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ON THE OSCILLATIONS AND THE STABILITY OF ROTATING GASEOUS MASSES. II. THE HOMOGENEOUS, COMPRESSIBLE MODEL

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ABSTRACT

The pulsation frequencies of rotating, gaseous masses of uniform density, i.e., of the Maclaurin spheroids, are found as functions of the angular momentum M and the ratio of the specific heats γ. Numerical calculations for the pulsation frequencies and normal modes are given for $\gamma = 1.3$, $\frac{4}{3}$, 1.4, 1.5, 1.6, and $\frac{5}{3}$. One finds that the value of γ at which dynamical instability sets in is reduced from $\gamma = \frac{4}{3}$ by rotation. One also finds that, when $\gamma = 1.6$, the normal modes of oscillation one obtains in the limit $M \to 0$ are both very far from being radial.

I. INTRODUCTION

The theory of the radial pulsations of spherical, gaseous masses, in its simplest form, predicts the following approximate formula for the frequency σ of the fundamental mode of pulsation:

$$\sigma^2 = (3\gamma - 4)\frac{|\mathfrak{W}|}{I}, \tag{1}$$

where $\mathfrak{W}$ is the gravitational energy and I the moment of inertia of the spherical configuration of equilibrium about which the pulsations (assumed small in amplitude) take place; the ratio of the specific heats, γ, appears because of the assumption that the pulsations take place adiabatically. This formula is known to be a good approximation if the central condensation is not too high (Ledoux and Pekeris 1941); it becomes exact in the limit in which the density is uniform.[1]

The assumption that the equilibrium configuration is spherical implies that rotation is absent. We now wish to drop this assumption and to find the effect of rotation on the pulsation frequency. For this purpose, one may ask whether there is a generalization of equation (1), valid in the presence of rotation, that reduces to equation (1) in the limit of vanishing rotation. The answer is in the affirmative (Chandrasekhar and Lebovitz 1962*a*, eq. [88]; this paper will be referred to as "Paper I"), and this result provides the solution to the problem in principle.

In practice, however, it is necessary to know the distribution of mass throughout the equilibrium configuration to find the pulsation frequencies, for only then can one evaluate $\mathfrak{W}$, I, and the further coefficients that appear in the formulae of Paper I. Such information is largely unavailable for rotating configurations, but there are exceptions: (1) the mass distribution in slowly rotating polytropes has been worked out (Chandrasekhar 1933); and (2) if one *assumes* that the mass is distributed uniformly, the equilibrium configurations are the well-known Maclaurin spheroids. This paper will be devoted to finding the oscillation frequencies of the Maclaurin spheroids.[2] The oscillations of slowly rotating polytropes are treated in the following paper (Chandrasekhar and Lebovitz 1962*c*).

II. THE VIRIAL EQUATIONS

Let the equilibrium figure, a Maclaurin spheroid, be rotating about the x_3-axis, and let the semiaxes in the directions of the x_1-, x_2-, and x_3-co-ordinate axes be a_1, a_2 ($= a_1$), and

[1] A gaseous configuration of uniform density will be called a "homogeneous, compressible model."

[2] The Maclaurin spheroids are not the only such models: for sufficiently large angular momentum the Jacobi ellipsoids provide another.

a_3. In order to find the oscillation frequencies, we shall use the tensor virial equations, as adapted to treat small perturbations of the equilibrium configuration (Paper I, eq. [13]); they may be written

$$\frac{d^2}{dt^2}\int_V \rho x_i \xi_j dx = 2\Omega\epsilon_{jl3}\frac{d}{dt}\int_V \rho x_i \xi_l dx + \Omega^2\int_V \rho(x_i\xi_j + x_j\xi_i)\,dx$$
$$-\Omega^2\delta_{j3}\int_V \rho(x_3\xi_i + x_i\xi_3)\,dx + \delta_{ij}(\gamma-1)\int_V \xi_l\frac{\partial p}{\partial x_l}\,dx - \int_V \rho\xi_l\frac{\partial \mathfrak{B}_{ij}}{\partial x_l}\,dx. \tag{2}$$

The results and equations of Paper I were obtained by replacing the Lagrangian displacement $\boldsymbol{\xi}$ appearing in equation (2) with the linear form

$$\xi_j = X_{jk}x_k e^{\lambda t}. \tag{3}$$

The justification for the substitution (3) lies, in part, in that the results so obtained become exact in the limit of uniform density and should therefore be good approximations if the central condensation is not too high.

If the substitution (3) is used in equation (2), the result is (Paper I, eq. [17])

$$\lambda^2 X_{jl}I_{li} = 2\lambda\Omega\epsilon_{jl3}X_{lm}I_{mi} + \Omega^2(X_{jl}I_{li} + X_{il}I_{lj})$$
$$-\Omega^2\delta_{j3}(X_{3l}I_{li} + X_{il}I_{l3}) + JX_{rr}\delta_{ij} - X_{lr}\mathfrak{W}_{rl;ij}, \tag{4}$$

where

$$J = -(\gamma-1)\int_V p\,dx = (\gamma-1)\mathfrak{W}_{33};$$

and the precise assertion is that the characteristic frequencies[3] determined by equation (4) become exact if the coefficients (I_{ij}, $\mathfrak{W}_{pq;ij}$, and J) are taken to be those appropriate to the homogeneous, compressible model. It is clearly important to prove this assertion for two reasons: (1) to establish the validity of the results of Paper I and (2) to emphasize that the results of the present paper, in which we use the equations of Paper I, are exact (i.e., involve no assumption as to the nature of $\boldsymbol{\xi}$, the Lagrangian displacement). We now turn to this proof.

III. THE EXACTNESS OF THE RESULTS FOR THE HOMOGENEOUS, COMPRESSIBLE MODEL

If a tensor N_{ij} is defined by the equation (cf. Lebovitz 1961, eq. [65]; this paper will be referred to as "Paper II")

$$N_{ij} = \int_V \rho x_i \xi_j dx, \tag{5}$$

then all the terms of equation (2), with the exception of the last two, manifestly involve the tensor N_{ij} linearly. Further, if $\partial p/\partial x_l$ and $\partial\mathfrak{B}_{ij}/\partial x_l$ should be linear in the co-ordinates, the last two terms would also be linear combinations of the N_{ij}'s. But, for the homogeneous, compressible model, these quantities are, in fact, linear in the co-ordinates: the equilibrium pressure gradient is

$$\frac{\partial p}{\partial x_l} = \rho\frac{\partial}{\partial x_l}\left[\mathfrak{B} + \tfrac{1}{2}|\boldsymbol{\Omega}\times\boldsymbol{x}|^2\right], \tag{6}$$

where (cf. Chandrasekhar and Lebovitz 1962*b*, eq. [47]; this paper will be referred to as "Paper III")

$$\mathfrak{B} = \pi G\rho a_1^2 a_3[I - A_1(x_1^2 + x_2^2) - A_3 x_3^2]; \tag{7}$$ [4]

[3] The frequencies are given by $\sigma = i\lambda$, where $i = \sqrt{-1}$.

[4] In this and subsequent formulae, the constants A_i and A_{ij} appearing in the expressions for $\mathfrak{B}$ and $\mathfrak{B}_{ij}$ are those defined in Paper III; this represents a departure from the notation of Paper II.

and for $\mathfrak{B}_{ij}$ we have the formula (Paper III, eq. [51])

$$\mathfrak{B}_{ij} = \pi G\rho a_1^2 a_3 \left[2x_i x_j (A_j - a_i^2 A_{ij}) + a_i^2 \delta_{ij} \left(A_i - \sum_{l=1}^{3} A_{il} x_l^2 \right) \right]. \tag{8}$$

In equation (8) and in the rest of this paper the summation convention is suspended.

If the foregoing formulae are used in equation (2), the result is

$$\begin{aligned} \frac{d^2}{dt^2} N_{ij} = & \sum_{l=1}^{3} 2\Omega\epsilon_{jl3} \frac{dN_{il}}{dt} + \Omega^2 (N_{ij} + N_{ji}) - \Omega^2 \delta_{j3} (N_{3i} + N_{i3}) \\ & - \delta_{ij} (\gamma - 1) [(N_{11} + N_{22}) (2\pi G\rho a_1^2 a_3 A_1 - \Omega^2) + 2\pi G\rho a_1^2 a_3 N_{33}] \\ & - \left[2 (A_j - a_i^2 A_{ij}) (N_{ij} + N_{ji}) - 2\delta_{ij} \sum_{l=1}^{3} a_i^2 A_{il} N_{ll} \right] (\pi G\rho a_1^2 a_3). \end{aligned} \tag{9}$$

Equation (9), which is exact (i.e., no assumption concerning the nature of ξ has been made), represents a system of nine linear differential equations with constant coefficients in nine unknowns; its solutions are therefore of the form

$$N_{ij}(t) = N_{ij}(0) e^{\lambda t}. \tag{10}$$

The substitution (10) reduces equation (9) to a system whose characteristic equation determines the frequencies; these frequencies are unaltered by the linear transformation

$$N_{ij}(0) = \sum_{m=1}^{3} X_{jm} I_{mi} = X_{ji} I_{ii}. \tag{11}$$

This substitution puts all but the last two terms of equation (9) in manifest agreement with the corresponding terms of equation (4). We wish to show that the last two terms agree also.

The fourth term on the right-hand side of equation (9) becomes

$$- \delta_{ij}(\gamma - 1)[(X_{11} + X_{22})(2\pi G\rho a_1^2 a_3 A_1 - \Omega^2) I_{11} + 2\pi G\rho a_1^2 a_3 A_3 I_{33}]; \tag{12}$$

and, since for the Maclaurin spheroid (Paper III, eq. [75])

$$(2\pi G\rho a_1^2 a_3 A_1 - \Omega^2) I_{11} = 2\pi G\rho a_1^2 a_3 A_3 I_{33}, \tag{13}$$

expression (12) becomes (Paper III, eq. [57] and Paper I, eq. [20])

$$- \delta_{ij} (\gamma - 1) (\pi G\rho a_1^2 a_3) 2A_3 I_{33} \sum_{l=1}^{3} X_{ll} = \delta_{ij} (\gamma - 1) \mathfrak{W}_{33} \sum_{l=1}^{3} X_{ll} = \delta_{ij} J \sum_{l=1}^{3} X_{ll}. \tag{14}$$

This is the same as the fourth term on the right-hand side of equation (4).

To verify that the final terms of equations (4) and (9) agree, we note that, according to equation (60) of Paper III together with equation (8) above,

$$\begin{aligned} - \sum_{r=1}^{3} \sum_{l=1}^{3} X_{lr} \mathfrak{W}_{rl;\,ij} = & - \pi G\rho a_1^2 a_3 \Big[2 (A_j - a_i^2 A_{ij}) (I_{ii} X_{ji} + I_{jj} X_{ji}) \\ & - 2a_i^2 \delta_{ij} \sum_{l=1}^{3} A_{il} I_{ll} X_{il} \Big]. \end{aligned} \tag{15}$$

This shows that the exact equation (9) is equivalent to the "approximate" equation (4).

IV. THE TRANSVERSE SHEAR MODES

The nine equations represented by equation (4) separate into smaller sets of equations. One such set consists of four equations for X_{13}, X_{31}, X_{23}, and X_{32} (Paper I, Sec. V), the other X_{ij}'s being set equal to zero to satisfy the remaining five equations. The corresponding normal modes represent a relative shearing of the northern and southern hemispheres and are therefore called "transverse shear modes." The equations governing them (Paper I, eq. [46]) can be shown, with the aid of the explicit formulae for the supermatrix elements (Paper III, eqs. [62]–[65]), to be the same as those already discussed in the incompressible case (Paper II, eqs. [141]–[144]). The oscillation frequencies have been tabulated (Paper II, Table 1) and have been found to correspond to stable oscillations.

V. THE TOROIDAL MODES

A second set, comprising two equations, yields the "toroidal modes," in which the motions are restricted to planes parallel to the equatorial plane. The equations, which involve only the combinations $(X_{11} - X_{22})$ and $(X_{12} + X_{21})$, yield characteristic roots (Paper I, eq. [76]) that can once again be shown to be identical with those found in the incompressible case (Paper II, eq. [189]). These are the modes that lead to neutral stability at the point of bifurcation where the Jacobi ellipsoids branch off from the Maclaurin spheroids (at an eccentricity $e = 0.8127$) and to instability when $e = 0.9529$. That these modes are unchanged in the homogeneous, compressible model means that the occurrence of neutral and unstable points along the sequence of Maclaurin spheroids is independent of the assumption that the fluid is intrinsically incompressible.

VI. THE PULSATION MODES

The transverse shear modes and the toroidal modes account for six of the nine modes and the associated characteristic values of the root λ^2 for six of the nine characteristic values that satisfy equation (4). Further, one of the equations represented by equation (4) is

$$\lambda^2(X_{21} - X_{12}) = -2\lambda\Omega(X_{11} + X_{22}) , \tag{16}$$

which can be satisfied by taking $\lambda^2 = 0$, $X_{12} = -X_{21} \neq 0$, and setting the remaining X_{ij}'s equal to zero (Paper I, eqs. [79] and [82]).

Seven modes are now accounted for. The remaining two are called "pulsation modes" because they represent the generalization to the case when rotation is present of the radial pulsation. They satisfy the equations (Paper I, eq. [87])

$$[\lambda^2 I_{11} + (\Omega^2 I_{11} - \alpha - \beta)](X_{11} + X_{22}) - 2\alpha X_{33} = 0 \tag{17}$$

and

$$-\alpha(X_{11} + X_{22}) + (\lambda^2 I_{33} - \beta)X_{33} = 0 , \tag{18}$$

where

$$\alpha = (\gamma - 1)\mathfrak{W}_{33} - \mathfrak{W}_{33;11} \quad \text{and} \quad \beta = (\gamma - 1)\mathfrak{W}_{33} - \mathfrak{W}_{33;33} . \tag{19}$$

Equations (17) and (18) are different from any of the equations in Paper II. Only in these equations does the effect of compressibility appear (in particular, through the coefficients α and β). They lead to (Paper I, eq. [88])

$$I_{11}I_{33}\lambda^4 - [\beta I_{11} + (\beta + \alpha - \Omega^2 I_{11})I_{33}]\lambda^2 - \Omega^2 I_{11}\beta + (\beta + 2\alpha)(\beta - \alpha) = 0 . \tag{20}$$

Let the roots of equation (20) be λ_1^2 and λ_2^2. There will be associated with these roots certain normal modes, which can be specified by the ratio of $X_{11} + X_{22}$ to X_{33}. Let

$$R = \frac{X_{11} + X_{22}}{X_{33}} = 2\,\frac{X_{11}}{X_{33}} \qquad (\text{since } X_{11} = X_{22}) \tag{21}$$

be this ratio; then two numbers, R_1 and R_2, specify the normal modes that correspond to λ_1^2 and λ_2^2. There is a simple identity relating R_1 and R_2, which we shall now derive.

In view of equations (17) and (18) and the foregoing definitions, we may write

$$[\lambda_1^2 I_{11} + (\Omega^2 I_{11} - a - \beta)] R_1 = 2a \tag{22}$$

and

$$a R_2 = (\lambda_2^2 I_{33} - \beta) . \tag{23}$$

Multiplication of each side of equation (22) by R_2 leads to

$$[\lambda_1^2 I_{11} + (\Omega^2 I_{11} - a - \beta)] R_1 R_2 = 2(\lambda_2^2 I_{33} - \beta) , \tag{24}$$

where equation (23) has been used. On account of equation (20), the roots λ_1^2 and λ_2^2 must satisfy

$$\lambda_1^2 + \lambda_2^2 = \frac{\beta}{I_{33}} + \frac{\beta + a - \Omega^2 I_{11}}{I_{11}}$$

or

$$\lambda_1^2 I_{11} + (\Omega^2 I_{11} - a - \beta) = -\frac{I_{11}}{I_{33}} (\lambda_2^2 I_{33} - \beta) . \tag{25}$$

We infer from equations (24) and (25) that

$$R_1 R_2 = -2 \frac{I_{33}}{I_{11}} , \tag{26}$$ [5]

which is the required identity.

a) *The Case When Rotation Is Absent*

If $\Omega^2 = 0$ and the equilibrium configuration is spherical, the roots λ_1^2 and λ_2^2 are (Paper I, eq. [95])

$$\lambda_1^2 = (3\gamma - 4) \frac{\mathfrak{W}}{I} \quad \text{and} \quad \lambda_2^2 = \frac{4}{5} \frac{\mathfrak{W}}{I} . \tag{27}$$

These roots are distinct if $\gamma \neq \frac{8}{5}$, and the normal modes can then be found. They are

$$R_1 = 2 \quad \text{and} \quad R_2 = -1 . \tag{28}$$

It is easy to see that, for a spherically symmetrical disturbance, $X_{11} = X_{22} = X_{33}$, and therefore $R = 2$. Hence the root λ_1^2 corresponds to a radial pulsation. The root λ_2^2 can be shown to belong to that second-order spherical harmonic which is symmetric about the x_3-axis. Such a mode is volume-conserving; this is reflected in $R_2 = -1$, which, according to the substitution (3), implies that div $\boldsymbol{\xi} = 0$.

If $\gamma = \frac{8}{5}$, then $\lambda_1^2 = \lambda_2^2$, and the normal modes are consequently unspecified.

If $\mathfrak{W}$ and I are evaluated for the homogeneous, compressible model, equation (27) becomes

$$\lambda_1^2 = -(3\gamma - 4)\tfrac{4}{3}\pi G \rho \quad \text{and} \quad \lambda_2^2 = -\tfrac{16}{15}\pi G \rho . \tag{29}$$

b) *The Case When Rotation Is Present*

With the exception of equation (29), the equations and remarks of this section have been general, applying to centrally condensed, as well as to uniform, mass distributions. We now want to restrict our attention to the homogeneous, compressible model. For this purpose it is convenient to make the followings definitions:

$$l^2 = (4\pi G \rho)^{-1} \lambda^2 , \quad f = -(4\pi G \rho I_{11})^{-1} (a + \beta - \Omega^2 I_{11}) ,$$
$$g = -(4\pi G \rho I_{33})^{-1} \beta , \quad \text{and} \quad k = -(4\pi G \rho I_{33})^{-1} a . \tag{30}$$

[5] In deriving this identity we have benefited by a discussion with Dr. Alar Toomre.

Equations (17) and (18) now become

$$(l^2+f)(N_{11}+N_{22})+2kN_{33}=0 \tag{31}$$

and

$$(1-e^2)k(N_{11}+N_{22})+(l^2+g)N_{33}=0\,, \tag{32}$$

where we have used equation (11), suppressing the argument, and have introduced the eccentricity e through the formula

$$\frac{I_{33}}{I_{11}}=\frac{a_3^2}{a_1^2}=1-e^2. \tag{33}$$

The characteristic equation now becomes

$$l^4+(f+g)l^2+[fg-2(1-e^2)k^2]=0\,, \tag{34}$$

with roots

$$l^2=-\tfrac{1}{2}(f+g)\pm\tfrac{1}{2}[(f-g)^2+8(1-e^2)k^2]^{1/2}\,. \tag{35}$$

Let l_1^2 be that root which in the limit $\Omega^2 \to 0$ approaches $-(\gamma-\frac{4}{3})$, and l_2^2 the root that approaches $-\frac{4}{15}$ (cf. eq. [29]). The normal modes will be specified by the ratios r_1 and r_2, where $r=(N_{11}+N_{22})/N_{33}$. The identity (26) becomes

$$r_1 r_2=-\frac{2}{1-e^2}, \tag{36}$$

since

$$r=\frac{RI_{11}}{I_{33}}=\frac{R}{1-e^2}.$$

The quantities f, g, and k introduced in equation (30) can be written in terms of A_i and A_{ij} by means of the formulae of Paper III. The results are

$$\begin{aligned} f&=\tfrac{1}{2}a_1^2a_3\{A_1+a_3^2A_{13}+(1-e^2)[2(\gamma-1)A_3-2A_3]\}\,,\\ g&=\tfrac{1}{2}a_1^2a_3[2A_3-3a_3^2A_{33}+(\gamma-1)A_3]\,, \end{aligned} \tag{37}$$

and

$$k=\tfrac{1}{2}a_1^2a_3[-a_1^2A_{13}+(\gamma-1)A_3]\,.$$

Since the constants A_i and A_{ij} can be written in terms of the eccentricity e (Paper III, eqs. [18], [19], [34], and [35]), these equations give f, g, and k as functions of e.

VII. THE LIMIT OF VANISHING ANGULAR MOMENTUM

If the rotation is slow, so that the equilibrium configuration is only slightly oblate, the coefficients f, g, and k may be approximated by expansions in the eccentricity e. To terms of second order in e the results are

$$\begin{aligned} f&=[-\tfrac{2}{15}+\tfrac{2}{3}(\gamma-1)]+[\tfrac{26}{105}-\tfrac{2}{5}(\gamma-1)]e^2\,,\\ g&=[\tfrac{1}{15}+\tfrac{1}{3}(\gamma-1)]+[\tfrac{2}{21}+\tfrac{2}{15}(\gamma-1)]e^2\,, \end{aligned} \tag{38}$$

and

$$k=[-\tfrac{1}{5}+\tfrac{1}{3}(\gamma-1)]+[-\tfrac{4}{35}+\tfrac{2}{15}(\gamma-1)]e^2\,.$$

The expressions $f+g$ and $(f-g)^2+8(1-e^2)k^2$ appearing in equation (34) become, to the same order,

$$f+g=[-\tfrac{1}{15}+(\gamma-1)]+[\tfrac{12}{35}-\tfrac{4}{15}(\gamma-1)]e^2 \tag{39}$$

and

$$\begin{aligned}(f-g)^2+8(1-e^2)k^2&=9[-\tfrac{1}{5}+\tfrac{1}{3}(\gamma-1)]^2\\&+[-\tfrac{1}{5}+\tfrac{1}{3}(\gamma-1)][\tfrac{8}{105}-\tfrac{8}{5}(\gamma-1)]e^2\,.\end{aligned} \tag{40}$$

Note that the right-hand side of equation (40) vanishes identically if $\gamma=\frac{8}{5}$; this case must therefore be treated separately.

a) *The Case When* $\gamma\neq\frac{8}{5}$

In this case formulae (39) and (40) are correct to second order in e, and one finds from equation (35) that the roots are

$$l_1^2=-\tfrac{1}{3}(3\gamma-4)+\tfrac{4}{45}(3\gamma-5)e^2 \tag{41}$$

and

$$l_2^2=-\tfrac{4}{15}-\tfrac{52}{315}e^2\,. \tag{42}$$

It is clear from equation (41) that the effect of a small rotation is to stabilize the configuration with $\gamma=\frac{4}{3}$. In other words, the critical value of γ, γ_c, at which dynamic instability sets in, is reduced from the value $\frac{4}{3}$. The amount of the reduction for a given value of e is obtained by setting $l_1^2=0$ in equation (41). The result is

$$\gamma_c=\tfrac{4}{3}-\tfrac{4}{45}e^2\,. \tag{43}$$

With the aid of equations (41) and (42) we can find the normal modes r_1 and r_2. They are

$$r_1=2+\frac{4}{3(8-5\gamma)}\,e^2 \tag{44}$$

and

$$r_2=-1+\frac{15\gamma-22}{3(8-5\gamma)}\,e^2\,. \tag{45}$$

That r_2 is no longer precisely -1 means that this mode is no longer volume-preserving and hence is no longer a deformation involving only a second-order surface harmonic: rotation has "mixed" the modes.

If, in equations (41) and (42), the factor $4\pi G\rho$ is restored, $\Omega^2(=8\pi G\rho e^2/15)$ is used instead of e^2, and $\sigma^2=-\lambda^2$, these equations become

$$\sigma_1^2=(3\gamma-4)\tfrac{4}{3}\pi G\rho-\tfrac{2}{3}(3\gamma-5)\Omega^2 \tag{46}$$

and

$$\sigma_2^2=\tfrac{16}{15}\pi G\rho+\tfrac{26}{21}\Omega^2\,. \tag{47}$$

Equation (46) agrees with a formula found by Ledoux (1945, eq. [54]). (For a somewhat more general comparison, see Chandrasekhar and Lebovitz 1962*c*, Sec. VIII, *c*).

b) *The Case When* $\gamma=\frac{8}{5}$

If $\gamma=\frac{8}{5}$, equation (40) is not adequate for finding the frequencies correctly to second order in e, and it is necessary to return to equations (38), which, for the present case, become

$$f=\tfrac{4}{15}+\tfrac{4}{525}e^2\,,\qquad g=\tfrac{4}{15}+\tfrac{92}{525}e^2\,,\qquad\text{and}\qquad k=-\tfrac{6}{175}e^2\,. \tag{48}$$

The roots are easily found to be

$$l_1^2 = -\frac{4}{15} - \frac{48 - \sqrt{2584}}{525}\, e^2 \tag{49}$$

and

$$l_2^2 = -\frac{4}{15} - \frac{48 + \sqrt{2584}}{525}\, e^2 . \tag{50}$$

If these results are used together with equations (31) and (32), the normal modes, which were unspecified in the absence of rotation, may be found, now that rotation has "lifted the degeneracy" in the characteristic roots, but not to the same order in e: the normal modes are found in the limit as $e \rightarrow 0$. They are

$$r_1 = -\frac{36}{44 - \sqrt{2584}} = 5.2685 \qquad \text{(approximately)} \tag{51}$$

and

$$r_2 = -\frac{36}{44 + \sqrt{2584}} = -0.3796 \quad \text{(approximately)} . \tag{52}$$

Since $r = 2$ for a spherically symmetric perturbation, it is clear that neither mode is even approximately spherically symmetric. This last fact is not peculiar to the homogeneous compressible case: it is found for distorted polytropes in general (see Table 4*B* in Chandrasekhar and Lebovitz 1962*a*).

VIII. THE LIMIT OF LARGE ANGULAR MOMENTUM

When the angular momentum becomes large, the Maclaurin spheroid becomes highly flattened, and $e \rightarrow 1$. It is convenient in this case to expand the coefficients f, g, and k in powers, not of e, but of $\eta = (\pi/2 - \sin^{-1} e)$, which becomes small in the limit under consideration. Expansions of f, g, and k must be taken to the second order in η. The results are

$$\begin{aligned} f &= \tfrac{1}{4}\pi\eta - 2(2-\gamma)\eta^2 , \\ g &= \gamma - \tfrac{1}{2}\pi(\gamma+1)\eta + 2(\gamma+2)\eta^2 , \end{aligned} \tag{53}$$

and

$$k = (\gamma - 2) + \tfrac{1}{4}\pi(5 - 2\gamma)\eta - 2(3-\gamma)\eta^2 .$$

The characteristic roots and normal modes are easily found to be

$$l_1^2 = -\gamma \qquad \text{and} \qquad l_2^2 = -\tfrac{1}{4}\pi\eta \tag{54}$$

and

$$r_1 = -\frac{2(2-\gamma)}{\gamma} \qquad \text{and} \qquad r_2 = \frac{\gamma}{2-\gamma}\,\frac{1}{\eta^2} . \tag{55}$$

These equations are helpful in interpreting the tables and graphs of Section IX.

IX. NUMERICAL RESULTS

When the angular momentum M is neither very small nor very large, the only satisfactory way of finding the behavior of the pulsation frequencies and normal modes is numerically.

Equation (43) shows that, at least for small values of e, the critical value of γ, γ_c, below which dynamical instability occurs, is reduced by rotation. Table 1 shows that this is a trend that is maintained for all values of e: γ_c is a monotonically decreasing function of e. In addition, Ω^2 (in the unit $4\pi G\rho$) and M (in the unit $[G\mathfrak{M}a_1^2 a_3]^{1/2}$, where $\mathfrak{M}$ is the mass of the spheroid) are given in Table 1. In Figure 1, γ_c is plotted against M.

The frequencies of those modes that start, when $e = 0$, as radial pulsations are given in Table 2*A*, and their corresponding normal modes in terms of the ratio r_1 in Table 2*B*, for several values of γ. The frequencies are plotted against M in Figure 2. The frequency σ ($= il$) is in the unit $(4\pi G\rho)^{1/2}$.

TABLE 1

THE CRITICAL VALUE OF γ, Ω^2, AND M, AS FUNCTIONS OF e

e	Ω^2	M	γ_c	e	Ω^2	M	γ_c
0 .	0	0	1 3333	0 8	0 1816	0 2934	1 2318
0 1	0 0027	0 0255	1 3324	0 9.	2203	0 4000	1 1535
0 2.	0107	0 0514	1 3297	0 95	2213	0 5008	1 0578
0 3	0243	0 0787	1 3249	0 99.	1552	0 7120	0 7828
0 4	0436	0 1085	1 3176	0 995	1219	0 7968	0 6551
0 5	0690	0 1417	1 3071	0 999.	0627	0 9928	0 3879
0 6	1007	0 1804	1 2920	0 9999	0 0214	1 2198	0 1495
0 7	0 1387	0 2283	1 2693	1 0000	0	∞	0

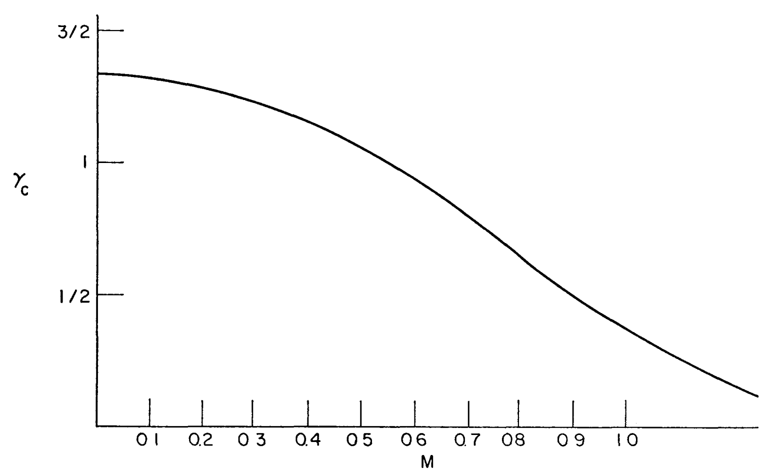

FIG. 1.—The critical value of γ, γ_c, as a function of the angular momentum M (the unit of M is $[G\mathfrak{M}a_1^2a_3]^{1/2}$, where $\mathfrak{M}$ is the mass of the spheroid).

In the same way, the frequencies of those modes that start, when $e = 0$, as second-order harmonics, are given in Table 3*A*, and their corresponding values of r_2 in Table 3*B*. The frequencies are also plotted against M in Figure 3.

Tables 2*B* and 3*B* for the normal modes are, in a sense, incomplete, for $N_{11} + N_{22}$ and N_{33} are not sufficient to specify the normal modes: $N_{12} - N_{21}$ is also needed (cf. Paper I, eq. [79]). It can, however, be determined from the information given in the tables through the equation

$$\lambda^2(N_{12} - N_{21}) = 2\lambda\Omega(N_{11} + N_{22}) . \tag{56}$$

TABLE 2*A*

THE FREQUENCIES OF THE MODE THAT STARTS AS A PURELY RADIAL PULSATION*

e	γ					
	1 3	$\frac{4}{3}$	1 4	1 5	1 6	$\frac{5}{3}$
0	i0 1825	0	0 2583	0 4083	0 5164	0 5773
0 1	i0 1800	0 0300	2596	4088	5163	0 5773
0 2	i0 1715	0600	2636	4104	5161	0 5775
0 3	i0 1559	0906	2706	4130	5157	0 5783
0 4	i0 1296	1225	2804	4167	5147	0 5816
0 5	i0 0812	1559	2929	4210	5128	0 5923
0 6	0 0843	1931	3084	4256	5092	0 6155
0 7	0 1594	2298	3268	4292	5023	0 6540
0 8	0 2238	2718	3464	4290	4881	0 7143
0 9	0 2855	3137	3604	4141	4538	0 8198
0 95	0 3068	3233	3517	3854	4111	0 9184
0 99	0.2722	2798	2855	2963	3051	1 0941
0 995	0 2433	2462	2512	2577	2632	1 1457
0 999	0 1761	1769	1786	1806	1822	1 2219
0 9999	0 1034	0 1034	0 1039	0 1039	0 1044	1 2685
1 0000	0	0	0	0	0	1 2910

* The factors of $i(=\sqrt{-1})$ appearing in the first column indicate that the corresponding mode is unstable; the numerical values of these entries therefore represent the growth rate

TABLE 2*B*

THE VALUES OF r $(=[N_{11}+N_{22}]/N_{33})$ FOR THE MODE THAT STARTS AS A PURELY RADIAL PULSATION

e	γ					
	1 3	$\frac{4}{3}$	1 4	1 5	1 6	$\frac{5}{3}$
0	2 0000	2 0000	2 0000	2 0000		+2 0000
0 1	2 0090	2 0101	2 0135	2 0270	5 30	+1 9589
0 2	2 0378	2 0425	2 0454	2 1117	5 48	+1 8201
0 3	2 0918	2 1031	2 1366	2 2664	5 75	+1 5538
0 4	2 1828	2 2050	2 2704	2 5172	6 21	+1 1111
0 5	2 3334*	2 3732	2 4899	2 9148	6 91	+0 5852
0 6	2 5911*	2 6820	2 8627	3 5634	8 02	+0 2051
0 7	3 0714	3 1963	3 5529	4 7142	9 92	−0 0090
0 8	4 1371	4 3845	5 0728	7 1142	1 372 $\times10^1$	−0 1384
0 9	7 7273	8 3751	1 0088$\times10^1$	1 4515$\times10^1$	2 500 $\times10^1$	−0 2356
0 95	1 5615$\times10^1$	1 7074$\times10^1$	2 0766$\times10^1$	2 9465$\times10^1$	4 669 $\times10^1$	−0 2866
0 99	8 4556$\times10^1$	1 0880$\times10^2$	1 1047$\times10^2$	1 4943$\times10^2$	2 1434$\times10^2$	−0 3486
0 995	1 7358$\times10^2$	1 8864$\times10^2$	2 2454$\times10^2$	2 9942$\times10^2$	4 1995$\times10^2$	−0 3634
0 999	9 0344$\times10^2$	9 7438$\times10^2$	1 1470$\times10^3$	1 4994$\times10^3$	2 0419$\times10^3$	−0 3834
0 9999	9 1956$\times10^3$	9 9442$\times10^3$	1 1604$\times10^4$	1 4999$\times10^4$	2 0029$\times10^4$	−0 3947
1 0000.	∞	∞	∞	∞	∞	−0 4000

* Between $e = 0$ 5 and $e = 0$ 6 the "frequency" vanishes (see Table 2*A*) One can show that at this point, although the limiting ratio r is finite, $N_{11}+N_{22}=N_{33}=0$, and only $N_{12}-N_{21}\neq 0$ (cf eq [56]) The interpretation of this is that the mode of neutral stability is one of pure rotation.

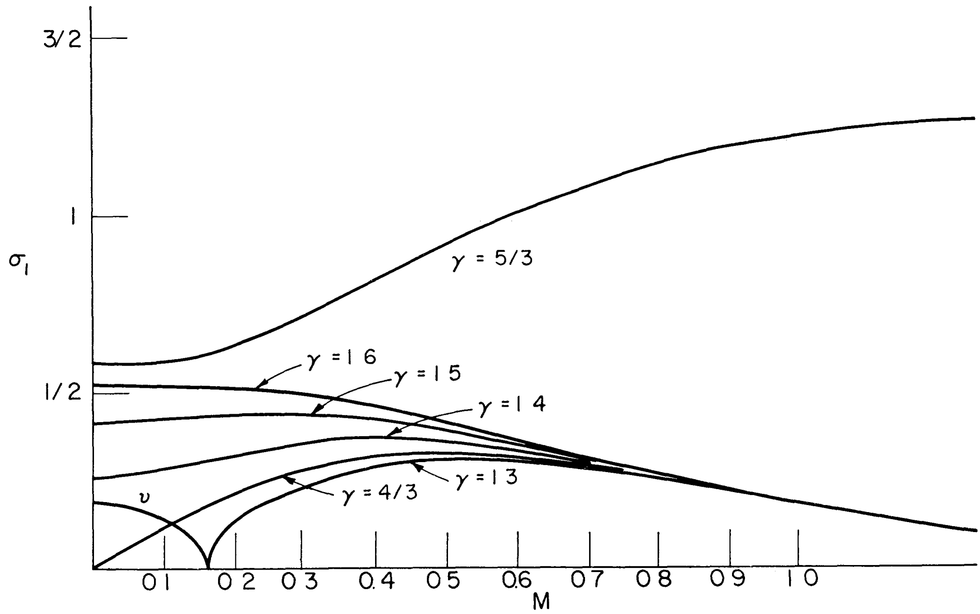

FIG. 2.—The frequencies of the modes that are radial pulsations when $M = 0$ When $\gamma = 1.3$, the radial pulsation starts out being unstable with growth rate ν. Stabilization occurs at $M = 0.165$; for $M > 0$ 165, the curve labeled "$\gamma = 1$ 3" gives the pulsation frequency.

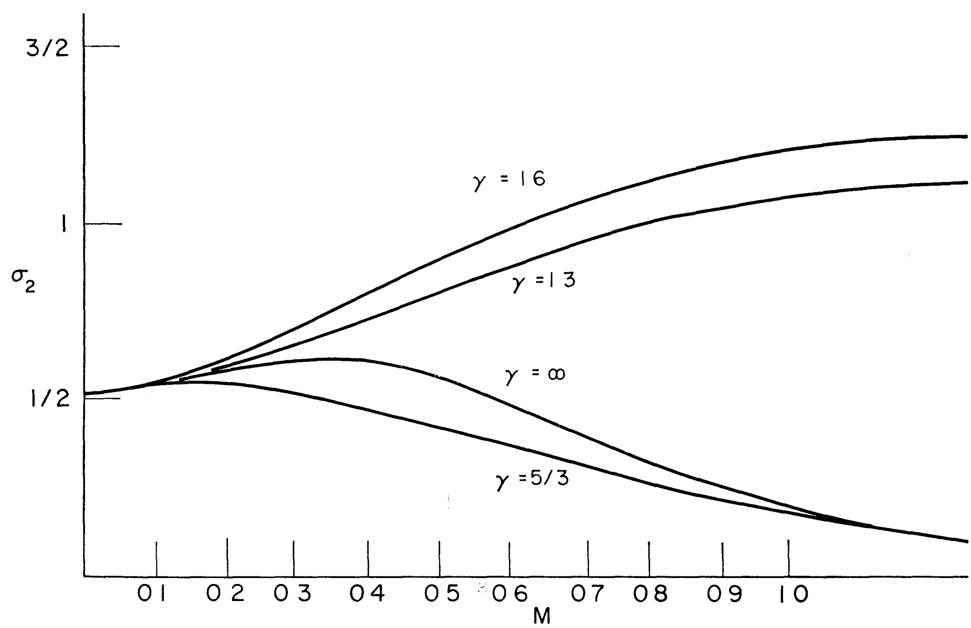

FIG 3 —The frequencies of the modes that correspond to second-order harmonics when $M = 0$. The curves for $\gamma = \frac{4}{3}$, 1 4, and 1 5 lie between those for $\gamma = 1.3$ and $\gamma = 1.6$ and are similar in appearance; they are therefore not shown The curve labeled "$\gamma = \infty$" is that for the incompressible case (cf. Paper II, Fig 1, the curve labeled "σ_3").

TABLE 3*A*

THE FREQUENCIES OF THE MODE THAT STARTS AS A SECOND-ORDER HARMONIC DEFORMATION

e	γ					
	1 3	$\frac{4}{3}$	1 4	1 5	1 6	$\frac{5}{3}$
0	0 5164	0 5164	0 5164	0 5164	0 5164	0 5164
0 1	0 5180	0 5180	0 5180	0 5180	0 5183	5180
0 2	0 5229	0 5229	0 5229	0 5229	0 5238	5227
0 3	0 5312	0 5312	0 5312	0 5314	0 5336	5303
0 4	0 5432	0 5433	0 5435	0 5441	0 5480	5392
0 5	0 5597	0 5599	0 5604	0 5619	0 5683	5453
0 6	0 5816	0 5821	0 5833	0 5867	0.5966	.5443
0 7	0 6111	0 6123	0 6151	0 6221	0 6365	.5358
0 8	0 6537	0 6563	0 6626	0 6760	0 6962	5170
0 9	0 7279	0 7341	0 7479	0 7719	0 7996	.4744
0 95	0 8029	0 8125	0 8324	0 8637	0 8963	.4252
0 99	0 9537	0 9670	0 9933	1 0319	1 0696	3102
0 995	1 0012	1 0151	1 0423	1 0820	1 1206	2665
0 999.	1 0734	1 0877	1 1158	1 1567	1 1963	1833
0 9999	1 1182	1 1327	1 1611	1 2025	1 2424	0 1044
1 0000	1 1402	1 1547	1 1832	1 2247	1 2649	0

TABLE 3*B*

THE VALUES OF r ($= [N_{11} + N_{22}]/N_{33}$) FOR THE MODE THAT STARTS AS A SECOND-ORDER HARMONIC DEFORMATION

e	γ					
	1.3	$\frac{4}{3}$	1 4	1 5	1 6	$\frac{5}{3}$
0	−1 0000	−1 0000	−1 0000	−1 0000	. . . .	−1 0000
0 1	−1 0056	−1 0050	−1 0033	−0 9966	−0 380	−1 0313
0 2	−1 0223	−1 0200	−1 0130	−0 9866	− 380	−1 1428
0 3	−1 0507	−1 0450	−1 0293	−0 9698	− 382	−1 4145
0 4	−1 0908	−1 0798	−1 0487	−0 9459	− 383	−2 1429
0 5.	−1 1428	−1.1237	−1 0710	−0 9149	− 386	−4 5559
0 6	−1 2060	−1 1745	−1 0916	−0 8769	− 390	$-1\ 5234\times10^{1}$*
0 7	−1 2768	−1 2269	−1 1038	−0 8319	− 395	$+4\ 3536\times10^{2}$*
0 8	−1 3428	−1 2671	−1 0951	−0 7809	− 405	$+4\ 0137\times10^{1}$
0 9.	−1 3622	−1 2568	−1 0435	−0 7252	− 423	$+4\ 4685\times10^{1}$
0 95	−1 3137	−1 2014	−0 9878	−0 6962	− 439	$+7\ 1537\times10^{1}$
0 99	−1 1886	−1 0905	−0 9098	−0 6726	− 469	$+2\ 8829\times10^{2}$
0 995	−1 1551	−1 0629	−0 8930	−0 6696	− .477	$+5\ 5181\times10^{2}$
0 999.	−1 1245	−1 0266	−0 8723	−0 6673	− 489	$+2\ 6098\times10^{3}$
0 9999.	−1 0875	−1 0084	−0 8618	−0 6667	− 497	$+2\ 5338\times10^{4}$
1 0000	−1 0769	−1 0000	−0 8571	−0 6667	−0 500	$+\ \infty$

* Between $e = 0\ 6$ and $e = 0\ 7$, the quantity $1/r$ changes continuously from negative to positive values

X. CONCLUDING REMARKS

Perhaps the only unexpected result of this work is that, when $\gamma = 1.6$, none of the normal modes becomes spherically symmetric as $M \to 0$. Even this might have been foreseen because the occurrence of the degeneracy was known (Paper I, Sec. IV), so there was no reason to believe that the normal modes when $\gamma = 1.6$ bear any resemblance to those when $\gamma \neq 1.6$.

This result is not, of course, restricted to the homogeneous model (cf. Chandrasekhar and Lebovitz 1962*a*, Sec. VIII). It is expressed in a more general way in another paper in this series (Chandrasekhar and Lebovitz 1962*d*), where it is applied to the question of double periods in the light- and velocity-curves of the β Canis Majoris stars.

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