1962ApJ...136.1048C

ON THE POINT OF BIFURCATION ALONG THE SEQUENCE OF THE JACOBI ELLIPSOIDS

S. Chandrasekhar
University of Chicago
Received June 16, 1962

ABSTRACT

In this paper, the known point of bifurcation along the sequence of the Jacobi ellipsoids is isolated by a new method based on equilibrium considerations only. The method consists in finding an integral property (or, more generally, a functional) of the configuration which vanishes as a condition of equilibrium. The first variation of such a functional will vanish at a point of bifurcation (and *only* at a point of bifurcation) for a Lagrangian displacement which deforms the body from the shape it has along an equilibrium sequence to the shape it will have in the sequence following bifurcation. For finding a functional J with the requisite properties, an equation for the third-order virial (namely, $\int \rho u_i x_j x_k dx$) is first established. And from an examination of the conditions, which follow from this equation, for equilibrium, it is found that

$$J = \int_V \rho \left[x_3 \mathfrak{B}_{13} + x_2 \mathfrak{B}_{12} + x_1 (\mathfrak{B}_{33} - \mathfrak{B}_{22}) \right] dx$$

(where $\mathfrak{B}_{ij}$ is the tensor potential of the gravitational field) has all the necessary properties The first variation of J, for the Lagrangian displacement which deforms a Jacobi ellipsoid into a pear-shaped object, is then evaluated, and it is shown that its vanishing determines the point of bifurcation along the Jacobian sequence, in agreement with Darwin's result.

I. INTRODUCTION

It is well known from the classical investigations of Jacobi, Poincaré, and Darwin that points of bifurcation occur as we follow the sequence of the possible forms of equilibrium of rotating incompressible masses. The first of these points of bifurcation occurs along the sequence of the Maclaurin spheroids when ellipsoids with three unequal axes (the ellipsoids of Jacobi) first become possible as figures of equilibrium. A further point of bifurcation (isolated by Darwin 1901) occurs along the sequence of the Jacobi ellipsoids when pear-shaped configurations, similarly, become possible as figures of equilibrium. The occurrence of these (and other) points of bifurcation was accounted for by Poincaré in terms of an analytical theory of the stability of these configurations (for an account of this theory see Lyttleton 1953). While Poincaré's theory may be considered as leaving nothing to be explained, it may yet be felt that the *occurrence* of the points of bifurcation should be made manifest by considerations pertaining to the equilibrium itself (as distinct from considerations pertaining to their stability). Thus Jacobi's discovery, that ellipsoidal forms are possible as figures of equilibrium for rotating incompressible masses, is described by Thomson and Tait in their *Natural Philosophy* (1883, p. 330) as a "curious theorem," even though the bearing of this "theorem" on the stability of the Maclaurin spheroids was well known to them (*ibid.*, pp. 332–333). And more recently it has been observed that "there is an element of surprise about Jacobi's result in view of the symmetry that might be expected to be associated with any form produced by a rotational field" (Lyttleton 1953, p. 1). Similar comments might perhaps be made when a rotating body loses even its triplanar symmetry and allows a pear-shaped form (as it does at the point of bifurcation on the Jacobian series).

It is in the spirit of the foregoing remarks that the criterion which has recently been given (Chandrasekhar and Lebovitz 1962*a;* this paper will be referred to hereafter as "Paper I") for the occurrence of the point of bifurcation, along the sequence of axisymmetric forms (not necessarily spheroidal forms) which a rotating mass (not necessarily an incompressible mass) can assume, has to be understood. The criterion in question (see Sec. II below) was derived from certain general conditions which follow from the second-order virial equations and which must obtain in equilibrium. However, the particular derivation given in Paper I left no room for accounting for the point of bifurcation on the Jacobian series whose determination, according to Jeans (1919, 1929), is an "arduous piece of work." In this paper we shall isolate the point of bifurcation along the Jacobian sequence by a method different from Darwin's and based on equilibrium considerations only. However, this new method requires us to consider the virial equations of the *third order;* the reasons for this are stated in Section III after some preliminary remarks in Section II pertaining to the first point of bifurcation.

II. THE POINT OF BIFURCATION ALONG THE MACLAURIN SEQUENCE

As we have stated in Section I, the occurrence of the first point of bifurcation along the sequence of the Maclaurin spheroids was made manifest in Paper I from general equilibrium considerations. We shall here review the arguments with some amplifications.

Now the tensor form of the virial theorem applied to a uniformly rotating mass (with no restrictions as to its nature or its constitution) characterizes the state of equilibrium by the equations

$$\mathfrak{W}_{11}+\Omega^2 I_{11}=\mathfrak{W}_{22}+\Omega^2 I_{22}=\mathfrak{W}_{33}\,, \tag{1}$$[1]

where $\mathfrak{W}_{ij}$ and I_{ij} are the potential-energy and the moment of inertia tensors and the orientation of the co-ordinate axes is so chosen that the x_3-axis is along the direction of $\mathbf{\Omega}$. An alternative form of equations (1) (which is equally general) is (Paper I, eq. [79])

$$-\mathfrak{W}_{12;12}+\Omega^2 I_{11}=-\mathfrak{W}_{21;12}+\Omega^2 I_{22}=\mathfrak{W}_{33}-\mathfrak{W}_{11;22}\,, \tag{2}$$

where

$$\mathfrak{W}_{pq;ij}=\int_V \rho x_p \frac{\partial \mathfrak{B}_{ij}}{\partial x_q}\,d\mathbf{x} \tag{3}$$

is the supermatrix which has been considered in detail in Paper I (Sec. IV).

The first thing to observe about equations (1) and (2) is that they provide no substance to the common expectation that symmetry about the rotational axis should be "associated with any form produced by a rotational field." However, when $\Omega^2 \to 0$ and the configuration becomes spherical, the terms on either side of the first equality in equations (2) are both negative; and an obvious way in which the equality can be satisfied *identically* is by requiring axisymmetry, in which case

$$\mathfrak{W}_{12;12}=\mathfrak{W}_{21;12} \qquad \text{and} \qquad I_{11}=I_{22}\,. \tag{4}$$

As Ω^2 increases and the configuration departs from sphericity, $-\mathfrak{W}_{12;12}+\Omega^2 I_{11}$ will increase; and if a point should be reached when the quantity vanishes, then it would become possible for the *first time* to satisfy the equations without the assumption of axisymmetry. The general condition, then, for the occurrence of a point of bifurcation along the sequence of the axisymmetric configurations is

$$\Omega^2 I_{11}=\mathfrak{W}_{12;12}\,. \tag{5}$$

A detailed consideration of the stability of rotating gaseous masses (Chandrasekhar and Lebovitz 1962*b*) establishes the same condition for the occurrence of a neutral mode of oscillation, again implying that we have arrived, here, at a point of bifurcation.

[1] The viral theorem also gives

$$\mathfrak{W}_{13}=\mathfrak{W}_{23}=0 \qquad \text{and} \qquad \mathfrak{W}_{12}+\Omega^2 I_{12}=0\,. \tag{1'}$$

For objects with triplanar symmetry, these equations are trivially satisfied.

The foregoing considerations are quite general. If the assumption that the configurations are homogeneous ellipsoids is now made (an assumption which will be valid only for incompressible masses), then the origin of the point of bifurcation along the Maclaurin sequence can be made even more explicit. For, if we should seek to satisfy the conditions expressed in equations (2) without the assumption of axisymmetry, then we should infer that

$$\Omega^2 = \frac{\mathfrak{W}_{12;12} - \mathfrak{W}_{21;12}}{I_{11} - I_{22}}. \tag{6}$$

But for a homogeneous ellipsoid, we have the *identity*

$$\frac{\mathfrak{W}_{12;12}}{I_{11}} = \frac{\mathfrak{W}_{21;12}}{I_{22}} \tag{7}$$

(cf. Chandrasekhar and Lebovitz 1962*d*, eq. [66]; this paper will be referred to hereafter as "Paper III"). From equations (6) and (7) it follows that *the only way in which the first equality in equations* (*2*) *can be satisfied, without the assumption of axisymmetry, is for each side of the equality to vanish separately*. But as $\Omega^2 \to 0$, this cannot be. Hence, *as* $\Omega^2 \to 0$, *the first equality in equations* (*2*) *can be satisfied only identically, under conditions of axisymmetry, when equations* (*4*) *obtain.* And, finally, *at the point of bifurcation and beyond, along the sequence of the triaxial ellipsoids,* Ω^2 *is determined by*

$$\Omega^2 = \frac{\mathfrak{W}_{12;12}}{I_{11}} = \frac{\mathfrak{W}_{21;12}}{I_{22}}; \tag{8}$$

and we must also have

$$\mathfrak{W}_{33} = \mathfrak{W}_{11;22}. \tag{9}$$

Equation (9) provides a single-valued relation between a_2/a_1 and a_3/a_1; and equation (8) determines the value of Ω^2 which is to be associated with each pair $(a_2/a_1, a_3/a_1)$ consistent with equation (9). The configurations we so obtain are, indeed, the ellipsoids of Jacobi. (See Appendix I for a further enumeration of the properties of these Jacobi ellipsoids.)

III. THE POINT OF BIFURCATION ALONG THE JACOBIAN SEQUENCE; HOW IT MIGHT BE ISOLATED

Now turning to the question of the occurrence of the point of bifurcation along the sequence of the Jacobi ellipsoids, we seek its exhibition and isolation along the following lines.

First, we remark that we should conclude, without any previous knowledge, that if a point of bifurcation occurs along the Jacobian sequence, then at such a point the figure of equilibrium must depart from the triplanar symmetry which it has preserved until then. And such a departure from triplanar symmetry can be achieved, in the first instance, only by a third harmonic deformation of the Jacobi ellipsoid. Since we are here concerned with ellipsoidal harmonics, it is in fact clear that the required deformation should be accomplished by a Lagrangian displacement of the form[2] (see Darwin 1901, p. 297; also Lyttleton 1953, p. 109)

$$\xi_j = \text{constant}\ \frac{\partial}{\partial x_j}\, x_1 \left(\frac{x_1^2}{a_1^2+\lambda} + \frac{x_2^2}{a_2^2+\lambda} + \frac{x_3^2}{a_3^2+\lambda} - 1\right), \tag{10}$$

[2] I am much indebted to Dr. Paul H. Roberts for clarifying discussions relative to these considerations.

where a_1, a_2, and a_3 are the semiaxes of the Jacobi ellipsoid and λ is the numerically larger of the two roots of the equation

$$\frac{3}{a_1^2+\lambda}+\frac{1}{a_2^2+\lambda}+\frac{1}{a_3^2+\lambda}=0. \tag{11}$$

Now if an arbitrary member of the Jacobian sequence is deformed by the application of the Lagrangian displacement (10), then the deformed configuration will not be one of equilibrium. *Only at the point of bifurcation* (*and nowhere else*) *will the deformation of a Jacobi ellipsoid by the application of the Lagrangian displacement* (10) *leave its equilibrium unaffected; and, in the same way, any integral property describing the equilibrium state will be unaffected by such a deformation only if it takes place at the point of bifurcation* (*and nowhere else*).

What we need, then, is some integral property or, more generally, some functional J of the configuration, which will vanish in the equilibrium state:

$$J=0 \text{ in equilibrium}. \tag{12}$$

If ξ now denotes a Lagrangian displacement which deforms a member of an equilibrium sequence into a shape appropriate to the sequence which might follow a point of bifurcation, then the first variation of J caused by such a deformation will vanish at the point of bifurcation (and nowhere else):

$$\delta J=0 \text{ at the point of bifurcation}. \tag{13}$$

Clearly, if δJ vanishes *identically* for all members of a given sequence, then the functional J cannot discriminate the point of bifurcation. If we wish, then, to exhibit and isolate the point of bifurcation along the Jacobian sequence, what we need is some functional J having the property (12) and whose first variation will not vanish, identically, for odd deformations such as the one specified by equation (10). More precisely, the kind of functional J we must seek is one which *might* vanish *trivially* for objects (such as the Jacobi ellipsoids) with triplanar symmetry and *will* vanish *non-trivially* (as a condition of equilibrium) for objects (such as the pear-shaped figures) which do not have such triplanar symmetry. Relations, required by equilibrium, among the tensors of the second rank, such as $\mathfrak{W}_{ij}$ and I_{ij}, are of no use in this connection, since the first variations of such relations will vanish identically for odd deformations on account of the even character of these tensors. We are, therefore, led to seek relations among tensors of higher rank; and, as we shall see in detail below, relations having the requisite properties can be deduced from the virial equations of the third order.

IV. THE THIRD-ORDER VIRIAL EQUATION

For the sake of definiteness, we shall consider the equations of motion appropriate to a gaseous, or a fluid, mass in a state of uniform rotation with an angular velocity $\boldsymbol{\Omega}$. In a frame of reference rotating with the angular velocity $\boldsymbol{\Omega}$, the equations of motion can be written in the form

$$\rho\frac{du_i}{dt}=-\frac{\partial p}{\partial x_i}+\rho\frac{\partial}{\partial x_i}\left(\tfrac{1}{2}\,|\boldsymbol{\Omega}\times\boldsymbol{x}|^2+\mathfrak{V}\right)+2\rho\epsilon_{ilm}u_l\Omega_m, \tag{14}$$

where the various symbols have their standard meanings.

In the derivation of the usual second-order virial equation, one multiplies the equation of motion by x_j and integrates over the volume V occupied by the fluid. Instead, we shall now multiply the equation by $x_j x_k$ and integrate over the same volume. We thus obtain

$$\int_V \rho x_j x_k \frac{du_i}{dt}\, d\boldsymbol{x} = -\int_V x_j x_k \frac{\partial p}{\partial x_i}\, d\boldsymbol{x} + \tfrac{1}{2}\int_V \rho x_j x_k \frac{\partial}{\partial x_i} |\boldsymbol{\Omega}\times\boldsymbol{x}|^2 d\boldsymbol{x} + \int_V \rho x_j x_k \frac{\partial \mathfrak{B}}{\partial x_i}\, d\boldsymbol{x} + 2\epsilon_{ilm}\Omega_m \int_V \rho x_j x_k u_l\, d\boldsymbol{x}\,; \tag{15}$$

and, considering in turn the different terms in this equation, we have

$$\int_V \rho x_j x_k \frac{du_i}{dt}\, d\boldsymbol{x} = \int_V \rho \frac{d}{dt}(x_j x_k u_i)\, d\boldsymbol{x} - \int_V \rho u_i (u_j x_k + u_k x_j)\, d\boldsymbol{x}\,, \tag{16}$$

$$-\int_V x_j x_k \frac{\partial p}{\partial x_i}\, d\boldsymbol{x} = \int_V p (x_j \delta_{ik} + x_k \delta_{ij})\, d\boldsymbol{x}\,, \tag{17}$$

$$\tfrac{1}{2}\int_V \rho x_j x_k \frac{\partial}{\partial x_i} |\boldsymbol{\Omega}\times\boldsymbol{x}|^2 d\boldsymbol{x} = \Omega^2 \int_V \rho x_i x_j x_k\, d\boldsymbol{x} - \Omega_i \Omega_l \int_V \rho x_l x_j x_k\, d\boldsymbol{x} \tag{18}$$

and

$$\begin{aligned}
\int_V \rho x_j x_k \frac{\partial \mathfrak{B}}{\partial x_i}\, d\boldsymbol{x}
&= G\int_V d\boldsymbol{x}\, \rho(\boldsymbol{x})\, x_j x_k \frac{\partial}{\partial x_i}\int_V d\boldsymbol{x}' \frac{\rho(\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|} \\
&= -G\int_V\int_V d\boldsymbol{x}\, d\boldsymbol{x}'\, \rho(\boldsymbol{x})\, \rho(\boldsymbol{x}') \frac{(x_i - x_i')\, x_j x_k}{|\boldsymbol{x}-\boldsymbol{x}'|^3} \\
&= -\tfrac{1}{2}G\int_V\int_V d\boldsymbol{x}\, d\boldsymbol{x}'\, \rho(\boldsymbol{x})\, \rho(\boldsymbol{x}') \frac{(x_i - x_i')(x_j x_k - x_j' x_k')}{|\boldsymbol{x}-\boldsymbol{x}'|^3} \\
&= -\tfrac{1}{2}G\int_V\int_V d\boldsymbol{x}\, d\boldsymbol{x}'\, \rho(\boldsymbol{x})\, \rho(\boldsymbol{x}') \frac{(x_i - x_i')\,[(x_k - x_k')\, x_j + (x_j - x_j')\, x_k']}{|\boldsymbol{x}-\boldsymbol{x}'|^3} \\
&= -\tfrac{1}{2}G\int_V d\boldsymbol{x}\, \rho(\boldsymbol{x})\, x_j \int_V d\boldsymbol{x}'\, \rho(\boldsymbol{x}') \frac{(x_i - x_i')(x_k - x_k')}{|\boldsymbol{x}-\boldsymbol{x}'|^3} \\
&\quad -\tfrac{1}{2}G\int_V d\boldsymbol{x}'\, \rho(\boldsymbol{x}')\, x_k' \int_V d\boldsymbol{x}\, \rho(\boldsymbol{x}) \frac{(x_i' - x_i)(x_j' - x_j)}{|\boldsymbol{x}-\boldsymbol{x}'|^3} \\
&= -\tfrac{1}{2}\int_V \rho(\boldsymbol{x})\, x_j \mathfrak{B}_{ik}\, d\boldsymbol{x} - \tfrac{1}{2}\int_V \rho(\boldsymbol{x})\, x_k \mathfrak{B}_{ij}\, d\boldsymbol{x}\,.
\end{aligned} \tag{19}$$

We now define the tensors:

$$\Pi_i = \int_V p x_i\, d\boldsymbol{x}\,, \tag{20}$$

$$I_{ijk} = \int_V \rho x_i x_j x_k\, d\boldsymbol{x}\,, \tag{21}$$

$$\mathfrak{T}_{ij;k} = \tfrac{1}{2}\int_V \rho u_i u_j x_k d\boldsymbol{x} \,, \tag{22}$$

$$\mathfrak{W}_{ij;k} = -\tfrac{1}{2}\int_V \rho \mathfrak{B}_{ij} x_k d\boldsymbol{x} \,, \tag{23}$$

and

$$V_{i;jk} = \int_V \rho u_i x_j x_k d\boldsymbol{x} \,. \tag{24}$$

(Note that an index after the semicolon indicates that a moment with respect to the associated space co-ordinate is involved.) In terms of the tensors we have defined, equation (15) now becomes

$$\begin{aligned}\frac{dV_{i;jk}}{dt} = 2(\mathfrak{T}_{ij;k}+\mathfrak{T}_{ik;j}) + \mathfrak{W}_{ij;k} + \mathfrak{W}_{ik;j} + \Omega^2 I_{ijk} - \Omega_i\Omega_l I_{ljk} \\ + \Pi_k\delta_{ij} + \Pi_j\delta_{ik} + 2\epsilon_{ilm} V_{l;jk}\Omega_m \,.\end{aligned} \tag{25}$$

This is the required virial equation of the third order. (The appropriate generalization of this equation to hydromagnetics will be found in Appendix III.)

V. THE THIRD-ORDER VIRIAL EQUATIONS GOVERNING EQUILIBRIUM

When no relative motions are present and hydrostatic equilibrium prevails,

$$\mathfrak{T}_{ij;k} \equiv 0 \quad \text{and} \quad V_{i;jk} \equiv 0 \,; \tag{26}$$

and equation (25) becomes

$$\mathfrak{W}_{ij;k} + \mathfrak{W}_{ik;j} + \Omega^2 I_{ijk} - \Omega_i\Omega_l I_{ljk} = -\Pi_k\delta_{ij} - \Pi_j\delta_{ik} \,. \tag{27}$$

It is now convenient to choose the 3-axis to be in the direction of $\boldsymbol{\Omega}$. Then

$$\Omega_j = \Omega\,\delta_{j3} \,, \tag{28}$$

and equation (27) takes the form

$$\mathfrak{W}_{ij;k} + \mathfrak{W}_{ik;j} + \Omega^2 I_{ijk} - \Omega^2\delta_{i3} I_{3jk} = -\Pi_k\delta_{ij} - \Pi_j\delta_{ik} \,. \tag{29}$$

The cases when $i = 3$ and $i \neq 3$ are clearly distinguished. Thus, when $i = 3$,

$$\mathfrak{W}_{3j;k} + \mathfrak{W}_{3k;j} = -\Pi_k\delta_{3j} - \Pi_j\delta_{3k} \,, \tag{30}$$

while, when $i = \varpi \neq 3$,

$$\mathfrak{W}_{\varpi j;k} + \mathfrak{W}_{\varpi k;j} + \Omega^2 I_{\varpi jk} = -\Pi_k\delta_{\varpi j} - \Pi_j\delta_{\varpi k} \qquad (\varpi = 1 \text{ or } 2) \tag{31}$$

Writing out the equations (29) and (30) explicitly, we have

$$\mathfrak{W}_{31;1}=0\,, \tag{32}$$
$$\mathfrak{W}_{32;2}=0\,, \tag{33}$$
$$\mathfrak{W}_{31;2}+\mathfrak{W}_{32;1}=0\,, \tag{34}$$
$$\mathfrak{W}_{31;3}+\mathfrak{W}_{33;1}=-\Pi_1\,, \tag{35}$$
$$\mathfrak{W}_{32;3}+\mathfrak{W}_{33;2}=-\Pi_2\,, \tag{36}$$
$$\mathfrak{W}_{33;3}=-\Pi_3\,, \tag{37}$$

$$2\mathfrak{W}_{11;1}+\Omega^2 I_{111}=-2\Pi_1\,, \tag{38}$$
$$2\mathfrak{W}_{12;2}+\Omega^2 I_{122}=0\,, \tag{39}$$
$$\mathfrak{W}_{11;2}+\mathfrak{W}_{12;1}+\Omega^2 I_{112}=-\Pi_2\,, \tag{40}$$
$$\mathfrak{W}_{11;3}+\mathfrak{W}_{13;1}+\Omega^2 I_{113}=-\Pi_3\,, \tag{41}$$
$$\mathfrak{W}_{12;3}+\mathfrak{W}_{13;2}+\Omega^2 I_{123}=0\,, \tag{42}$$
$$2\mathfrak{W}_{13;3}+\Omega^2 I_{133}=0\,, \tag{43}$$

$$2\mathfrak{W}_{21;1}+\Omega^2 I_{211}=0\,, \tag{44}$$
$$2\mathfrak{W}_{22;2}+\Omega^2 I_{222}=-2\Pi_2\,, \tag{45}$$
$$\mathfrak{W}_{21;2}+\mathfrak{W}_{22;1}+\Omega^2 I_{221}=-\Pi_1\,, \tag{46}$$
$$\mathfrak{W}_{21;3}+\mathfrak{W}_{23;1}+\Omega^2 I_{123}=0\,, \tag{47}$$
$$\mathfrak{W}_{22;3}+\mathfrak{W}_{23;2}+\Omega^2 I_{223}=-\Pi_3\,, \tag{48}$$
$$2\mathfrak{W}_{23;3}+\Omega^2 I_{233}=0\,. \tag{49}$$

The foregoing equations allow some important simplifications. Thus, by combining equations (42) and (47), we have

$$\Omega^2 I_{123} = -\mathfrak{W}_{12;3} - \mathfrak{W}_{13;2} = -\mathfrak{W}_{21;3} - \mathfrak{W}_{23;1} \,. \tag{50}$$

Hence

$$\mathfrak{W}_{13;2} = \mathfrak{W}_{23;1} \,. \tag{51}$$

From equations (34) and (51) it now follows that

$$\mathfrak{W}_{23;1} = \mathfrak{W}_{13;2} = 0 \,. \tag{52}$$

Next, from equations (39) and (46) and similarly from equations (40) and (44), we obtain, respectively,

$$\mathfrak{W}_{22;1} - \mathfrak{W}_{12;2} = -\Pi_1 \tag{53}$$

and

$$\mathfrak{W}_{11;2} - \mathfrak{W}_{12;1} = -\Pi_2 \,. \tag{54}$$

Finally, eliminating Π_1 and Π_2 from equations (35), (36), (53), and (54), we obtain the pair of relations

$$\mathfrak{W}_{31;3} + \mathfrak{W}_{12;2} + \mathfrak{W}_{33;1} - \mathfrak{W}_{22;1} = 0 \tag{55}$$

and

$$\mathfrak{W}_{32;3} + \mathfrak{W}_{12;1} + \mathfrak{W}_{33;2} - \mathfrak{W}_{11;2} = 0 \,. \tag{56}$$

With the simplifications achieved, the eighteen equations (32)–(49) group themselves into the following sets of nine, six, and three equations:

$$\Omega^2 I_{123} = -\mathfrak{W}_{12;3} \,, \tag{57}$$

$$\Omega^2 I_{112} = -2\mathfrak{W}_{21;1} \,; \qquad \Omega^2 I_{221} = -2\mathfrak{W}_{12;2} \,, \tag{58}$$

$$\Omega^2 I_{133} = -2\mathfrak{W}_{13;3} \,; \qquad \Omega^2 I_{233} = -2\mathfrak{W}_{23;3} \,, \tag{59}$$

$$\Omega^2 I_{113} = \mathfrak{W}_{33;3} - \mathfrak{W}_{11;3} \,; \qquad \Omega^2 I_{223} = \mathfrak{W}_{33;3} - \mathfrak{W}_{22;3} \,, \tag{60}$$

$$\Omega^2 I_{111} = 2\left(\mathfrak{W}_{22;1} - \mathfrak{W}_{12;2} - \mathfrak{W}_{11;1}\right) \,, \tag{61}$$

$$\Omega^2 I_{222} = 2\left(\mathfrak{W}_{11;2} - \mathfrak{W}_{12;1} - \mathfrak{W}_{22;2}\right) \,, \tag{62}$$

$$\mathfrak{W}_{31;1} = \mathfrak{W}_{32;1} = \mathfrak{W}_{31;2} = \mathfrak{W}_{32;2} = 0 \,, \tag{63}$$

$$\mathfrak{W}_{32;3} + \mathfrak{W}_{12;1} + \mathfrak{W}_{33;2} - \mathfrak{W}_{11;2} = 0 \,, \tag{64}$$

$$\mathfrak{W}_{31;3} + \mathfrak{W}_{12;2} + \mathfrak{W}_{33;1} - \mathfrak{W}_{22;1} = 0 \,, \tag{65}$$

$$\mathfrak{W}_{12;2} - \mathfrak{W}_{22;1} = \Pi_1 \,; \qquad \mathfrak{W}_{12;1} - \mathfrak{W}_{11;2} = \Pi_2 \,; \qquad \mathfrak{W}_{33} = -\Pi_3 \,. \tag{66}$$

Equations (63)–(65) provide relations exactly of the kind we had set out to find in Section III; for, written out explicitly, these equations express the integral properties:

$$\mathfrak{W}_{13;1} = -\tfrac{1}{2}\int_V \rho x_1 \mathfrak{B}_{13} d\boldsymbol{x} = 0; \qquad \mathfrak{W}_{13;2} = -\tfrac{1}{2}\int_V \rho x_2 \mathfrak{B}_{13} d\boldsymbol{x} = 0 \,, \tag{67}$$

$$\mathfrak{W}_{23;1} = -\tfrac{1}{2}\int_V \rho x_1 \mathfrak{V}_{23} d\boldsymbol{x} = 0 \; ; \qquad \mathfrak{W}_{23;2} = -\tfrac{1}{2}\int_V \rho x_2 \mathfrak{V}_{23} d\boldsymbol{x} = 0 \, , \tag{68}$$

$$J_1 = \int_V \rho \left[x_3 \mathfrak{V}_{23} + x_1 \mathfrak{V}_{12} + x_2 (\mathfrak{V}_{33} - \mathfrak{V}_{11}) \right] d\boldsymbol{x} = 0 \, , \tag{69}$$

and

$$J_2 = \int_V \rho \left[x_3 \mathfrak{V}_{13} + x_2 \mathfrak{V}_{12} + x_1 (\mathfrak{V}_{33} - \mathfrak{V}_{22}) \right] d\boldsymbol{x} = 0 \, . \tag{70}$$

Each of the quantities listed, being an integral property of the configuration, is a functional of the configuration, and their vanishing is a *requirement* of equilibrium. It is evident that these functionals vanish *identically* for configurations (such as the Jacobi ellipsoids) which have triplanar symmetry; but they are *required* to vanish even if the configurations (such as the pear-shaped configurations) do not have such symmetry. It is this latter *requirement* that enables us to bridge, via these functionals, configurations having triplanar symmetry and those which do not have this symmetry. Thus $\mathfrak{W}_{13;1}$, $\mathfrak{W}_{23;1}$, $\mathfrak{W}_{13;2}$, $\mathfrak{W}_{23;2}$, J_1, and J_2 are all functionals having the properties of the hypothetical "J" postulated in Section III. It remains to find out whether any of them will enable us to exhibit and isolate the point of bifurcation along the Jacobian sequence; this we now proceed to find out.

VI. THE FIRST VARIATION OF $\mathfrak{W}_{ij;k}$

In this section we shall establish a lemma which will be useful in our subsequent analysis.

Consider an infinitesimal deformation of a given configuration. Any such deformation can be thought of as the result of each element of mass of the configuration having been subject to a certain Lagrangian displacement $\boldsymbol{\xi}$. Kinematically, the only restriction on $\boldsymbol{\xi}$ is that required by the conservation of mass, namely,

$$\delta \int_V \rho \, d\boldsymbol{x} = 0 \; ; \tag{71}$$

and, for the present, we shall not restrict $\boldsymbol{\xi}$ any further.

LEMMA: *The first variation of $\mathfrak{W}_{ij;k}$ due to an infinitesimal deformation of a configuration is given by*

$$\begin{aligned} -2\,\delta \mathfrak{W}_{ij;k} &= \delta \int_V \rho x_k \mathfrak{V}_{ij} d\boldsymbol{x} \\ &= \int_V \rho \xi_k \mathfrak{V}_{ij} d\boldsymbol{x} + \int_V \rho x_k \xi_l \frac{\partial \mathfrak{V}_{ij}}{\partial x_l} d\boldsymbol{x} + \int_V \rho \xi_l \frac{\partial \mathfrak{D}_{ij;k}}{\partial x_l} d\boldsymbol{x} \, , \end{aligned} \tag{72}$$

where $\boldsymbol{\xi}$ is the Lagrangian displacement which induces the deformation, and $\mathfrak{D}_{ij;k}$ is the tensor

$$\mathfrak{D}_{ij;k}(\boldsymbol{x}) = G \int_V \rho(\boldsymbol{x}') \frac{(x_i - x_i')(x_j - x_j') x_k'}{|\boldsymbol{x} - \boldsymbol{x}'|^3} d\boldsymbol{x}' \, . \tag{73}$$

The tensor $\mathfrak{D}_{ij;k}$ has been defined in an earlier paper (Chandrasekhar and Lebovitz 1962*c;* this paper will be referred to hereafter as "Paper II"); and it has also been shown (in Paper III) how, for homogeneous ellipsoids, explicit expressions for the different components of this tensor can be written down.

PROOF: Making use of the condition (71), we have

$$\delta\int_V dx\,\rho(x)\,x_k\mathfrak{B}_{ij}(x) = G\delta\int_V\int_V dx\,dx'\rho(x)\,\rho(x')\frac{x_k(x_i-x_i')(x_j-x_j')}{|x-x'|^3}$$

$$= G\int_V\int_V dx\,dx'\rho(x)\,\rho(x')\Big[\xi_k\frac{(x_i-x_i')(x_j-x_j')}{|x-x'|^3}$$

$$+x_k\left(\xi_l\frac{\partial}{\partial x_l}+\xi_l'\frac{\partial}{\partial x_l'}\right)\frac{(x_i-x_i')(x_j-x_j')}{|x-x'|^3}\Big]$$

$$= G\int_V dx\,\rho(x)\,\xi_k\int_V dx'\rho(x')\frac{(x_i-x_i')(x_j-x_j')}{|x-x'|^3} \tag{74}$$

$$+G\int_V dx\,\rho(x)\,x_k\xi_l\frac{\partial}{\partial x_l}\int_V dx'\rho(x')\frac{(x_i-x_i')(x_j-x_j')}{|x-x'|^3}$$

$$+G\int_V dx'\rho(x')\,\xi_l'\frac{\partial}{\partial x_l'}\int_V dx\,\rho(x)\frac{x_k(x_i'-x_i)(x_j'-x_j)}{|x'-x|^3}$$

$$=\int_V\rho\,\xi_k\mathfrak{B}_{ij}dx+\int_V\rho\,x_k\xi_l\frac{\partial\mathfrak{B}_{ij}}{\partial x_l}dx+\int_V\rho\,\xi_l\frac{\partial\mathfrak{D}_{ij;k}}{\partial x_l}dx\,.$$

VII. THE FIRST VARIATIONS OF THE BASIC FUNCTIONALS FOR A LAGRANGIAN DISPLACEMENT WHICH DEFORMS THE JACOBI ELLIPSOID INTO A PEAR-SHAPED CONFIGURATION

Since our prime interest is to find a condition that will exhibit the point of bifurcation along the Jacobian sequence where the pear-shaped configurations first become possible, we shall consider the first variations of the six functionals[3] listed in Section V (eqs. [67]–[70]) when the initial configuration is a Jacobi ellipsoid and the Lagrangian displacement is one which deforms it into a pear-shaped object. The Lagrangian displacement to be considered is, therefore, the one which we have already specified in Section III (eqs. [10] and [11]). The components of this displacement can be written in the forms

$$\xi_1=(\alpha+\beta)\,x_1^2-\alpha x_2^2-\beta x_3^2-1\,, \qquad \xi_2=-2\alpha x_1x_2\,,\quad\text{and}\quad \xi_3=-2\beta x_1x_3\,, \tag{75}$$

where

$$\alpha=-\frac{1}{a_2^2+\lambda}\quad\text{and}\quad\beta=-\frac{1}{a_3^2+\lambda}, \tag{76}$$

and λ is the numerically larger of the two roots of the equation (cf. Darwin 1901, eqs. on p. 297):

$$5\lambda^2+2\lambda(a_1^2+2a_2^2+2a_3^2)+3a_2^2a_3^2+a_1^2(a_2^2+a_3^2)=0\,. \tag{77}$$

Consider, first, $\delta\mathfrak{W}_{13;1}$. According to the lemma of Section VI,

$$-2\,\delta\mathfrak{W}_{13;1}=\int_V\rho\,\xi_1\mathfrak{B}_{13}dx+\int_V\rho\,x_1\xi_l\frac{\partial\mathfrak{B}_{13}}{\partial x_l}dx+\int_V\rho\,\xi_l\frac{\partial\mathfrak{D}_{13;1}}{\partial x_l}dx\,. \tag{78}$$

[3] These are the only functionals that are available for our present purposes (indeed, as we shall see presently, only one of them, J_2, actually turns out to be useful); for the relations provided by the nine equations (57)–(62) are not usable as they stand, since Ω^2, which occurs in them, is not invariable and, moreover, none of the relations that can be obtained by eliminating Ω^2 from any pair of these equations is useful because their first variations will vanish identically.

The expressions relevant to the evaluation of $\delta\mathfrak{W}_{13;1}$ can all be found in Paper III; they are

$$\mathfrak{D}_1 = a_1^2 x_1 \Big(A_1 - \sum_{l=1}^{3} A_{1l} x_l^2 \Big), \tag{79}$$

$$\mathfrak{D}_{13} = a_1^2 a_3^2 x_1 x_3 \Big(A_{13} - \sum_{l=1}^{3} A_{13l} x_l^2 \Big), \tag{80}$$

$$\mathfrak{B}_{13} = 2 (A_1 - a_3^2 A_{31}) x_1 x_3 , \tag{81}$$

$$\begin{aligned}\mathfrak{D}_{13;1} &= - x_3 \frac{\partial \mathfrak{D}_1}{\partial x_1} + \frac{\partial \mathfrak{D}_{13}}{\partial x_1} \\ &= - a_1^2 x_3 \Big[\Big(A_1 - \sum_{l=1}^{3} A_{1l} x_l^2 \Big) - 2 A_{11} x_1^2 \Big] \\ &\quad + a_1^2 a_3^2 x_3 \Big[\Big(A_{13} - \sum_{l=1}^{3} A_{13l} x_l^2 \Big) - 2 A_{131} x_1^2 \Big] ,\end{aligned} \tag{82}$$

where we have suppressed a common factor $(\pi G \rho a_1 a_2 a_3)$ in all the expressions. On inserting the foregoing expressions in equation (78), we readily find that

$$\delta \mathfrak{W}_{13;1} \equiv 0 \tag{83}$$

for any Lagrangian displacement of the general form considered. And it can be verified that the same thing is true of all the other functionals except J_2. Thus

$$\delta \mathfrak{W}_{23;1} \equiv \delta \mathfrak{W}_{13;2} \equiv \delta \mathfrak{W}_{23;2} \equiv \delta J_1 \equiv 0 ; \tag{84}$$

but

$$\delta J_2 \not\equiv 0 . \tag{85}$$

Among the six functionals listed in Section V, J_2 is therefore the only one that has all the properties requisite for the exhibition and the isolation of the point of bifurcation along the Jacobian sequence. We shall, accordingly, consider its first variation in some detail.

Now

$$\delta J_2 = \delta \int_V \rho \left[x_3 \mathfrak{B}_{13} + x_2 \mathfrak{B}_{12} + x_1 (\mathfrak{B}_{33} - \mathfrak{B}_{22}) \right] d\boldsymbol{x} . \tag{86}$$

According to the expressions for the tensor potential given in Paper III (eqs. [51] and [52]),

$$\mathfrak{B}_{13} = 2 (A_1 - a_3^2 A_{31}) x_1 x_3 , \tag{87}$$

$$\mathfrak{B}_{12} = 2 (A_1 - a_2^2 A_{21}) x_1 x_2 , \tag{88}$$

and

$$\begin{aligned}\mathfrak{B}_{33} - \mathfrak{B}_{22} = a_3^2 A_3 - a_2^2 A_2 + (a_2^2 A_{21} - a_3^2 A_{31}) x_1^2 + (A_1 - a_2^2 A_{21} - 2 a_3^2 A_{32}) x_2^2 \\ - (A_1 - a_3^2 A_{31} - 2 a_2^2 A_{23}) x_3^2 ,\end{aligned} \tag{89}$$

where, again, a constant common factor, $\pi G \rho a_1 a_2 a_3$, in all the expressions has been suppressed. In obtaining the particular form for $\mathfrak{B}_{33} - \mathfrak{B}_{22}$ given in equation (89), we have made use of the relations among the symbols A_i and A_{ij} which are given in Paper III (eqs. [23] and [29]).

Making use of equations (87)–(89), we find

$$\int_V \xi_3 \mathfrak{V}_{13} dx + \int_V x_3 \xi_l \frac{\partial \mathfrak{V}_{13}}{\partial x_l} dx = 2(A_1 - a_3^2 A_{31}) [\langle \xi_1 x_3^2 \rangle + 2\langle \xi_3 x_1 x_3 \rangle] , \tag{90}$$

$$\int_V \xi_2 \mathfrak{V}_{12} dx + \int_V x_2 \xi_l \frac{\partial \mathfrak{V}_{12}}{\partial x_l} dx = 2(A_1 - a_2^2 A_{21}) [\langle \xi_1 x_2^2 \rangle + 2\langle \xi_2 x_1 x_2 \rangle] , \tag{91}$$

and

$$\begin{aligned} \int_V \xi_1 (\mathfrak{V}_{33} - \mathfrak{V}_{22}) dx &+ \int_V x_1 \xi_l \frac{\partial}{\partial x_l} (\mathfrak{V}_{33} - \mathfrak{V}_{22}) dx \\ &= (a_3^2 A_3 - a_2^2 A_2) \langle \xi_1 \rangle + 3(a_2^2 A_{21} - a_3^2 A_{31}) \langle \xi_1 x_1^2 \rangle \\ &+ (A_1 - a_2^2 A_{21} - 2a_3^2 A_{32}) [\langle \xi_1 x_2^2 \rangle + 2\langle \xi_2 x_1 x_2 \rangle] \\ &- (A_1 - a_3^2 A_{31} - 2a_2^2 A_{23}) [\langle \xi_1 x_3^2 \rangle + 2\langle \xi_3 x_1 x_3 \rangle] , \end{aligned} \tag{92}$$

where the angular brackets signify that the quantity inclosed is to be integrated over the volume of the ellipsoid.

By the lemma of Section VI and equations (90)–(92), the first variation of J_2 due to an arbitrary Lagrangian displacement is given by

$$\begin{aligned} \delta J_2 = (a_3^2 A_3 - a_2^2 A_2) \langle \xi_1 \rangle &+ 3(a_2^2 A_{21} - a_3^2 A_{31}) \langle \xi_1 x_1^2 \rangle \\ &+ (3A_1 - 3a_2^2 A_{21} - 2a_3^2 A_{32}) [\langle \xi_1 x_2^2 \rangle + 2\langle \xi_2 x_1 x_2 \rangle] \\ &+ (A_1 - a_3^2 A_{31} + 2a_2^2 A_{23}) [\langle \xi_1 x_3^2 \rangle + 2\langle \xi_3 x_1 x_3 \rangle] \\ &+ \int_V \xi_l \frac{\partial}{\partial x_l} (\mathfrak{D}_{13;3} + \mathfrak{D}_{12;2} + \mathfrak{D}_{33;1} - \mathfrak{D}_{22;1}) dx , \end{aligned} \tag{93}$$

where an additional common factor, ρ, has been ignored. To evaluate the last remaining integral on the right-hand side of equation (93), we require a knowledge of the different components of the tensor $\mathfrak{D}_{ij;k}$ which appear in the integrand. They can all be determined with the aid of the formula (Paper II, eq. [15])

$$\mathfrak{D}_{ij;k} = -x_i \frac{\partial \mathfrak{D}_k}{\partial x_j} + \frac{\partial \mathfrak{D}_{ik}}{\partial x_j} \tag{94}$$

in terms of $\mathfrak{D}_k$ and $\mathfrak{D}_{ik}$; and the expressions for the latter are

$$\mathfrak{D}_i = a_i^2 x_i \Big(A_i - \sum_{l=1}^{3} A_{il} x_l^2 \Big) \tag{95}$$

and

$$\mathfrak{D}_{ik} = a_i^2 a_k^2 x_i x_k \Big(A_{ik} - \sum_{l=1}^{3} A_{ikl} x_l^2 \Big) \qquad (i \neq k) . \tag{96}$$

(When $i = k$ the expression for $\mathfrak{D}_{ik}$ is more complicated; but we do not need it in the present context.) Making use of equations (94)–(96) we find

$$\begin{aligned} \mathfrak{D}_{13;3} = a_3^2 x_1 [-(A_3 - a_1^2 A_{13}) &+ (A_{31} - a_1^2 A_{131}) x_1^2 + (A_{32} - a_1^2 A_{132}) x_2^2 \\ &+ 3(A_{33} - a_1^2 A_{133}) x_3^2] , \end{aligned} \tag{97}$$

$$\mathfrak{D}_{12;2} = a_2^2 x_1 [-(A_2 - a_1^2 A_{12}) + (A_{21} - a_1^2 A_{121}) x_1^2 + 3(A_{22} - a_1^2 A_{122}) x_2^2 + (A_{23} - a_1^2 A_{123}) x_3^2] , \tag{98}$$

and

$$\begin{aligned}\mathfrak{D}_{33;1} - \mathfrak{D}_{22;1} = a_1^2 [& (a_3^2 A_{31} - a_2^2 A_{21}) x_1 + (a_2^2 A_{211} - a_3^2 A_{311}) x_1^3 \\ & - (2A_{12} + a_3^2 A_{312} - 3a_2^2 A_{212}) x_1 x_2^2 \\ & + (2A_{13} + a_2^2 A_{213} - 3a_3^2 A_{313}) x_1 x_3^2] .\end{aligned} \tag{99}$$

The remaining integral in equation (93) can be evaluated now; and we find, after some minor reductions and rearrangements,

$$\begin{aligned}\delta J_2 = {} & 2\langle \xi_1 \rangle (a_1^2 a_3^2 A_{13} - a_2^2 A_2) + 6\langle \xi_1 x_1^2 \rangle (a_2^2 A_{21} - a_1^2 a_3^2 A_{131}) \\ & + [\langle \xi_1 x_2^2 \rangle + 2\langle \xi_2 x_1 x_2 \rangle] [3A_1 + 3a_2^2 A_{22} - (2a_1^2 + 3a_2^2) A_{12} \\ & \qquad - a_3^2 A_{32} - 2a_1^2 a_3^2 A_{132}] \\ & + [\langle \xi_1 x_3^2 \rangle + 2\langle \xi_3 x_1 x_3 \rangle] [A_1 + 3a_3^2 A_{33} + (2a_1^2 - a_3^2) A_{13} \\ & \qquad + 3a_2^2 A_{23} - 6a_1^2 a_3^2 A_{133}] .\end{aligned} \tag{100}$$

The terms in the brackets in equation (100) can be simplified if use is made of the relations among the symbols A_i and A_{ij} given in Paper III. We find

$$\begin{aligned}\tfrac{1}{2}\delta J_2 = {} & \langle \xi_1 \rangle (a_1^2 a_3^2 A_{13} - a_2^2 A_2) + 3\langle \xi_1 x_1^2 \rangle (a_2^2 A_{21} - a_1^2 a_3^2 A_{131}) \\ & + [\langle \xi_1 x_2^2 \rangle + 2\langle \xi_2 x_1 x_2 \rangle] [3(A_1 - a_2^2 A_{12}) - a_3^2 (A_{32} + a_1^2 A_{132})] \\ & + [\langle \xi_1 x_3^2 \rangle + 2\langle \xi_3 x_1 x_3 \rangle] [(2A_3 + a_2^2 A_{23}) - 3a_1^2 a_3^2 A_{133}] .\end{aligned} \tag{101}$$

Equation (101) is entirely general and is applicable to any Lagrangian displacement. For the Lagrangian displacement of the particular form specified in equation (75), the moments of the components of $\boldsymbol{\xi}$ which occur in equation (101) can be readily written down with the aid of the following elementary formulae:

$$\begin{aligned}\langle x_i^4 \rangle &= 4\pi a_1 a_2 a_3 \frac{a_i^4}{35} ; \qquad \langle x_i^2 x_j^2 \rangle = 4\pi a_1 a_2 a_3 \frac{a_i^2 a_j^2}{105} \quad (i \neq j) \\ \langle x_i^2 \rangle &= 4\pi a_1 a_2 a_3 \frac{a_i^2}{15} ; \qquad \text{and} \qquad \langle 1 \rangle = \tfrac{4}{3}\pi a_1 a_2 a_3 .\end{aligned} \tag{102}$$

Thus

$$\begin{aligned}\langle \xi_1 \rangle &= 7[(a_1^2 - a_2^2)\alpha + (a_1^2 - a_3^2)\beta - 5] , \\ \langle \xi_1 x_1^2 \rangle &= a_1^2 [(3a_1^2 - a_2^2)\alpha + (3a_1^2 - a_3^2)\beta - 7] , \\ \langle \xi_1 x_2^2 \rangle &= a_2^2 [(a_1^2 - 3a_2^2)\alpha + (a_1^2 - a_3^2)\beta - 7] , \\ \langle \xi_1 x_3^2 \rangle &= a_3^2 [(a_1^2 - a_2^2)\alpha + (a_1^2 - 3a_3^2)\beta - 7] ,\end{aligned} \tag{103}$$

and

$$\langle \xi_2 x_1 x_2 \rangle = -2a_1^2 a_2^2 \alpha , \qquad \text{and} \qquad \langle \xi_3 x_1 x_3 \rangle = -2a_1^2 a_3^2 \beta ,$$

where a common factor $4\pi a_1 a_2 a_3/105$ has been suppressed.

For α and β given by equation (76)

$$
\begin{aligned}
-\frac{1}{7}\langle \xi_1 \rangle &= \frac{a_1^2-a_2^2}{a_2^2+\lambda}+\frac{a_1^2-a_3^2}{a_3^2+\lambda}+5 \\
&= \frac{1}{(a_2^2+\lambda)(a_3^2+\lambda)}\left[5\lambda^2+2\lambda(a_1^2+2a_2^2+2a_3^2)+3a_2^2a_3^2+a_1^2(a_2^2+a_3^2)\right] \qquad (104 \\
&= 0
\end{aligned}
$$

by virtue of equation (77) satisfied by λ.

Now substituting from equations (103) and (104) in equation (101), we finally obtain (restoring the common factors which have been suppressed at different stages)

$$
\begin{aligned}
\frac{105\,\delta J_2}{8(\pi\rho a_1a_2a_3)^2G} &= 3a_1^2[(3a_1^2-a_2^2)\alpha+(3a_1^2-a_3^2)\beta-7](a_2^2A_{21}-a_1^2a_3^2A_{131}) \\
&+a_2^2[-3(a_1^2+a_2^2)\alpha+(a_1^2-a_3^2)\beta-7][3(A_1-a_2^2A_{12})-a_3^2(A_{32}+a_1^2A_{132})] \qquad (105) \\
&+a_3^2[(a_1^2-a_2^2)\alpha-3(a_1^2+a_3^2)\beta-7][(2A_3+a_2^2A_{23})-3a_1^2a_3^2A_{133}]\,.
\end{aligned}
$$

VIII. THE EXHIBITION AND THE ISOLATION OF THE POINT OF BIFURCATION ALONG THE JACOBIAN SEQUENCE

As we have already pointed out in Section III, we should be able to exhibit and isolate a point of bifurcation along an equilibrium sequence, if a functional of the equilibrium state can be found which vanishes non-trivially at least on one side of the point of bifurcation. Then the first variation of such a functional must vanish for a Lagrangian displacement which deforms the body from the shape it has before the point of bifurcation to the shape it has after the point of bifurcation—and *only* at the point of bifurcation, since at any other point along the sequence the displacement will not carry the equilibrium configuration into another equilibrium configuration. Since J_2 is a functional having all the properties requisite for the isolation of the point of bifurcation along the sequence of the Jacobi ellipsoids, it is clear that a necessary condition for its occurrence is that

$$
\delta J_2 = 0 \qquad (106)
$$

for the particular Lagrangian displacement specified by equations (75)–(77). In other words, if the quantity on the right-hand side of equation (105) is evaluated along the sequence of the Jacobi ellipsoids, we must find that it vanishes at the point of bifurcation. With the constants of the Jacobi ellipsoids tabulated in Appendix I (Table 2), it is a simple matter to carry out the necessary calculations. The results are given in the accompanying tabulation (eq. [107]). This tabulation exhibits and isolates the point of

$\cos^{-1}a_3/a_1$	$\delta\mathfrak{J}^*$
68°	+0.00505
69°	+0 00034
69°8166	0.00000
70°	+0 00023
71°	+0 00263

(107)[4]

bifurcation where the pear-shaped figures branch off; and the calculations isolating it cannot certainly be described as "arduous."

[4] The quantity $\delta\mathfrak{J}^*$ listed is the value of the expression on the right-hand side equation (104) evaluated with the constants normalized as they are in Table 2 of Appendix I.

IX. CONCLUDING REMARKS

The relative simplicity with which we have been able to isolate the point of bifurcation along the Jacobian sequence suggests that, with the aid of the general virial equations of the third order derived in this paper, we should be able to treat the stability of the Jacobi ellipsoids with comparable simplicity. The method, which will be based on equation (25), will be similar to the one (based on the corresponding virial equation of the second order) by which the stability of the Maclaurin spheroids has been considered by Lebovitz (1961). The solution to the problem of the stability of the Jacobi ellipsoids along these lines will be presented soon.

The research reported in this paper has in part been supported by the Office of Naval Research under contract Nonr-2121(24) with the University of Chicago.

APPENDIX I

The Properties and the Constants of the Jacobi Ellipsoids

In 1887, Darwin computed the principal constants of the Jacobi ellipsoids for nine members of the series (including the spheroid common to this and the Maclaurin series). In 1901, after he had determined the point of bifurcation along the Jacobian series, he added the constants for this critical ellipsoid to his earlier list. And this information provided by Darwin has been reproduced in all books dealing with this topic since that time. However, the information Darwin provided is not complete even for his ten cases: the constants A_i in the expression for the internal potential of the ellipsoid are essential to many calculations pertaining to the Jacobi ellipsoids, and they cannot be deduced, simply, from the tabulated constants. For the calculations of the present paper, not only the symbols A_i but also the higher index symbols, A_{ij} and A_{ijk} (these are defined in Paper III) are needed; and they will be needed again when we come to consider the stability of the Jacobi ellipsoids. For these reasons, it seemed worthwhile to provide more complete and more adequate information than Darwin's on the constants of the Jacobi ellipsoids.

Before we present the results of the newer calculations, we may note that the equation determining the geometry of the Jacobi ellipsoids is

$$\mathfrak{W}_{33} - \mathfrak{W}_{11;22} = 0 \; ; \tag{AI, 1}$$

with the expressions for the tensor components $\mathfrak{W}_{33}$ and $\mathfrak{W}_{11;22}$ given in Paper III (eqs. [57] and [63]), the equation gives

$$a_3^2 A_3 - a_1^2 a_2^2 A_{12} = 0 \; . \tag{AI, 2}$$

Now, substituting for A_{12} its expression in terms of A_1 and A_2, we obtain

$$a_3^2 A_3 + \frac{a_1^2 a_2^2}{a_1^2 - a_2^2}(A_1 - A_2) = 0 \, . \tag{AI, 3}$$

We shall rewrite this equation in the form

$$\Delta D = \left(A_1 - \frac{a_3^2}{a_1^2} A_3\right) - \left(A_2 - \frac{a_3^2}{a_2^2} A_3\right) = 0 \, . \tag{AI, 4}$$

Using the known expressions for the constants A_1 (quoted in Paper III, eqs. [15]–[17]), we can reduce equation (AI, 4) to the following form which is convenient for numerical calculations:

$$\begin{aligned}[a_1^4(a_2^2 + a_3^2) + a_2^4(a_3^2 + a_1^2) - 4a_1^2 a_2^2 a_3^2]E(\theta, \phi) - 2a_1^2 a_2^2(a_2^2 - a_3^2)F(\theta, \phi) \\ = \frac{a_2 a_3}{a_1}(a_1^2 - a_2^2)(2a_1^2 - a_2^2)\sqrt{(a_1^2 - a_3^2)} \, .\end{aligned} \tag{AI, 5}$$

In equation (AI, 5), $E(\theta, \phi)$ and $F(\theta, \phi)$ are the standard elliptic integrals of the two kinds; and the arguments are

$$\theta = \sin^{-1}\sqrt{\frac{a_1^2 - a_2^2}{a_1^2 - a_3^2}} \quad \text{and} \quad \phi = \cos^{-1}\frac{a_3}{a_1}. \tag{AI, 6}$$

For various initially assigned values of ϕ (i.e., of a_3/a_1), equation (AI, 5) was solved for θ (i.e., for a_2/a_1) by a method of successive approximations. (The accuracy with which the equation has been solved can be judged from the entries against ΔD in Table 2.) For every pair of values $(a_2/a_1, a_3/a_1)$ determined consistently with equation (AI, 5) the various related quantities in which one is interested were calculated. The results of the calculation are summarized in Tables 1 and 2.

TABLE 1

THE PROPERTIES OF THE JACOBI ELLIPSOIDS*

$\cos^{-1}a_3/a_1$	$a_1/\bar{a}$	$a_2/\bar{a}$	$a_3/\bar{a}$	$\Omega^2/2\pi G\rho$	$-[\mathfrak{W}]$	$[T]$	$[M]$
54°3576	1 197234	1 197234	0 697657	0 18711	0 585054	0 080461	0 303751
55	1 216081	1 178918	697516	18701	584999	.080473	303876
56	1 246772	1 150440	697186	18663	584740	080565	.304541
57	1 278733	1 122876	696448	18585	.584235	080733	.305812
58	1 312155	1 096024	695336	18470	583484	080983	.307710
59	1 347190	1 069802	693854	18317	582481	081312	310248
60	1 383985	1 044160	691992	18126	581216	081722	313453
61	1 422706	1 019056	.689742	17897	579679	.082215	317357
62	1 463530	0 994460	687086	17628	.577859	082788	321993
63	1 506673	0 970321	684015	17321	575743	083443	327404
64	1 552368	0 946605	680513	16975	573317	084177	333635
65	1 600892	0 923268	676566	16590	570564	084989	340741
66	1 652556	0 900273	672155	16166	.567466	085877	348785
67	1 707707	0 877597	667254	15704	.564000	086837	357838
68	1 766796	0 855169	661853	15204	.560142	087867	367990
69 .	1 830295	0 832968	655919	14666	555866	088961	379337
69 8166	1 885826	0 814976	.650659	14200	552041	089897	389571
70	1 898788	0 810953	649424	14092	551139	090112	391996
71	1 972986	0 789060	642341	13482	545924	091315	406110
72	2 053705	0 767258	634630	12838	540179	092559	421841
75	2 346331	0 701819	607275	10716	519239	096413	480943
77..	2 600213	0 657497	.584921	09166	501479	098897	53344
80	3 129265	0 588092	543391	06699	466969	101873	64274
83	4 006116	0 511280	0 488223	0 04187	0 418240	0 102437	0 81750

* In the second to fourth columns $\bar{a} = (a_1a_2a_3)^{1/3}$ For the meanings of the headings of the last three columns see text.

In Table 1 we list the principal physical characteristics of 24 members of the Jacobian series, including the critical ellipsoid at the point of bifurcation (at $\phi = 69°8166$) and the limiting spheroid (at $\phi = 54°3576$) with which the series begins. In addition to the principal axes in the unit $\bar{a} = (a_1a_2a_3)^{1/3}$ (introduced by Darwin in this connection), the quantities tabulated are

$$\frac{\Omega^2}{2\pi G\rho} = a_1a_2a_3(A_2 - a_1^2 A_{12}), \tag{AI, 7}$$

$$-[\mathfrak{W}] = -\frac{\mathfrak{W}}{GM^2/\bar{a}} = \frac{3}{10}(a_1a_2a_3)^{1/3}\sum_{i=1}^{3} A_i a_i^2, \tag{AI, 8}$$

$$[T] = \frac{\text{kinetic energy}}{GM^2/\bar{a}} = \frac{3}{20}\,\frac{a_1^2 + a_2^2}{(a_1a_2a_3)^{2/3}}\,\frac{\Omega^2}{2\pi G\rho}, \tag{AI, 9}$$

TABLE 2

THE CONSTANTS OF THE JACOBI ELLIPSOIDS†

$\cos^{-1}a_3/a_1$	54°35762	55°	56°	57°	58°	59°
a_2	1 000000	0 969440	0 922735	0 878116	0 835285	0 794099
a_3	0 582724	0 573576	0 559193	0 544639	0 529919	0 515038
A^*_1	0 515890	0 505637	0 489423	0 473126	0 456723	0 440213
A^*_2	0 515890	0 526042	0 542255	0 558410	0 574584	0 590789
A^*_3	0 968219	0 968321	0 968322	0 968463	0 968693	0 968997
I^*	1 360557	1 318586	1.253912	1.190986	1 129634	1.069801
ΔD‡	0	$-3\ 6\times10^{-6}$	-9×10^{-7}	$+1\times10^{-7}$	$+6\times10^{-7}$	-1×10^{-7}
A^*_{11}	0 328776	0 323812	0 315864	0 307734	0 299414	0 290896
A^*_{22}	0 328776	0 354994	0 400827	0 452306	0 510437	0 576249
A^*_{33}	1 506684	1 555203	1 636114	1 724624	1 821634	1 928235
A^*_{12}	0 328776	0 339029	0 355629	0 372561	0 389883	0 407617
A^*_{23}	0 684897	0 724070	0 790854	0 864260	0 945363	1 035253
A^*_{31}	0 684897	0 689534	0 696780	0 704236	0 711875	0 719693
A^*_{111}	0 243462	0 240428	0 235622	0 230616	0 225444	0 220085
A^*_{222}	0 243462	0 278974	0 346124	0 429424	0 533302	0 663334
A^*_{333}	2 971302	3 165530	3 503720	3 893135	4 343486	4 866799
A^*_{112}	0 243462	0 25283	0 26767	0 28320	0 299270	0 315967
A^*_{113}	0 539225	0 545032	0 554218	0 563718	0 573512	0 583607
A^*_{221}	0 243462	0 26526	0 30424	0 34836	0 398791	0 456494
A^*_{223}	0 539225	0 604227	0 723957	0 868267	1 043271	1 256414
A^*_{331}	1 244315	1 290097	1 366694	1 450717	1 543078	1 644866
A^*_{332}	1 244315	1 360674	1 568949	1 813371	2 101942	2 44432
A^*_{123}	0 539225	0 573824	0 633235	0 699064	0 772374	0 854234

$\cos^{-1}a_3/a_1$	60°	61°	62°	63°	64°	65°
a_2	0 754459	0 716280	0 679494	0 644016	0 609782	0 576721
a_3	0 500000	0 484810	0 469472	0 453990	0 438371	0 422618
A^*_1	0 423606	0 406915	0 390154	0 373333	0 356468	0 339571
A^*_2	0 607019	0 623260	0 639494	0 655714	0 671903	0 688053
A^*_3	0 969375	0 969824	0 970352	0 970953	0 971629	0 972376
I^*	1 011471	0 954627	0 899284	0.845415	0 793021	0.742095
ΔD‡	-6×10^{-7}	-4×10^{-7}	-6×10^{-7}	-8×10^{-7}	-5×10^{-7}	$+7\times10^{-7}$
A^*_{11}	0 282184	0 273279	0 264187	0 254908	0 245450	0 235815
A^*_{22}	0 650907	0 735783	0 832466	0 942917	1 069449	1 214901
A^*_{33}	2 045713	2 175584	2 319644	2 480018	2 659248	2 860392
A^*_{12}	0 425757	0 444294	0 463210	0 482503	0 502153	0 522153
A^*_{23}	1 135169	1 246557	1 371102	1 510854	1 668250	1 846240
A^*_{31}	0 727691	0 735868	0 744229	0 752772	0 761497	0 770403
A^*_{111}	0 214543	0 208815	0 202902	0 196799	0 190508	0 184026
A^*_{222}	0 826630	1 032434	1 29274	1 62354	2 045896	2 588220
A^*_{333}	5 478028	6 195868	7 043898	8 052018	9 258584	10 713159
A^*_{112}	0 333276	0 351200	0 369734	0 388888	0 408655	0 429041
A^*_{113}	0 594009	0 604723	0 615758	0 627116	0 638806	0 650831
A^*_{221}	0 522645	0 598611	0 685980	0 786706	0 903097	1 037991
A^*_{223}	1 51707	1 83720	2 23215	2 72197	3 33289	4 09957
A^*_{331}	1 757363	1 882081	2 020808	2 175667	2 349194	2 544442
A^*_{332}	2 85250	3 34162	3 93084	4 64495	5 51583	6 585352
A^*_{123}	0 945883	1 048765	1 164566	1 295328	1.443492	1 611999

† The unit of distance adopted in this table is the semimajor axis a_1 For the meaning of the asterisks to the symbols see text

‡ This is the quantity defined in eq (AI, 4)

TABLE 2—*Continued*

$\cos^{-1}a_3/a_1$	66°	67°	68°	69°	69°8166	70°
a_2	0 544776	0 513904	0 484022	0 455101	0 432159	0 427090
a_3.	0 406737	0 390731	0 374607	0 358368	0 345026	0 342020
A^*_1	0 322660	0 305755	0 288865	0 272016	0 258301	0 255229
A^*_2	0 704145	0 720151	0 736082	0 751902	0 764728	0 767592
A^*_3	0 973195	0 974094	0 975052	0 976082	0 976971	0 977179
I^* .	0 692637	0 644660	0 598142	0 553104	0 517424	0 509550
ΔD‡	$+2\ 5\times10^{-6}$	$-1\ 2\times10^{-6}$	$+1\ 7\times10^{-6}$	$+1\ 8\times10^{-6}$	$+1\ 4\times10^{-6}$	$-1\ 7\times10^{-6}$
A^*_{11}	0 226009	0 216043	0 205919	0 195649	0 187163	0 185246
A^*_{22}	1 382698	1 576944	1 803057	2 067406	2 317308	2 378194
A^*_{33}	3 087160	3 344097	3 636750	3 972075	4 283544	4 358786
A^*_{12}	0 542484	0 563112	0 584045	0 605241	0 622728	0 626672
A^*_{23}	2 048395	2 279029	2 543662	2 848941	3 134220	3 203317
A^*_{31}	0 779489	0 788760	0 798198	0 807811	0 815783	0 817590
A^*_{111}	0 177353	0 170494	0 163444	0 156210	0 150168	0 148796
A^*_{222}	3 288762	4 199280	5 39280	6 96939	8 63854	9 07148
A^*_{333}	12 480529	14 646359	17 32486	20 671042	24 044408	24 897542
A^*_{112}	0 450038	0 471624	0 493816	0 516585	0 535593	0 539909
A^*_{113}	0 663196	0 675909	0 688962	0 702364	0 713565	0 716113
A^*_{221}	1 19481	1 377671	1 591976	1 844111	2 083742	2 142287
A^*_{223}	5 06825	6 30090	7 88317	9 93196	12 0635	12 6112
A^*_{331}	2 765117	3 015755	3 301909	3 630523	3 936358	4 010314
A^*_{332}	7 908586	9 558505	11 63509	14 27309	16 97224	17 66020
A^*_{123}	1 804425	2 025089	2 279499	2 574312	2 850867	2 917983

$\cos^{-1}a_3/a_1$	71°	72°	75°	77°	80°	83°
a_2	0 399932	0 373597	0 299113	0 252863	0 187933	0 127625
a^3.	0 325568	0 309017	0 258819	0 224951	0 173648	0 121869
A^*_1	0 238521	0 221920	0 173043	0 141562	0 0968573	0 0566418
A^*_2	0 783150	0 798541	0 843499	0 872191	0 912638	0 948784
A^*_3	0 978329	0 979538	0 983454	0 986249	0 990512	0 994588
I^*.	0 467480	0 426914	0 314389	0 247237	0 158958	0.0868674
ΔD‡	$+2\ 3\times10^{-6}$	$+2\ 4\times10^{-6}$	-6×10^{-7}	0	$+1\ 0\times10^{-5}$	-8×10^{-6}
A^*_{11}	0 174717	0 164080	0 131690	0 109918	0 0776373	0 0470000
A^*_{22}	2 746075	3 184424	5 130827	7 315576	13 568	29 995
A^*_{33}	4 807890	5 333626	7 587490	10 02713	16 776	33 9370
A^*_{12}	0 648327	0 670158	0 736334	0 780536	0 845648	0 906914
A^*_{23}	3 617688	4 105793	6 22533	8 55225	15 077	31 898
A^*_{31}	0 827521	0 837601	0 868596	0 889709	0 921440	0 952086
A^*_{111}	0 141203	0 133440	0 109226	0 0924442	0 0667666	0 041423
A^*_{222}	11 9050	15 7681	39 26868	77 89900	259 58	1237
A^*_{333}	30 30087	37 30162	75 58220	132 167	370 88	1523
A^*_{112}	0 563784	0 588171	0 664057	0 716426	0 796129	0 874152
A^*_{113}	0 730201	0 744626	0 789813	0 821353	0 870038	0 918731
A^*_{221}	2 497158	2 922121	4 826296	6 981430	13 188	29 570
A^*_{223}	16 1556	20 9006	48 6844	92 7278	292 2	1325
A^*_{331}	4 452288	4 970682	7 20129	9 62444	16 348	33 4822
A^*_{332}	22 0607	27 8525	60 5902	110 589	328 9	1420
A^*_{123}	3 32141	3 79834	5 88312	8 18597	14 674	31 458

and

$$[M]=\frac{\text{angular momentum}}{(GM^3\bar{a})^{1/2}}=\left[\frac{3}{50}\frac{(a_1^2+a_2^2)^2}{(a_1a_2a_3)^{4/3}}\frac{\Omega^2}{2\pi G\rho}\right]^{1/2}. \tag{AI, 10}$$

Table 2 is devoted to the symbols A_i, A_{ij}, and A_{ijk}. These symbols are defined in Paper III and occur in many calculations pertaining to the Jacobi ellipsoids. In tabulating these symbols, the semimajor axis, a_1, was adopted as the unit of distance. Also, the constants A_i^* which are tabulated differ from the A_i's (as they have been defined) by the constant factor $a_1a_2a_3$ so that

$$\sum_{i=1}^{3}A_i^*=2 \quad \text{instead of} \quad \sum_{i=1}^{3}A_i=\frac{2}{a_1a_2a_3}. \tag{AI, 11}$$

The two and the three index symbols are derived from the A_i^*'s with the aid of the formulae given in Paper III; they are distinguished by asterisks to emphasize that they differ from the symbols without the asterisks by the factor $a_1a_2a_3$ and that their values depend on the adopted unit of distance (a_1).

I am indebted to Miss Donna D. Elbert for her assistance with the calculations summarized in Tables 1 and 2.

APPENDIX II

The Second Point of Bifurcation along the Maclaurin Sequence

It is clear that a sequence of pear-shaped configurations must branch off from the Maclaurin sequence even as one branches off from the Jacobian sequence. An infinitesimal Lagrangian displacement, corresponding to that given by equation (10) for the Jacobi ellipsoid, which will deform a Maclaurin spheroid into a pear-shaped object must be of the form

$$\xi_j=\text{constant}\,\frac{\partial}{\partial x_j}\,x_1(x_1^2+x_2^2+\beta x_3^2-\kappa), \tag{AII, 1}$$

where the constants β and κ, determined by the conditions

$$\operatorname{div}\boldsymbol{\xi}=0 \quad \text{and} \quad \langle\boldsymbol{\xi}\rangle=0, \tag{AII, 2}$$

have the values

$$\beta=-4 \quad \text{and} \quad \kappa=\tfrac{4}{5}(a_1^2-a_3^2). \tag{AII, 3}$$

The components of $\boldsymbol{\xi}$, apart from an arbitrary multiplicative constant, are, therefore,

$$\xi_1=3x_1^2+x_2^2-4x_3^2-\tfrac{4}{5}(a_1^2-a_3^2),$$

$$\xi_2=2x_1x_2 \quad \text{and} \quad \xi_3=-8x_1x_3. \tag{AII, 4}$$

For this displacement, we readily find that (cf. eqs. [102] and [103])

$$\langle\xi_1x_1^2\rangle=\langle\xi_1x_2^2\rangle+2\langle\xi_2x_1x_2\rangle=\tfrac{2}{5}a_1^2(11a_1^2+4a_3^2)$$

and

$$\langle\xi_1x_3^2\rangle+2\langle\xi_3x_1x_3\rangle=-\tfrac{8}{5}a_3^2(11a_1^2+4a_3^2). \tag{AII, 5}$$

Inserting the foregoing averages in equation (101) and simplifying appropriately for this case ($a_1 = a_2$), we find that

$$\tfrac{1}{2}\delta J_2 = \tfrac{2}{5}\,(11a_1^2 + 4a_3^2)\,(3a_1^2 A_1 - 8a_3^2 A_3 - 5a_1^2 a_3^2 A_{13} - 4a_1^4 a_3^2 A_{113} + 12a_1^2 a_3^4 A_{133})\,. \tag{AII, 6}$$

Hence, at the point of bifurcation, the condition

$$3a_1^2 A_1 - 8a_3^2 A_3 - 5a_1^2 a_3^2 A_{13} - 4a_1^4 a_3^2 A_{113} + 12a_1^2 a_3^4 A_{133} = 0 \tag{AII, 7}$$

must hold. Remembering that along the Maclaurin sequence

$$\frac{\Omega^2}{\pi G\rho a_1^2 a_3} = \frac{2}{a_1^2}(a_1^2 A_1 - a_3^2 A_3)\,, \tag{AII, 8}$$

we can rewrite the condition (AII, 7) in the form

$$\frac{4\Omega^2}{\pi G\rho a_1^2 a_3} = 5A_1 + 5a_3^2 A_{13} + 4a_1^2 a_3^2 A_{113} - 12a_3^4 A_{133}\,. \tag{AII, 9}$$

It is found that this condition is satisfied when the eccentricity along the Maclaurin sequence attains the value

$$e = 0.96937\,. \tag{AII, 10}$$

At this point, then, the Maclaurin sequence has its second point of bifurcation. It will be shown in a later paper that at this second point of bifurcation the Maclaurin spheroid becomes neutrally stable with respect to a particular mode of third harmonic oscillation while the Jacobi ellipsoid becomes unstable at the corresponding point along the Jacobian sequence.

APPENDIX III

The Third-Order Virial Equations in Hydromagnetics

The extension of the analysis of Section IV to hydromagnetics is straightforward. We start with the equation of motion in its standard form, namely,

$$\rho\frac{du_i}{dt} = -\frac{\partial}{\partial x_i}\left(p + \frac{|\boldsymbol{H}|^2}{8\pi}\right) + \rho\frac{\partial \mathfrak{B}}{\partial x_i} + \frac{1}{4\pi}\frac{\partial}{\partial x_l}H_i H_l\,, \tag{AIII, 1}$$

(where $\boldsymbol{H}$ denotes the intensity of the prevailing magnetic field) and proceed in the same way. However, the specification of the volume over which the integration is to be effected, after multiplication by $x_j\, x_k$, requires some consideration, since the magnetic field may not vanish in the regions in which the material density vanishes. On this account, if the integration is effected only over those regions in which the material density is non-vanishing, then the virial equations (obtained after some of the integrals have been transformed by integration by parts) will include surface integrals over the non-vanishing magnetic fields on the boundary. The occurrence of such surface integrals can be formally avoided by extending the integration over those regions in which the magnetic field is also non-vanishing. Adopting this latter procedure, we readily obtain from equation (AIII, 1) the required third-order equation:

$$\frac{d\mathfrak{B}_{i;jk}}{dt} = 2(\mathfrak{T}_{ij;k} + \mathfrak{T}_{ik;j} - \mathfrak{M}_{ij;k} - \mathfrak{M}_{ik;j}) + \mathfrak{W}_{ij;k} + \mathfrak{W}_{ik;j} + (\Pi_k + \mathfrak{M}_{ll;k})\,\delta_{ij} + (\Pi_j + \mathfrak{M}_{ll;j})\,\delta_{ik}\,, \tag{AIII, 2}$$

where

$$\mathfrak{M}_{ij;k} = \frac{1}{8\pi}\int_V H_i H_j x_k d\boldsymbol{x}\,, \tag{AIII, 3}$$

and the remaining tensors have the same meanings as in § IV.

APPENDIX IV

THE TENSOR $\mathfrak{W}_{ij;k}$ IN POLYTROPIC EQUILIBRIUM

In a recent note (Chandrasekhar 1961) it has been shown how, under conditions of polytropic equilibrium, we can express, with the aid of the virial equations of the second order, the components of the potential energy tensor[5] $\mathfrak{W}_{ij}$ in terms of the mass M, the moment of inertia tensor I_{ij}, and the gravitational potential $\mathfrak{B}_0$ at the pole of the configuration. It will appear that, with the aid of the virial equations of the third order, we can similarly express, under the same conditions, the components of the tensor $\mathfrak{W}_{ij;k}$ in terms of $\mathfrak{B}_0$ and the moment of inertia tensors of the odd ranks 1 and 3.

By taking the x_k-moment of the equation

$$(n+1)p = \rho[\mathfrak{B} + \tfrac{1}{2}\Omega^2(x_1^2 + x_2^2) - \mathfrak{B}_0] \,, \tag{AIV, 1}$$

which obtains in polytropic equilibrium (cf. Chandrasekhar 1961, eq. [13]), we have

$$(n+1)\Pi_k = -2\mathfrak{W}_{jj;k} + \tfrac{1}{2}\Omega^2(I_{11k} + I_{22k}) - \mathfrak{B}_0 I_k \,, \tag{AIV, 2}$$

where

$$I_k = \int_V \rho x_k dx \,. \tag{AIV, 3}$$

Supplementing the eighteen equations (57)–(66) by the three equations provided by equation (AIV, 2), we have a set of equations which will enable us to express all the eighteen different components of $\mathfrak{W}_{ij;k}$ in terms of the components of I_{ijk} and I_k.

Note added in proof.—The statement in the footnote on page 1057 that the nine equations (57–62) cannot be used to isolate neutral points and points of bifurcation is not strictly correct: Ω^2 can be considered as invariable for deformations belonging to the third harmonic. On this last account, equations (57–62) can in fact be used to isolate *all* configurations neutrally stable with respect to third harmonic deformations along the Maclaurin and the Jacobian sequences without even specifying the form of the Lagrangian displacement. These matters are considered in a paper now in preparation.

REFERENCES

Chandrasekhar, S. 1961, *Ap. J.*, **134,** 662.
Chandrasekhar, S., and Lebovitz, N. R. 1962*a*, *Ap. J.*, **135,** 238 (referred to as Paper I).
———. 1962*b*, *ibid*, p. 248.
———. 1962*c*, *ibid.*, **136,** 1032 (referred to as Paper II).
———. 1962*d*, *ibid*, p. 1037 (referred to as Paper III).
Darwin, G. H. 1887, *Proc R. Soc. London, A*, **41,** 319; also *Scientific Papers*, Vol **3** (Cambridge: Cambridge University Press, 1910), p. 119.
———. 1901, *Phil. Trans R. Soc. London, A*, **198,** 301; also *Scientific Papers*, Vol. **3** (Cambridge: Cambridge University Press, 1910), p 288.
Jeans, J. H. 1919, *Problems of Cosmogony and Stellar Dynamics* (Cambridge: Cambridge University Press), pp. 82–85.
———. 1929, *Astronomy and Cosmogony* (Cambridge: Cambridge University Press), pp. 216–219.
Lebovitz, N. R. 1961, *Ap. J.*, **134,** 500.
Lyttleton, R. A. 1953, *The Stability of Rotating Liquid Masses* (Cambridge: Cambridge University Press), pp. 1 and 109.
Thomson, W., and Tait, P. G. 1883, *Treatise on Natural Philosophy* (Cambridge: Cambridge University Press), Part II, pp. 324–335 (later reprint editions).

[5] In the note, expressions were written down explicitly only for the diagonal components; the nondiagonal components follow from the general relations (see n. 1 on p. 1049)

$$\mathfrak{W}_{13} = \mathfrak{W}_{23} = 0 \quad \text{and} \quad \mathfrak{W}_{12} = -\Omega^2 I_{12} \,.$$