 \delta I_{\varpi jk} + \delta_{\varpi j} \delta \Pi_k + \delta_{\varpi k} \delta \Pi_j , \tag{25}$$

where ϖ^* ($\neq \varpi$) = 2 or 1 (when $\varpi = 1$ or 2, respectively) and $[\varpi, \varpi^*, 3]$ is the sign of the permutation $(\varpi, \varpi^*, 3)$.

Equations (24) and (25) together represent a total of eighteen equations. These eighteen equations fall into two non-combining groups of ten and eight equations, respectively, distinguished by their parity (i.e., even or odd) with respect to the index 3. It is convenient to have these equations written out explicitly. The even equations are

$$\lambda^2 V_{3;13} - \delta \mathfrak{W}_{13;3} - \delta \mathfrak{W}_{33;1} = \delta \Pi_1 , \tag{26}$$

$$\lambda^2 V_{3;23} - \delta \mathfrak{W}_{23;3} - \delta \mathfrak{W}_{33;2} = \delta \Pi_2 , \tag{27}$$

$$\lambda^2 V_{1;11} - 2\lambda\Omega V_{2;11} - \Omega^2 \delta I_{111} - 2\delta \mathfrak{W}_{11;1} = 2\delta \Pi_1 , \tag{28}$$

$$\lambda^2 V_{2;22} + 2\lambda\Omega V_{1;22} - \Omega^2 \delta I_{222} - 2\delta \mathfrak{W}_{22;2} = 2\delta \Pi_2 , \tag{29}$$

$$\lambda^2 V_{1;22} - 2\lambda\Omega V_{2;22} - \Omega^2 \delta I_{122} - 2\delta \mathfrak{W}_{12;2} = 0 , \tag{30}$$

$$\lambda^2 V_{2;11} + 2\lambda\Omega V_{1;11} - \Omega^2 \delta I_{211} - 2\delta \mathfrak{W}_{12;1} = 0 , \tag{31}$$

$$\lambda^2 V_{1;12} - 2\lambda\Omega V_{2;12} - \Omega^2 \delta I_{112} - \delta \mathfrak{W}_{11;2} - \delta \mathfrak{W}_{12;1} = \delta \Pi_2 , \tag{32}$$

$$\lambda^2 V_{2;12} + 2\lambda\Omega V_{1;12} - \Omega^2 \delta I_{122} - \delta \mathfrak{W}_{22;1} - \delta \mathfrak{W}_{12;2} = \delta \Pi_1 , \tag{33}$$

$$\lambda^2 V_{1;33} - 2\lambda\Omega V_{2;33} - \Omega^2 \delta I_{133} - 2\delta \mathfrak{W}_{13;3} = 0 , \tag{34}$$

$$\lambda^2 V_{2;33} + 2\lambda\Omega V_{1;33} - \Omega^2 \delta I_{233} - 2\delta \mathfrak{W}_{23;3} = 0 . \tag{35}$$

And the odd equations are

$$\lambda^2 V_{3;11} - 2\delta \mathfrak{W}_{13;1} = 0\,, \tag{36}$$

$$\lambda^2 V_{3;22} - 2\delta \mathfrak{W}_{23;2} = 0\,, \tag{37}$$

$$\lambda^2 V_{3;12} - \delta \mathfrak{W}_{13;2} - \delta \mathfrak{W}_{23;1} = 0\,, \tag{38}$$

$$\lambda^2 V_{3;33} - 2\delta \mathfrak{W}_{33;3} = 2\delta \Pi_3\,, \tag{39}$$

$$\lambda^2 V_{1;13} - 2\lambda\Omega V_{2;13} - \Omega^2 \delta I_{113} - \delta \mathfrak{W}_{11;3} - \delta \mathfrak{W}_{13;1} = \delta \Pi_3\,, \tag{40}$$

$$\lambda^2 V_{2;23} + 2\lambda\Omega V_{1;23} - \Omega^2 \delta I_{223} - \delta \mathfrak{W}_{22;3} - \delta \mathfrak{W}_{23;2} = \delta \Pi_3\,, \tag{41}$$

$$\lambda^2 V_{1;23} - 2\lambda\Omega V_{2;23} - \Omega^2 \delta I_{123} - \delta \mathfrak{W}_{12;3} - \delta \mathfrak{W}_{13;2} = 0\,, \tag{42}$$

$$\lambda^2 V_{2;13} + 2\lambda\Omega V_{1;13} - \Omega^2 \delta I_{123} - \delta \mathfrak{W}_{21;3} - \delta \mathfrak{W}_{23;1} = 0\,. \tag{43}$$

These equations must be further supplemented by the three equations (17) expressing the stationariness of the center of mass. Thus we have, altogether, twenty-one equations to consider.

IV. THE REDUCTION OF THE EQUATIONS

The use of equations (26)–(43) for the determination of the characteristic values of λ^2 will depend on our ability to express δI_{ijk}, $\delta \mathfrak{W}_{ij;k}$, and $\delta \Pi_k$, which occur in these equations, in terms of the virials $V_{i;jk}$.

The expression of δI_{ijk} in terms of the virials is, of course, immediate:

$$\begin{aligned} \delta I_{ijk} &= \delta \int_V \rho x_i x_j x_k d\boldsymbol{x} = \int_V \rho (\, \xi_i x_j x_k + \xi_j x_k x_i + \xi_k x_i x_j)\, d\boldsymbol{x} \\ &= V_{i;jk} + V_{j;ki} + V_{k\cdot ij}\,. \end{aligned} \tag{44}$$

It is convenient to have a symbol for the symmetric combination of the virials which occurs in the expression for δI_{ijk}; we shall denote it by V_{ijk} (without the semicolon):

$$V_{ijk} = V_{i;jk} + V_{j;ki} + V_{k;ij}\,. \tag{45}$$

We can then write

$$\delta I_{ijk} = V_{ijk}\,. \tag{46}$$

Turning to $\delta \mathfrak{W}_{ij;k}$ and $\delta \Pi_k$, we shall find that, for homogeneous ellipsoids (in which we are presently interested), the $\delta \mathfrak{W}_{ij;k}$'s can also be expressed in terms of these symmetrized virials. The treatment of $\delta \Pi_k$ will, however, depend on whether the fluid is considered incompressible or compressible. In the former case, the $\delta \Pi_k$'s should be eliminated from equations (26)–(43); and the remaining fifteen equations should be supplemented by three further equations which express the incompressibility of the fluid. These additional equations will be obtained in Section VI. But if the fluid should be considered compressible, then the $\delta \Pi_k$'s must also be expressed in terms of the virials; as to how one might accomplish this will be illustrated in another paper. In this paper, we shall be concerned only with the incompressible case. Nevertheless, it will be useful to eliminate the $\delta \Pi_k$'s from equations (26)–(43) and carry out the reduction of these equations to minimal sets without making any restrictive assumptions.

a) The Even Equations

As the dependent variables we shall choose $V_{1;22}$, $V_{1;33}$, $V_{2;11}$, and $V_{2;33}$ besides the six symmetric virials

$$V_{111} = 3V_{1,11}\,, \quad V_{122} = V_{1;22} + 2V_{2;12}\,, \quad V_{133} = V_{1;33} + 2V_{3;13}\,,$$

and

$$V_{222} = 3V_{2;22}\,, \quad V_{112} = V_{2;11} + 2V_{1;12}\,, \quad V_{233} = V_{2;33} + 2V_{3;23}\,. \tag{47}$$

In accordance with equation (46) we first write V_{ijk} in place of δI_{ijk}. We then combine the equations appropriately so that λ^2 occurs multiplied by the virials we are presently considering as the dependent variables. In this manner we obtain the following set of equations:

$$(\tfrac{1}{3}\lambda^2 - \Omega^2)V_{111} - 2\lambda\Omega V_{2;11} - 2\delta\mathfrak{W}_{11;1} = 2\delta\Pi_1\,, \tag{48}$$

$$(\lambda^2 - 3\Omega^2)V_{122} + 4\lambda\Omega V_{1;12} - \tfrac{2}{3}\lambda\Omega V_{222} - 4\delta\mathfrak{W}_{21;2} - 2\delta\mathfrak{W}_{22;1} = 2\delta\Pi_1\,, \tag{49}$$

$$(\lambda^2 - \Omega^2)V_{133} - 2\lambda\Omega V_{2;33} - 4\delta\mathfrak{W}_{13;3} - 2\delta\mathfrak{W}_{33;1} = 2\delta\Pi_1\,, \tag{50}$$

$$(\tfrac{1}{3}\lambda^2 - \Omega^2)V_{222} + 2\lambda\Omega V_{1;22} - 2\delta\mathfrak{W}_{22;2} = 2\delta\Pi_2\,, \tag{51}$$

$$(\lambda^2 - 3\Omega^2)V_{112} - 4\lambda\Omega V_{2;12} + \tfrac{2}{3}\lambda\Omega V_{111} - 4\delta\mathfrak{W}_{21;1} - 2\delta\mathfrak{W}_{11;2} = 2\delta\Pi_2\,, \tag{52}$$

$$(\lambda^2 - \Omega^2)V_{233} + 2\lambda\Omega V_{1;33} - 4\delta\mathfrak{W}_{23;3} - 2\delta\mathfrak{W}_{33;2} = 2\delta\Pi_2\,, \tag{53}$$

$$\lambda^2 V_{1;22} - \tfrac{2}{3}\lambda\Omega V_{222} - \Omega^2 V_{122} - 2\delta\mathfrak{W}_{12;2} = 0\,, \tag{54}$$

$$\lambda^2 V_{1;33} - 2\lambda\Omega V_{2;33} - \Omega^2 V_{133} - 2\delta\mathfrak{W}_{13;3} = 0\,, \tag{55}$$

$$\lambda^2 V_{2;11} + \tfrac{2}{3}\lambda\Omega V_{111} - \Omega^2 V_{112} - 2\delta\mathfrak{W}_{21;1} = 0\,, \tag{56}$$

$$\lambda^2 V_{2;33} + 2\lambda\Omega V_{1;33} - \Omega^2 V_{233} - 2\delta\mathfrak{W}_{23;3} = 0\,. \tag{57}$$

Next, we eliminate $\delta\Pi_1$ from equations (49) and (50) by making use of equation (48); and similarly we eliminate $\delta\Pi_2$ from equations (52) and (53) by making use of equation (51). We write the resulting four equations in the forms:

$$(\lambda^2 - 3\Omega^2)(V_{122} - \tfrac{1}{3}V_{111}) + 2\lambda\Omega(V_{112} - \tfrac{1}{3}V_{222}) + \delta S_{122} = 0\,, \tag{58}$$

$$(\lambda^2 - 3\Omega^2)(V_{112} - \tfrac{1}{3}V_{222}) - 2\lambda\Omega(V_{122} - \tfrac{1}{3}V_{111}) + \delta S_{112} = 0\,, \tag{59}$$

$$(\lambda^2 - \Omega^2)V_{133} - (\tfrac{1}{3}\lambda^2 - \Omega^2)V_{111} - 2\lambda\Omega(V_{2;33} - V_{2;11}) + \delta S_{133} = 0\,, \tag{60}$$

$$(\lambda^2 - \Omega^2)V_{233} - (\tfrac{1}{3}\lambda^2 - \Omega^2)V_{222} + 2\lambda\Omega(V_{1;33} - V_{1;22}) + \delta S_{233} = 0\,, \tag{61}$$

where we have introduced the abbreviation

$$\delta S_{ijj} = -4\delta\mathfrak{W}_{ij;j} - 2\delta\mathfrak{W}_{jj;i} + 2\delta\mathfrak{W}_{ii;i} \tag{62}$$

(no summation over repeated indices).

Introducing the further abbreviation,

$$\delta Q_{ijj} = \Omega^2 V_{ijj} + 2\delta\mathfrak{W}_{ij;j} \tag{63}$$

(no summation over repeated indices),

we can rewrite the remaining four equations (54)–(57) more conveniently in the forms

$$\lambda^2 V_{1;33} - 2\lambda\Omega V_{2;33} - \delta Q_{133} = 0\,, \tag{64}$$

$$\lambda^2 V_{2;33} + 2\lambda\Omega V_{1;33} - \delta Q_{233} = 0\,, \tag{65}$$

$$\lambda^2 V_{1;22} - \tfrac{2}{3}\lambda\Omega V_{222} - \delta Q_{122} = 0\,, \tag{66}$$

$$\lambda^2 V_{2;11} + \tfrac{2}{3}\lambda\Omega V_{111} - \delta Q_{112} = 0\,. \tag{67}$$

By subtracting equation (66) from equation (64) and similarly equation (67) from equation (65), we obtain

$$\lambda^2(V_{1;33} - V_{1;22}) - 2\lambda\Omega V_{2;33} + \tfrac{2}{3}\lambda\Omega V_{222} - \delta Q_{133} + \delta Q_{122} = 0\,, \tag{68}$$

$$\lambda^2(V_{2;33} - V_{2;11}) + 2\lambda\Omega V_{1;33} - \tfrac{2}{3}\lambda\Omega V_{111} - \delta Q_{233} + \delta Q_{112} = 0\,. \tag{69}$$

Next, we eliminate $(V_{2;33} - V_{2;11})$ and $(V_{1;33} - V_{1;22})$ from equations (60) and (61) by making use of equations (68) and (69) and obtain

$$\lambda[(\lambda^2 - \Omega^2)V_{133} - \tfrac{1}{3}(\lambda^2 + \Omega^2)V_{111} + \delta S_{133}] + 4\Omega^2\lambda V_{1;33} - 2\Omega(\delta Q_{233} - \delta Q_{112}) = 0\,, \tag{70}$$

$$\lambda[(\lambda^2 - \Omega^2)V_{233} - \tfrac{1}{3}(\lambda^2 + \Omega^2)V_{222} + \delta S_{233}] + 4\Omega^2\lambda V_{2;33} + 2\Omega(\delta Q_{133} - \delta Q_{122}) = 0\,. \tag{71}$$

On the other hand, from equations (64) and (65) we have

$$\lambda(\lambda^2 + 4\Omega^2)V_{1;33} - 2\Omega\delta Q_{233} - \lambda\delta Q_{133} = 0\,, \tag{72}$$

$$\lambda(\lambda^2 + 4\Omega^2)V_{2;33} + 2\Omega\delta Q_{133} - \lambda\delta Q_{233} = 0\,. \tag{73}$$

It will be noticed that equations (72) and (73) (unlike eqs. [64] and [65] from which they were derived) are not linearly independent when $\lambda^2 = -4\Omega^2$. We shall return to the consequences of this linear dependence presently; but meantime, using equations (72) and (73) to eliminate $V_{1;33}$ and $V_{2;33}$ from equations (70) and (71), we obtain

$$\lambda(\lambda^2 + 4\Omega^2)[(\lambda^2 - \Omega^2)V_{133} - \tfrac{1}{3}(\lambda^2 + \Omega^2)V_{111} + \delta S_{133}] + 4\Omega^2(+2\Omega\delta Q_{233} + \lambda\delta Q_{133}) - 2\Omega(\lambda^2 + 4\Omega^2)(\delta Q_{233} - \delta Q_{112}) = 0\,, \tag{74}$$

$$\lambda(\lambda^2 + 4\Omega^2)[(\lambda^2 - \Omega^2)V_{233} - \tfrac{1}{3}(\lambda^2 + \Omega^2)V_{222} + \delta S_{233}] + 4\Omega^2(-2\Omega\delta Q_{133} + \lambda\delta Q_{233}) + 2\Omega(\lambda^2 + 4\Omega^2)(\delta Q_{133} - \delta Q_{122}) = 0\,. \tag{75}$$

Thus, after the elimination of $\delta\Pi_1$ and $\delta\Pi_2$, the remaining eight even equations have been reduced to the following four:

$$(\lambda^2 - 3\Omega^2)(V_{122} - \tfrac{1}{3}V_{111}) + 2\lambda\Omega(V_{112} - \tfrac{1}{3}V_{222}) + \delta S_{122} = 0\,, \tag{76}$$

$$(\lambda^2 - 3\Omega^2)(V_{112} - \tfrac{1}{3}V_{222}) - 2\lambda\Omega(V_{122} - \tfrac{1}{3}V_{111}) + \delta S_{112} = 0\,, \tag{77}$$

$$\lambda(\lambda^2 + 4\Omega^2)[(\lambda^2 - \Omega^2)V_{133} - \tfrac{1}{3}(\lambda^2 + \Omega^2)V_{111} + \delta S_{133}] + 2\Omega(\lambda^2 + 4\Omega^2)(\Omega^2 V_{112} + 2\delta\mathfrak{W}_{12;1}) - 2\Omega\lambda^2(\Omega^2 V_{233} + 2\delta\mathfrak{W}_{23;3}) + 4\Omega^2\lambda(\Omega^2 V_{133} + 2\delta\mathfrak{W}_{13;3}) = 0\,, \tag{78}$$

$$\lambda(\lambda^2+4\Omega^2)[(\lambda^2-\Omega^2)V_{233}-\tfrac{1}{3}(\lambda^2+\Omega^2)V_{222}+\delta S_{233}]$$
$$-2\Omega(\lambda^2+4\Omega^2)(\Omega^2 V_{122}+2\delta\mathfrak{W}_{12;2})+2\Omega\lambda^2(\Omega^2 V_{133}+2\delta\mathfrak{W}_{13;3}) \tag{79}$$
$$+4\Omega^2\lambda(\Omega^2 V_{233}+2\delta\mathfrak{W}_{23;3})=0\,.$$

We have already remarked that equations (72) and (73) are not linearly independent when $\lambda^2=-4\Omega^2$. Specifically, when $\lambda^2=-4\Omega^2$, equations (72) and (73) reduce to a pair of identical equations:

$$\delta Q_{133}:\delta Q_{233}=-2\Omega:\lambda=\lambda:2\Omega\,. \tag{80}$$

Therefore, the characteristic determinant of a system of equations, which includes equations (72) and (73), will vanish when $\lambda^2=-4\Omega^2$. In other words, the roots $\lambda=\pm 2i\Omega$ will appear spuriously among the characteristic roots.

An alternative manner of reduction which will avoid introducing spurious roots is the following. Let

$$V_{\varpi 33}=V_{133}+iV_{233}\,;\qquad \delta Q_{\varpi 33}=\delta Q_{133}+i\delta Q_{233}\,,$$
$$V_{\varpi\varpi\varpi}=V_{111}+iV_{222}\,;\qquad \delta Q_{\varpi\varpi^*\varpi}=\delta Q_{122}+i\delta Q_{112}\,, \tag{81}$$
$$V_{\varpi;33}=V_{1;33}+iV_{2;33}\,;\qquad \delta S_{\varpi 33}=\delta S_{133}+i\delta S_{233}\,.$$

In terms of these variables, equations (64) and (65) and equations (70) and (71) can be combined to give, respectively,

$$\lambda(\lambda+2i\Omega)V_{\varpi;33}-\delta Q_{\varpi 33}=0 \tag{82}$$

and

$$\lambda[(\lambda^2-\Omega^2)V_{\varpi 33}-\tfrac{1}{3}(\lambda^2+\Omega^2)V_{\varpi\varpi\varpi}+\delta S_{\varpi 33}]+4\Omega^2\lambda V_{\varpi;33}$$
$$+2i\Omega(\delta Q_{\varpi 33}-\delta Q_{\varpi\varpi^*\varpi^*})=0\,. \tag{83}$$

The elimination of $V_{\varpi;33}$ from equations (82) and (83) gives

$$\lambda(\lambda+2i\Omega)[(\lambda^2-\Omega^2)V_{\varpi 33}-\tfrac{1}{3}(\lambda^2+\Omega^2)V_{\varpi\varpi\varpi}+\delta S_{\varpi 33}]$$
$$+2i\Omega(\lambda+2i\Omega)(\delta Q_{\varpi 33}-\delta Q_{\varpi\varpi^*\varpi^*})+4\Omega^2\delta Q_{\varpi 33}=0\,. \tag{84}$$

In place of equations (74) and (75), we now have equation (84) and its "conjugate" (obtained by writing $-i$ wherever i occurs *explicitly* in eq. [84] and in the definitions [81]). It is clear that a system of equations which includes equation (84) and its "conjugate" will lead to a characteristic equation for λ^2 that is two degrees lower than the one which will be obtained from a system which includes equations (74) and (75) (or eqs. [78] and [79]) in their place. The roots $\lambda=\pm 2i\Omega$, which will spuriously occur by the inclusion of the latter pair of equations, can thus be avoided.

b) The Odd Equations

As the dependent variables we now choose

$$V^{\dagger}_{123}=V_{1;23}-V_{2;13}\,,\qquad V_{3;11}+V_{3;22}\,,\qquad \text{and}\qquad V_{3;11}-V_{3;22} \tag{85}$$

besides the four odd symmetric virials

$$V_{123}\,,\qquad V_{113}\,,\qquad V_{223}\,,\qquad \text{and}\qquad V_{333}\,. \tag{86}$$

By making use of equation (46) and suitably combining equations (36)–(43), we readily obtain the following equivalent set of eight equations:

$$\tfrac{1}{3}\lambda^2 V_{333} - 2\delta\mathfrak{W}_{33;3} = 2\delta\Pi_3 , \tag{87}$$

$$\lambda^2(V_{3;11} + V_{3;22}) = 2\delta\mathfrak{W}_{13;1} + 2\delta\mathfrak{W}_{23;2} , \tag{88}$$

$$\lambda^2(V_{3;11} - V_{3;22}) = 2\delta\mathfrak{W}_{13;1} - 2\delta\mathfrak{W}_{23;2} , \tag{89}$$

$$\lambda^2 V_{3;12} = \delta\mathfrak{W}_{31;2} + \delta\mathfrak{W}_{32;1} , \tag{90}$$

$$(\lambda^2 - 2\Omega^2)V_{113} - 2\lambda\Omega(V_{123} - V\dagger_{123} - V_{3;12}) - 2\delta\mathfrak{W}_{11;3} - 4\delta\mathfrak{W}_{13;1} = 2\delta\Pi_3 , \tag{91}$$

$$(\lambda^2 - 2\Omega^2)V_{223} + 2\lambda\Omega(V_{123} + V\dagger_{123} - V_{3;12}) - 2\delta\mathfrak{W}_{22;3} - 4\delta\mathfrak{W}_{23;2} = 2\delta\Pi_3 , \tag{92}$$

$$(\lambda^2 - 2\Omega^2)V_{123} + \lambda\Omega[V_{113} - V_{223} - (V_{3;11} - V_{3;22})] + \delta S_{123} = 0 , \tag{93}$$

$$\lambda^2 V\dagger_{123} - \lambda\Omega[V_{113} + V_{223} - (V_{3;11} + V_{3;22})] - \delta\mathfrak{W}_{13;2} + \delta\mathfrak{W}_{23;1} = 0 , \tag{94}$$

where (cf. eq. [62])

$$\delta S_{123} = -2\delta\mathfrak{W}_{12;3} - 2\delta\mathfrak{W}_{23;1} - 2\delta\mathfrak{W}_{31;2} . \tag{95}$$

We first eliminate $\delta\Pi_3$ from equations (91) and (92) by making use of equation (87) to obtain the pair of equations

$$(\lambda^2 - 2\Omega^2)V_{113} - \tfrac{1}{3}\lambda^2 V_{333} - 2\lambda\Omega(V_{123} - V\dagger_{123} - V_{3;12}) + \delta S_{113} = 0 , \tag{96}$$

$$(\lambda^2 - 2\Omega^2)V_{223} - \tfrac{1}{3}\lambda^2 V_{333} + 2\lambda\Omega(V_{123} + V\dagger_{123} - V_{3;12}) + \delta S_{223} = 0 , \tag{97}$$

where δS_{113} and δS_{223} have meanings in accordance with the general definition (62); and then combine the two equations to give

$$(\lambda^2 - 2\Omega^2)\,(V_{113} + V_{223}) - \tfrac{2}{3}\lambda^2 V_{333} + 4\lambda\Omega V\dagger_{123} + \delta S_{113} + \delta S_{223} = 0 , \tag{98}$$

$$(\lambda^2 - 2\Omega^2)\,(V_{113} - V_{223}) - 4\lambda\Omega V_{123} + 4\lambda\Omega V_{3;12} + \delta S_{113} - \delta S_{223} = 0 . \tag{99}$$

We now eliminate $V_{3;11}$ and $V_{3;22}$ from equations (93) and (94) by making use of equations (88) and (89); and we obtain

$$\lambda^3 V\dagger_{123} - \lambda^2\Omega(V_{113} + V_{223}) - \lambda(\delta\mathfrak{W}_{13;2} - \delta\mathfrak{W}_{23;1}) + 2\Omega(\delta\mathfrak{W}_{13;1} + \delta\mathfrak{W}_{23;2}) = 0 , \tag{100}$$

$$\lambda(\lambda^2 - 2\Omega^2)V_{123} + \lambda^2\Omega(V_{113} - V_{223}) + \lambda\delta S_{123} - 2\Omega(\delta\mathfrak{W}_{13;1} - \delta\mathfrak{W}_{23;2}) = 0 . \tag{101}$$

Finally, we eliminate $V\dagger_{123}$ and $V_{3;12}$ from equations (98) and (99) by making use of equations (100) and (90), respectively. We are then left with

$$\begin{aligned}\lambda^4(V_{113} + V_{223} - \tfrac{2}{3}V_{333}) + \lambda^2[2\Omega^2(V_{113} + V_{223}) + \delta S_{113} + \delta S_{223}]\\ + 4\lambda\Omega(\delta\mathfrak{W}_{13;2} - \delta\mathfrak{W}_{23;1}) - 8\Omega^2(\delta\mathfrak{W}_{13;1} + \delta\mathfrak{W}_{23;2}) = 0 ,\end{aligned} \tag{102}$$

$$\begin{aligned}\lambda^3(V_{113} - V_{223}) - 4\lambda^2\Omega V_{123} + \lambda[-2\Omega^2(V_{113} - V_{223}) + \delta S_{113} - \delta S_{223}]\\ + 4\Omega(\delta\mathfrak{W}_{13;2} + \delta\mathfrak{W}_{23;1}) = 0 .\end{aligned} \tag{103}$$

Thus, after the elimination of $\delta\Pi_3$, the remaining seven odd equations have been reduced to the three equations (101)–(103).

V. THE EXPANSION OF $\delta\mathfrak{W}_{ij;k}$ IN TERMS OF THE VIRIALS V_{ijk}

First, we recall that the variation in $\mathfrak{W}_{ij;k}$ due to a general Lagrangian displacement, ξ, is given by (Paper I, eq. [72])

$$-2\,\delta\mathfrak{W}_{ij;k}=\int_V \rho\mathfrak{B}_{ij}\,\xi_k d\boldsymbol{x}+\int_V \rho\,\xi_l\,\frac{\partial\mathfrak{B}_{ij}}{\partial x_l}\,x_k d\boldsymbol{x}+\int_V \rho\,\xi_l\,\frac{\partial\mathfrak{D}_{ij;k}}{\partial x_l}\,d\boldsymbol{x}, \tag{104}$$

where $\mathfrak{D}_{ij;k}$ denotes the tensor

$$\mathfrak{D}_{ij;k}=G\int_V \rho(\boldsymbol{x}')\,\frac{(x_i-x'_i)(x_j-x'_j)\,x'_k}{|\boldsymbol{x}-\boldsymbol{x}'|^3}\,d\boldsymbol{x}'. \tag{105}$$

Now we have shown in another paper (Chandrasekhar and Lebovitz 1962*a*) that the tensor $\mathfrak{D}_{ij;k}$ can be deduced from a knowledge of the Newtonian potentials $\mathfrak{D}_k$ and $\mathfrak{D}_{ik}$ due to the fictitious "density distributions" ρx_k and $\rho x_i x_k$; thus

$$\mathfrak{D}_{ij;k}=-x_i\frac{\partial\mathfrak{D}_k}{\partial x_j}+\frac{\partial\mathfrak{D}_{ik}}{\partial x_j}. \tag{106}$$

The foregoing equations, as well as the analysis in the preceding sections, are quite general: they do not depend on any restrictive assumptions on the equation of state or otherwise. But from now on the analysis will be restricted to homogeneous ellipsoids.

In an earlier paper (Chandrasekhar and Lebovitz 1962*b*; this paper will be referred to hereafter as "Paper II") explicit expressions are given for the various potentials and superpotentials of homogeneous ellipsoids. The expressions for $\mathfrak{D}_i$, $\mathfrak{B}_{ij}$, and $\mathfrak{D}_{ij}$ given in that paper (Paper II, eqs. [49], [51], [68], and [70]) all contain $\pi G\rho a_1 a_2 a_3$ as a common factor. For brevity, this common factor will be suppressed in the remainder of this section. And the summation convention will also be suspended: *summation over a repeated index is not to be understood unless it is so explicitly indicated.* With these understandings, we can write

$$\mathfrak{D}_i=a_i^2 x_i\Big(A_i-\sum_{l=1}^{3}A_{il}x_l^2\Big), \tag{107}$$

$$\mathfrak{B}_{ij}=2B_{ij}x_i x_j+a_i^2\delta_{ij}\Big(A_i-\sum_{l=1}^{3}A_{il}x_l^2\Big), \tag{108}$$

$$\mathfrak{D}_{ij}=a_i^2 a_j^2 x_i x_j\Big(A_{ij}-\sum_{l=1}^{3}A_{ijl}x_l^2\Big)\qquad (i\neq j), \tag{109}$$

and

$$\mathfrak{D}_{ii}=a_i^4 x_i^2\Big(A_{ii}-\sum_{l=1}^{3}A_{iil}x_l^2\Big)+\tfrac{1}{4}a_i^2\Big(B_i-2\sum_{l=1}^{3}B_{il}x_l^2+\sum_{l=1}^{3}\sum_{m=1}^{3}B_{ilm}x_l^2x_m^2\Big), \tag{110}$$

where, in transcribing the formulae given in Paper II, we have introduced the further symbols

$$B_i=I-a_i^2A_i,\qquad B_{ij}=A_j-a_i^2A_{ij}=A_i-a_j^2A_{ij}, \tag{111}$$

and

$$B_{ilm}=A_{lm}-a_i^2A_{ilm}=A_{mi}-a_l^2A_{lmi}=A_{il}-a_m^2A_{mil}. \tag{112}$$

The new symbols $B_{ijk\ldots}$, like the symbols $A_{ijk\ldots}$ introduced in Paper II, are completely symmetric in their indices and are, moreover, related to $A_{ijk\ldots}$ by

$$B_{ijk\ldots} = A_{jk\ldots} - a_i^2 A_{ijk\ldots} . \tag{113}$$

The symmetry of the symbol B_{ijk} as well as the existence of the relation (113) are apparent from the alternative definition,

$$B_{ijk\ldots} = \int_0^\infty \frac{u\,du}{\Delta(a_i^2+u)(a_j^2+u)(a_k^2+u)\ldots} . \tag{114}$$

Now inserting the expressions for $\mathfrak{D}_k$ and $\mathfrak{D}_{ik}$ in equation (106), we find (distinguishing the different cases)

$$\mathfrak{D}_{ij;k} = 2a_k^2 B_{ijk} x_i x_j x_k \qquad (i \neq j \neq k), \tag{115}$$

$$\mathfrak{D}_{ij;j} = a_j^2 x_i (-B_{ij} + B_{iij} x_i^2 + 3B_{ijj} x_j^2 + B_{ijk} x_k^2) \qquad (i \neq j \neq k), \tag{116}$$

$$\mathfrak{D}_{ii;j} = a_j^2 x_j \left[2B_{iij} x_i^2 + a_i^2 \left(A_{ij} - \sum_{l=1}^{3} A_{ijl} x_l^2 \right) \right] \qquad (i \neq j), \tag{117}$$

$$\mathfrak{D}_{ii;i} = a_i^2 x_i \Big[(a_i^2 A_{ii} - 2B_{ii}) + (4B_{iii} - a_i^2 A_{iii}) x_i^2 + \sum_{l \neq i} (2B_{iil} - a_i^2 A_{iil}) x_l^2 \Big] . \tag{118}$$

With the explicit expressions for $\mathfrak{B}_{ij}$ and $\mathfrak{D}_{ij;k}$ which we now have, it is a straightforward matter to evaluate $\delta \mathfrak{W}_{ij;k}$ in accordance with equation (104); and we find (again distinguishing the different cases and letting $i \neq j \neq k$)

$$-2\delta\mathfrak{W}_{ij;k} = 2B_{ij;k} V_{ijk} \tag{119}$$

$$-2\delta\mathfrak{W}_{ij;j} = a_j^2 B_{iij} V_{iii} + (2B_{ij} + 3a_j^2 B_{ijj}) V_{ijj} + a_j^2 B_{ijk} V_{ikk} - a_j^2 B_{ij} V_i , \tag{120}$$

$$-2\delta\mathfrak{W}_{ii;j} = -a_i^2 A_{ij;j} V_{jjj} - a_i^2 A_{ik;j} V_{jkk} + (2B_{ii;j} - a_i^2 A_{ii;j}) V_{iij} + a_i^2 (A_i + a_j^2 A_{ij}) V_j , \tag{121}$$

$$-2\delta\mathfrak{W}_{ii;i} = [2(B_{ii} + 2a_i^2 B_{iii}) - a_i^2 A_{ii;i}] V_{iii} + a_i^2 (2B_{iij} - A_{ij;i}) V_{ijj} + a_i^2 (2B_{iik} - A_{ik;i}) V_{ikk} + a_i^2 (A_i + a_i^2 A_{ii} - 2B_{ii}) V_i , \tag{122}$$

where, for brevity, we have introduced the further abbreviations

$$A_{ij;k} = A_{ij} + a_k^2 A_{ijk} \quad \text{and} \quad B_{ij;k} = B_{ij} + a_k^2 B_{ijk} . \tag{123}$$

If the Lagrangian displacement is referred in a frame in which the center of mass is at rest, then we can set (as we have seen in Sec. II)

$$V_i = 0 \tag{124}$$

in equations (119)–(122). The result is that in all cases the $\delta\mathfrak{W}_{ij;k}$'s are expressible in terms of the symmetric virials only. Equations (76)–(79) and (101)–(103), therefore, involve only these virials.

In working with formulae (119)–(122) it is convenient to have in an explicit form the coefficients of the virials in the expansions of the different $\delta\mathfrak{W}_{ij;k}$'s. Table 1 provides these coefficients.

Now equations (76)–(79) and (101)–(103) involve, in addition to particular $\delta\mathfrak{W}_{ij;k}$'s, certain linear combinations of them which we have denoted by δS_{ijj} (see eq. [62]). It will therefore be convenient to have explicitly the coefficients of the virials in the expansions of these δS_{ijj}'s also. The required coefficients can be obtained by suitably combining the ones given in Table 1. Thus, considering δS_{311}, we have

$$\delta S_{311} = -4\delta\mathfrak{W}_{31;1} - 2\delta\mathfrak{W}_{11;3} + 2\delta\mathfrak{W}_{33;3}\ ; \tag{125}$$

and expressing this in the form

$$\delta S_{311} = \langle 311|113\rangle V_{113} + \langle 311|223\rangle V_{223} + \langle 311|333\rangle V_{333}\,, \tag{126}$$

we find

$$\langle 311|113\rangle = 3[B_{11} + B_{13} + (2a_1^2 + a_3^2)B_{113} - a_3^2 B_{133}]\,,$$

$$\langle 311|223\rangle = (a_1^2 + 2a_3^2)B_{123} - 3a_3^2 B_{233}\,, \tag{127}$$

$$\langle 311|333\rangle = (a_1^2 + 2a_3^2)B_{133} - 5a_3^2 B_{333} - 2B_{33}\,.$$

TABLE 1

THE COEFFICIENTS OF THE VIRIALS IN THE EXPANSIONS OF $\delta\mathfrak{W}_{ij;k}$

Element	V_{111}	V_{122}	V_{133}
$-2\delta\mathfrak{W}_{11;1}$	$2(B_{11}+2a_1^2B_{111})-a_1^2A_{11;1}$	$a_1^2(2B_{112}-A_{12;1})$	$a_1^2(2B_{113}-A_{13;1})$
$-2\delta\mathfrak{W}_{22;1}$ . . .	$-a_2^2A_{12;1}$	$2B_{22;1}-a_2^2A_{22;1}$	$-a_2^2A_{23;1}$
$-2\delta\mathfrak{W}_{33;1}$	$-a_3^2A_{13;1}$	$-a_3^2A_{23;1}$	$2B_{33;1}-a_3^2A_{33;1}$
$-2\delta\mathfrak{W}_{12;2}$	$+a_2^2B_{112}$	$2B_{12}+3a_2^2B_{122}$	$a_2^2B_{123}$
$-2\delta\mathfrak{W}_{13;3}$ . . .	$+a_3^2B_{113}$	$a_3^2B_{123}$	$2B_{13}+3a_3^2B_{133}$

Element	V_{222}	V_{211}	V_{233}
$-2\delta\mathfrak{W}_{22;2}$	$2(B_{22}+2a_2^2B_{222})-a_2^2A_{22;2}$	$a_2^2(2B_{122}-A_{12;2})$	$a_2^2(2B_{223}-A_{23;2})$
$-2\delta\mathfrak{W}_{11;2}$	$-a_1^2A_{12;2}$	$2B_{11;2}-a_1^2A_{11;2}$	$-a_1^2A_{13;2}$
$-2\delta\mathfrak{W}_{33;2}$	$-a_3^2A_{32;2}$	$-a_3^2A_{13;2}$	$2B_{33;2}-a_3^2A_{33;2}$
$-2\delta\mathfrak{W}_{12;1}$	$+a_1^2B_{122}$	$2B_{12}+3a_1^2B_{112}$	$a_1^2B_{123}$
$-2\delta\mathfrak{W}_{23;3}$	$+a_3^2B_{223}$	$a_3^2B_{123}$	$2B_{23}+3a_3^2B_{233}$

Element	V_{333}	V_{113}	V_{223}
$-2\delta\mathfrak{W}_{11;3}$	$-a_1^2A_{13;3}$	$2B_{11;3}-a_1^2A_{11;3}$	$-a_1^2A_{12;3}$
$-2\delta\mathfrak{W}_{22;3}$	$-a_2^2A_{23;3}$	$-a_2^2A_{12;3}$	$2B_{22;3}-a_2^2A_{22;3}$
$-2\delta\mathfrak{W}_{33;3}$	$2(B_{33}+2a_3^2B_{333})-a_3^2A_{33;3}$	$a_3^2(2B_{331}-A_{31;3})$	$a_3^2(2B_{332}-A_{32;3})$
$-2\delta\mathfrak{W}_{13;1}$	$+a_1^2B_{133}$	$2B_{13}+3a_1^2B_{311}$	$a_1^2B_{123}$
$-2\delta\mathfrak{W}_{23;2}$	$+a_2^2B_{233}$	$a_2^2B_{123}$	$2B_{23}+3a_2^2B_{322}$
$-2\delta\mathfrak{W}_{ij;k}$	$2B_{ij;k}V_{ijk}$ $(i\neq j\neq k)$		

The coefficients in the expansions of the other δS_{ijj}'s can be obtained by cyclically permuting the indices in equations (127).

We may also note here that, according to equations (95), (98), and (119),

$$\delta S_{123} = 2[B_{12} + B_{23} + B_{31} + (a_1^2 + a_2^2 + a_3^2)B_{123}]V_{123} . \tag{128}$$

Again, for convenience in use, we list in Table 2 the coefficients of the virials in the expansions of the different δS_{ijj}'s and $\delta \mathfrak{W}_{ij;k}$'s which occur in equations (76)–(79) and (101)–(103).

VI. THE DIVERGENCE CONDITION

Equations (76)–(79) provide four relations among the six even virials; and equations (101)–(103) provide three relations among the four odd virials. Clearly, we need to supplement equations (76)–(79) by two additional relations and equations (101)–(103) by one additional relation among the respective sets of virials. These additional relations can be obtained by making use of the solenoidal character of the Lagrangian displacement required by the incompressibility of the fluid. Thus, consider

$$\begin{aligned} \frac{V_{111}}{a_1^2}+\frac{V_{122}}{a_2^2}+\frac{V_{133}}{a_3^2} &= \int_V \rho \left[\frac{3}{a_1^2}\,\xi_1 x_1^2+\frac{1}{a_2^2}(\xi_1 x_2^2+2\,\xi_2 x_1 x_2)+\frac{1}{a_3^2}(\xi_1 x_3^2+2\,\xi_3 x_1 x_3)\right] d x \\ &= \int_V \rho\, \boldsymbol{\xi}\cdot \text{grad}\left[x_1\left(\frac{x_1^2}{a_1^2}+\frac{x_2^2}{a_2^2}+\frac{x_3^2}{a_3^2}\right)\right] d x . \end{aligned} \tag{129}$$

By an integration by parts, the last integral becomes (since ρ is a constant)

$$\begin{aligned} \int_S \rho x_1\left(\frac{x_1^2}{a_1^2}+\frac{x_2^2}{a_2^2}+\frac{x_3^2}{a_3^2}\right)\boldsymbol{\xi}\cdot d\boldsymbol{S}-\int_V \rho x_1\left(\frac{x_1^2}{a_1^2}+\frac{x_2^2}{a_2^2}+\frac{x_3^2}{a_3^2}\right) \text{div}\ \boldsymbol{\xi}\, d x \\ = \int_S \rho x_1 \boldsymbol{\xi}\cdot d\boldsymbol{S}, \end{aligned} \tag{130}$$

since $x_1^2/a_1^2 + x_2^2/a_2^2 + x_3^2/a_3^2 = 1$ on the surface ($\boldsymbol{S}$) of the ellipsoid and div $\boldsymbol{\xi} = 0$ in the interior. On transforming the last surface integral back into a volume integral by Gauss's theorem, we obtain

$$\begin{aligned} \frac{V_{111}}{a_1^2}+\frac{V_{122}}{a_2^2}+\frac{V_{133}}{a_3^2} &= \int_V \rho\ \text{div}(x_1 \boldsymbol{\xi})\, d x \\ &= \int_V \rho\, \xi_1 d x+\int_V \rho x_1\ \text{div}\ \boldsymbol{\xi} d x = V_1 . \end{aligned} \tag{131}$$

But we have seen in Section II that we can set $V_i = 0$. Accordingly,

$$\frac{V_{111}}{a_1^2}+\frac{V_{122}}{a_2^2}+\frac{V_{133}}{a_3^2}=0 .$$

Similarly,

$$\frac{V_{222}}{a_2^2}+\frac{V_{233}}{a_3^2}+\frac{V_{211}}{a_1^2}=0 ,$$

and

$$\frac{V_{333}}{a_3^2}+\frac{V_{311}}{a_1^2}+\frac{V_{322}}{a_2^2}=0 , \tag{134}$$

Equations (132)–(134) provide the three needed additional relations.

TABLE 2

THE COEFFICIENTS OF THE VIRIALS IN THE EXPANSIONS OF δS_{ij} AND $\delta \mathfrak{W}_{ij;k}$

a) FOR THE EVEN EQUATIONS (76)–(79)

Element	V_{111}	V_{122}	V_{133}
δS_{122}	$(a_2^2+2a_1^2)B_{112}-5a_1^2B_{111}-2B_{11}$	$3[B_{22}+B_{12}+(2a_2^2+a_1^2)B_{122}-a_1^2B_{112}]$	$(a_2^2+2a_1^2)B_{123}-3a_1^2B_{113}$
δS_{133}	$(a_3^2+2a_1^2)B_{113}-5a_1^2B_{111}-2B_{11}$	$(a_3^2+2a_1^2)B_{123}-3a_1^2B_{112}$	$3[B_{33}+B_{13}+(2a_3^2+a_1^2)B_{133}-a_1^2B_{113}]$
$-2\delta\mathfrak{W}_{12;2}$	$a_2^2B_{112}$	$2B_{12}+3a_2^2B_{122}$	$a_2^2B_{123}$
$-2\delta\mathfrak{W}_{13;3}$	$a_3^2B_{113}$	$a_3^2B_{123}$	$2B_{13}+3a_3^2B_{133}$

Element	V_{222}	V_{112}	V_{233}
δS_{112}	$(a_1^2+2a_2^2)B_{122}-5a_2^2B_{222}-2B_{22}$	$3[B_{11}+B_{12}+(2a_1^2+a_2^2)B_{112}-a_2^2B_{122}]$	$(a_1^2+2a_2^2)B_{123}-3a_2^2B_{223}$
δS_{233}	$(a_3^2+2a_2^2)B_{223}-5a_2^2B_{222}-2B_{22}$	$(a_3^2+2a_2^2)B_{123}-3a_2^2B_{122}$	$3[B_{33}+B_{23}+(2a_3^2+a_2^2)B_{233}-a_2^2B_{223}]$
$-2\delta\mathfrak{W}_{12;1}$	$a_1^2B_{122}$	$2B_{12}+3a_1^2B_{112}$	$a_1^2B_{123}$
$-2\delta\mathfrak{W}_{23;3}$	$a_3^2B_{223}$	$a_3^2B_{123}$	$2B_{23}+3a_3^2B_{233}$

b) FOR THE ODD EQUATIONS (101)–(103)

Element	V_{333}	V_{113}	V_{223}
δS_{113}	$(a_1^2+2a_3^2)B_{133}-5a_3^2B_{333}-2B_{33}$	$3[B_{11}+B_{13}+(2a_1^2+a_3^2)B_{113}-a_3^2B_{133}]$	$(a_1^2+2a_3^2)B_{123}-3a_3^2B_{233}$
δS_{223}	$(a_2^2+2a_3^2)B_{233}-5a_3^2B_{333}-2B_{33}$	$(a_2^2+2a_3^2)B_{123}-3a_3^2B_{133}$	$3[B_{22}+B_{23}+(2a_2^2+a_3^2)B_{223}-a_3^2B_{233}]$
$-2\delta\mathfrak{W}_{13;1}$	$a_1^2B_{133}$	$2B_{13}+3a_1^2B_{113}$	$a_1^2B_{123}$
$-2\delta\mathfrak{W}_{23;2}$	$a_2^2B_{233}$	$a_2^2B_{123}$	$2B_{23}+3a_2^2B_{223}$

$-2\delta\mathfrak{W}_{13;2} = 2(B_{13} + a_2^2B_{123})V_{123}$; $-2\delta\mathfrak{W}_{23;1} = 2(B_{23} + a_1^2B_{123})V_{123}$; $\delta S_{123} = 2[B_{12} + B_{23} + B_{31} + (a_1^2 + a_2^2 + a_3^2)B_{123}]V_{123}$.

VII. THE CHARACTERISTIC FREQUENCIES OF OSCILLATION OF THE JACOBI ELLIPSOIDS BELONGING TO THE THIRD HARMONICS

Equations (76)–(79), supplemented by the two divergence conditions (132) and (133), provide a system of six linear homogeneous equations for the six even virials; and the vanishing of the determinant of the system leads (as can be verified) to a characteristic equation of degree *seven* in λ^2. For reasons which have been explained in Section IV*a*, the characteristic equation obtained in the manner described will allow a spurious root $\lambda^2 = -4\Omega^2$ (since the system considered includes eqs. [78] and [79]).[3] The occurrence of the spurious root can be avoided by using equation (84) and its "conjugate" (obtained by writing $-i$ wherever i occurs explicitly) in place of equations (78) and (79) (in which

TABLE 3

THE SQUARES OF THE CHARACTERISTIC FREQUENCIES BELONGING TO THE SIX EVEN MODES

(σ^2 Is Listed in the Unit $\pi G\rho$)

$\cos^{-1} a_3/a_1$	σ_1^2	σ_2^2	σ_3^2	σ_4^2	σ_5^2	σ_6^2
54°3576	+0 005994	0 10863	1 1148	2 0844	2 4120	2 8523
56	+ 005949	.10743	1 1119	2 0784	2 3968	2 8681
57	+ 005873	10553	1 1076	2 0657	2 3783	2 8908
58	+ 005764	10271	1 1010	2 0492	2 3517	2 9212
59	+ 005621	09901	1 0921	2 0276	2 3194	2 9584
60	+ 005441	09442	1 0812	1 9983	2 2884	2 9984
61	+ 005228	08898	1 0679	1 9533	2 2684	3 0380
62	+ .004979	.08271	1 0524	1 9042	2 2464	3 0805
63	+ 004694	07566	1 0346	1 8455	2 2298	3 1229
64	+ 004373	06788	1 0146	1 7824	2 2138	3 1662
65	+ 004009	05943	0 9920	1 7107	2 2025	3 2087
66	+ 003593	05039	0 9673	1 6384	2 1888	3 2519
67	+ .003100	04087	0 9404	1 5615	2 1774	3 2944
68	+ 002468	03102	0 9110	1 4812	2 1666	3 3365
69	+ 001504	02115	0 8794	1 3992	2 1559	3 3782
69 8166	0	01360	0 8519	1 3306	2 1474	3 4118
70	− 000548	01211	0 8456	1 3151	2 1456	3 4192
71	− 006002	00634	0 8096	1 2295	2 1355	3 4598
72	− .01500	00415	0 7713	1 1437	2 1252	3 4997
75	− 04471	00205	0 6446	0 8854	2 0953	3 6144
77	−0 06172	0 00133	0 5517	0 7178	2 0763	3 6882

case the resulting characteristic equation will only be of degree *six* in λ^2). However, it is more convenient to use equations (78) and (79), since they avoid manipulation with complex numbers. In all cases, it is clear that there are exactly *six even modes* of oscillation belonging to six different characteristic roots.

With the constants of the Jacobi ellipsoids given in Paper I (Appendix I, Table 2) the characteristic equation for λ^2 which follows from equations (76)–(79), (132), and (133) was solved for all of its roots, for twenty-one different members of the Jacobian sequence (including the first member, which is a Maclaurin spheroid). The results of the calculations are given in Table 3.

Equations (101)–(103), supplemented by the divergence condition (134), provide a system of four linear homogeneous equations for the four odd virials; and the vanishing of the determinant of the system leads (as can be verified) to a characteristic equation of degree five in λ^2. There are thus *five odd modes*. The characteristic roots belonging to

[3] It can be shown that all the others are genuine roots.

these odd modes were also calculated for the same twenty-one members of the Jacobian sequence. And the results of the calculations are given in Table 4.

In Figures 1, 2, and 3 the variation of the squares of the different characteristic frequencies along the Jacobian sequence is illustrated. Particular interest attaches to the "lowest" mode which we have designated by "1"; it is the one with respect to which the

TABLE 4

THE SQUARES OF THE CHARACTERISTIC FREQUENCIES BELONGING TO THE FIVE ODD MODES

(σ^2 Is Listed in the Unit $\pi G\rho$)

$\cos^{-1} a_3/a_1$	σ_7^2	σ_8^2	σ_9^2	σ_{10}^2	σ_{11}^2
54°3576	0 33494	0 44920	1 1288	2 6389	2 9145
56	.33444	44811	1 1254	2 6260	2 9287
57.	.33365	44637	1 1198	2 6075	2 9492
58	33247	44382	1 1116	2 5838	2 9758
59	.33087	44043	1 1007	2 5570	3 0066
60	32883	43621	1 0871	2 5283	3 0402
61	.32632	43119	1 0707	2 4987	3 0759
62	.32331	42535	1 0516	2 4687	3 1129
63	.31976	41873	1 0298	2 4388	3 1510
64	31561	.41134	1 0052	2 4093	3 1896
65	31082	40321	0 9780	2 3804	3 2288
66	.30533	39439	0 9482	2 3523	3 2682
67	29906	38491	0 9158	2 3250	3 3078
68	29198	37479	0 8810	2 2987	3 3474
69	28397	36412	0 8440	2 2734	3 3869
69 8166	27674	35498	0 8121	2 2535	3 4190
70	27505	35285	0 8048	2 2491	3 4262
71.	26513	34103	0 7636	2 2259	3 4652
72	25417	32872	0 7207	2 2038	3 5040
75	21546	28797	0 5835	2 1439	3 6178
77	0.18531	0 25704	0.4875	2 1093	3 6887

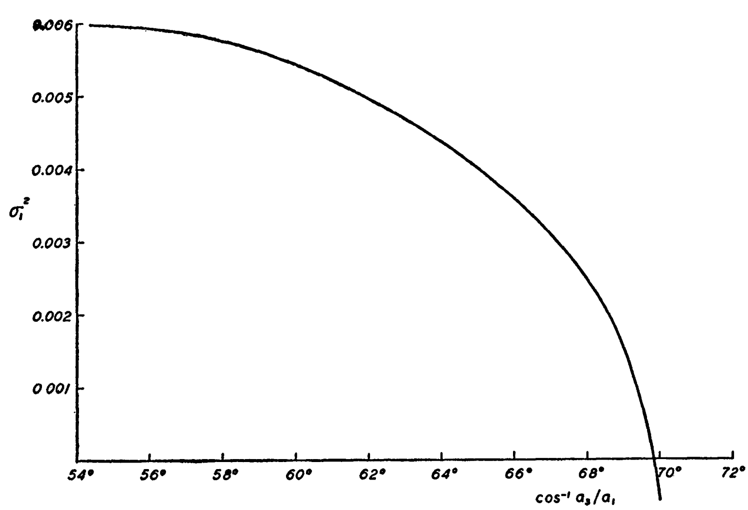

FIG. 1.—The square of the characteristic frequency belonging to the mode with respect to which the Jacobi ellipsoid becomes unstable at the point of bifurcation.

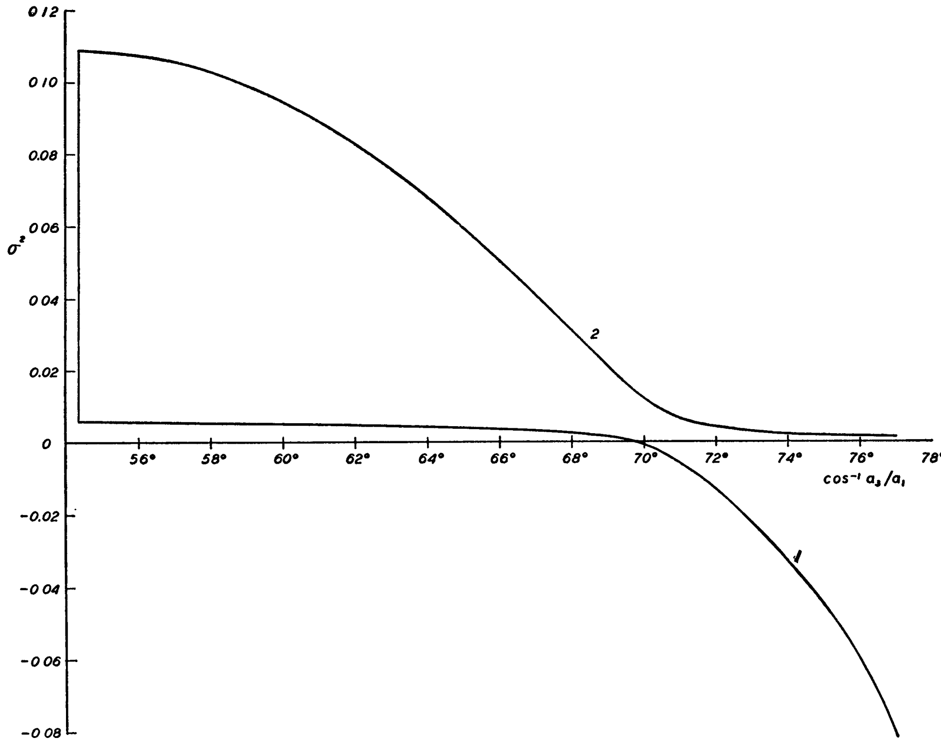

FIG. 2.—The square of the characteristic frequencies belonging to the two lowest modes

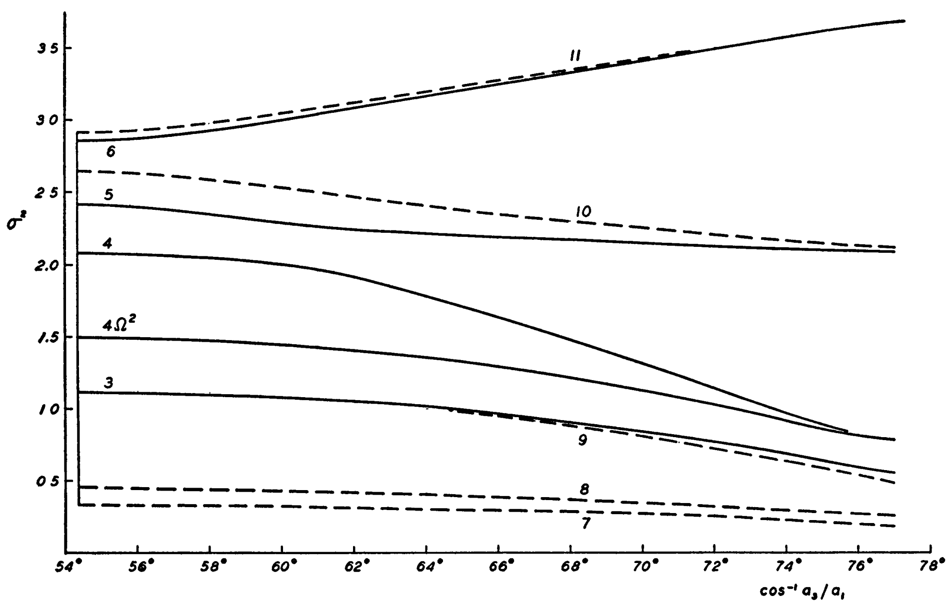

FIG. 3.—The square of the characteristic frequencies of the remaining nine modes. The curves are labeled to correspond with the enumeration in Tables 3 and 4, and the curves belonging to the odd modes are dashed.

Jacobi ellipsoids become unstable at the point of bifurcation. There is, however, nothing "spectacular" about its behavior prior to instability; its behavior, after instability, relative to the mode which we have designated by "2" is more noteworthy (see Fig. 2).

We are greatly indebted to Miss Donna Elbert for having carried out the numerical solution of all the characteristic equations. By her efforts the problem at which so many have labored for so long is finally completed.

The work of the first author was supported in part by the Office of Naval Research under contract Nonr-2121(24) with the University of Chicago. The work of the second author was supported in part by the United States Air Force under contract AF-49(638)-42 monitored by the Air Force Office of Scientific Research of the Air Research and Development Command.

APPENDIX I

The Oscillations of the Jacobi Ellipsoids Belonging to the Second Harmonics

The characteristic frequencies of oscillation of the Jacobi ellipsoid belonging to the second harmonics have been derived in an earlier paper (Lebovitz 1961, Sec. VIII, eqs. [253] and [269]); but the information on the constants of the Jacobi ellipsoid available at that time was insufficient to permit their evaluation. With the data given in Paper I (Appendix I, Tables 1 and 2) the frequencies can now be calculated and their variation along the Jacobian sequence ascertained.

Of the five principal modes of oscillation belonging to the second harmonics, there is one which is neutral throughout the sequence; this is the same one that starts as a neutral mode at the point of bifurcation with the Maclaurin sequence. The characteristic roots belonging to the remaining four modes occur in pairs as roots of quadratic equations. The equation determining the first pair is, in our present notation,

$$\sigma^2(\sigma^2+\Omega^2-4B_{13})(\sigma^2+\Omega^2-4B_{23})-4\Omega^2(\sigma^2-2B_{13})(\sigma^2-2B_{23})=0\,. \quad \text{(AI, 1)}$$

Besides the root $\sigma^2=\Omega^2$, which the equation clearly allows, we have

$$\sigma^2=2B_{13}+2B_{23}+\tfrac{1}{2}\Omega^2\pm[(2B_{13}+2B_{23}+\tfrac{1}{2}\Omega^2)^2-16B_{13}B_{23}]^{1/2}\,. \quad \text{(AI, 2)}$$

The equation determining the second pair of roots is

$$\left\|\begin{matrix}\Omega^2+3B_{11}-B_{13}-\frac{1}{2}\sigma^2 & B_{12}-B_{23} & \Omega^2+3(B_{11}-B_{33})+B_{12}-B_{23}\\ B_{12}-B_{13} & \Omega^2+3B_{22}-B_{23}-\frac{1}{2}\sigma^2 & \Omega^2+3(B_{22}-B_{33})+B_{12}-B_{13}\\ \dfrac{1}{a_1^2} & \dfrac{1}{a_2^2} & \dfrac{1}{a_1^2}+\dfrac{1}{a_2^2}+\dfrac{1}{a_3^2}\end{matrix}\right\|=0\,. \quad \text{(AI, 3)}$$

The roots determined in accordance with equations (AI, 2) and AI, 3) are listed in Table 5.

APPENDIX II

The Independence of the Normal Modes on the First-Order Virials

In Section II, physical arguments were presented for putting the first-order virials equal to zero. The independence of the derived characteristic roots on the choice of the first-order virials can be established more formally.

The eighteen equations (26)–(43) which are provided by the third-order virial equation are eighteen linear homogenous equations for the eighteen third-order virials $V_{i;jk}$ and the three first-order virials V_i. The equations are of the forms

$$\sum_{l;mn}\langle i;jk\,|\,l;mn\rangle V_{l;mn}+\sum_{s}\langle i;jk\,|\,s\rangle V_s=0\,. \quad \text{(AII, 1)}$$

Actually, equations (26)–(43) involve the $\delta\Pi_k$'s. In the incompressible case, these $\delta\Pi_k$'s are eliminated; but the remaining fifteen equations are supplemented by three divergence conditions which are again linear equations of the same form. In the more general case, when the gas laws appropriate to adiabatic changes are applicable, the relation (cf. Chandrasekhar and Lebovitz 1963)

$$\delta\Pi_k = (\gamma - 1)\int_V \xi_j \frac{\partial p}{\partial x_j} x_k d\boldsymbol{x} + \gamma \int_V p\, \xi_k d\boldsymbol{x} \qquad \text{(AII, 2)}$$

(where γ denotes the ratio of the specific heats and p is the pressure in the unperturbed configuration) enables its expression in terms of the virials; so that in all cases we have eighteen equations of the form (AII, 1).

TABLE 5

THE SQUARES OF THE CHARACTERISTIC FREQUENCIES BELONGING TO THE SECOND HARMONICS*

(σ^2 Is Listed in the Unit $\pi G\rho$)

$\cos^{-1} a_3/a_1$	σ_1^2	σ_2^2	σ_3^2	σ_4^2
54°35762....	1.99799	0 64281	1 54602	1.49206
55.........	1 99786	.64263	1 55090	1 48638
56	1 99780	.64146	1 57474	1 45946
57	1.99753	.63925	1 60278	1.42528
58...... ..	1 99712	.63600	1 63264	1 38634
59 . . .	1.99659	.63166	1 66013	1 34683
60. .	1 99595	62622	1 68259	1 30945
61....	1 99520	61964	1 70699	1 26726
62 .	1 99436	61189	1 72918	1 22445
63 ..	1 99345	.60296	1 75219	1 17808
64..	1 99248	59281	1 77302	1 13123
65	1 99149	.58141	1 79270	1 08300
66	1 99048	.56875	1 81124	1 03348
67........ .	1 98947	55479	1 82866	0 98282
68	1 98850	.53951	1 84503	0 93110
69	1 98759	.52289	1 86036	0 87852
69 8166	1 98691	50832	1 87213	0 83505
70...... ..	1 98677	50493	1 87468	0 82523
71	1 98606	48560	1 88805	0 77140
72...... ..	1 98548	46492	1 90047	0 71723
75..... ...	1 98481	.39488	1 93250	0 55482
77	1 98526	34190	1 94999	0 44906
80..... ...	1.98765	25466	1 97076	0 29955
83	1 99167	0 16219	2 02252	0 13164

* The roots σ_1^2 and σ_2^2 are those derived from eq (AI, 2), while the roots σ_3^2 and σ_4^2 are derived from eq (AI, 3)

The eighteen equations (AII, 1) must be supplemented by the three equations (15) and (16) governing the first-order virials. These equations are of the form

$$\sum_s \langle i \mid s \rangle V_s = 0 . \qquad \text{(AII, 3)}$$

The secular equation for the complete set of equations provided by (AII, 1) and (AII, 3) is given by

$$\|\langle i;jk \mid l;mn \rangle\| \cdot \|\langle i \mid s \rangle\| = 0 . \qquad \text{(AII, 4)}$$

The required characteristic roots are, therefore, the roots of the two independent secular equations

$$\|\langle i;jk|l;mn\rangle\| = 0 \quad \text{and} \quad \|\langle i|s\rangle\| = 0 . \qquad \text{(AII, 5)}$$

The first of these two secular equations is the same as one would obtain by setting the V_i's equal to zero (as we have done); and the roots of the second equation (namely, $\sigma^2 = 0$ and $\sigma^2 = \Omega^2$) can be ignored, since they are in no way dependent on the construction or the constitution of the particular system which may be under consideration.

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