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ON SUPERPOTENTIALS IN THE THEORY OF NEWTONIAN GRAVITATION. II. TENSORS OF HIGHER RANK

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ABSTRACT

In addition to the tensors considered in the earlier paper, the following tensors are defined and studied:

$$\mathfrak{D}_{ij}(x) = G\int_V \rho(x')\frac{x'_i x'_j}{|x-x'|}\,dx',$$

$$\mathfrak{D}_{ij;k}(x) = G\int_V \rho(x')\frac{(x_i-x'_i)(x_j-x'_j)x'_k}{|x-x'|^3}\,dx',$$

and

$$\mathfrak{B}_{ijkl}(x) = G\int_V \rho(x')\frac{(x_i-x'_i)(x_j-x'_j)(x_k-x'_k)(x_l-x'_l)}{|x-x'|^5}\,dx'.$$

These tensors are useful in problems (such as the stability of the Jacobi ellipsoids) in which it is necessary to examine the effects of perturbations belonging to the third harmonic.

I. INTRODUCTION

In earlier papers (Chandrasekhar 1960, 1961; Lebovitz 1961; Chandrasekhar and Lebovitz 1962*a*, *b*) we have shown the utility of defining the quantities

$$\mathfrak{D}_i(x) = G\int_V \rho(x')\frac{x'_i}{|x-x'|}\,dx' \tag{1}$$

and

$$\mathfrak{B}_{ij}(x) = G\int_V \rho(x')\frac{(x_i-x'_i)(x_j-x'_j)}{|x-x'|^3}\,dx' \tag{2}$$

where G is the constant of gravitation, $dx' = dx'_1\,dx'_2 dx'_3$, and the integration is effected over the whole volume V occupied by the fluid. The vector $\mathfrak{D}_i$ and the tensor potential $\mathfrak{B}_{ij}$ are related between themselves and with the Newtonian potential,

$$\mathfrak{B}(x) = G\int_V \frac{\rho(x')}{|x-x'|}\,dx', \tag{3}$$

and the superpotential

$$\chi(x) = -G\int_V \rho(x')\,|x-x'|\,dx', \tag{4}$$

by the relations

$$\mathfrak{B}_{ij} = -x_i\frac{\partial\mathfrak{B}}{\partial x_j}+\frac{\partial\mathfrak{D}_i}{\partial x_j} \tag{5}$$

$$= \mathfrak{B}\,\delta_{ij}+\frac{\partial^2\chi}{\partial x_i\partial x_j}, \tag{6}$$

and

$$\mathfrak{D}_i = x_i\mathfrak{B}+\frac{\partial\chi}{\partial x_i}. \tag{7}$$

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Further, $\mathfrak{D}_i$ and χ are governed by the "Poisson" equations,

$$\nabla^2 \mathfrak{D}_i = -4\pi G \rho x_i \tag{8}$$

and

$$\nabla^2 \chi = -2\mathfrak{B} . \tag{9}$$

In the further developments of the theory (cf. Chandrasekhar 1962) we have found that it is necessary to introduce the following tensors of still higher rank:

$$\mathfrak{D}_{ij}(x) = G\int_V \rho(x') \frac{x_i' x_j'}{|x - x'|} dx' , \tag{10}$$

$$\mathfrak{D}_{ij;k}(x) = G\int_V \rho(x') \frac{(x_i - x_i')(x_j - x_j')x_k'}{|x - x'|^3} dx' , \tag{11}$$

and

$$\mathfrak{B}_{ijkl}(x) = G\int_V \rho(x') \frac{(x_i - x_i')(x_j - x_j')(x_k - x_k')(x_l - x_l')}{|x - x'|^5} dx' . \tag{12}$$

On contraction, the tensors $\mathfrak{D}_{ij;k}$ and $\mathfrak{B}_{ijkl}$ reduce to the corresponding ones of lower order:

$$\mathfrak{D}_{ii;k} = \mathfrak{D}_k \quad \text{and} \quad \mathfrak{B}_{iikl} = \mathfrak{B}_{kl} . \tag{13}$$

II. THE TENSORS $\mathfrak{D}_{ij}$ AND $\mathfrak{D}_{ij;k}$

Clearly, $\mathfrak{D}_{ij}$ is the Newtonian potential induced by the "density distribution" $\rho x_i x_j$; accordingly, it can be deduced from Poisson's equation

$$\nabla^2 \mathfrak{D}_{ij} = -4\pi G \rho x_i x_j . \tag{14}$$

The tensor $\mathfrak{D}_{ij;k}$ is the proper tensorial generalization of $\mathfrak{D}_k$ even as $\mathfrak{B}_{ij}$ is the proper tensorial generalization of $\mathfrak{B}$: for $\mathfrak{D}_{ij;k}$ is the "$\mathfrak{B}_{ij}$" induced by ρx_k even as $\mathfrak{D}_k$ is the "$\mathfrak{B}$" induced by ρx_k. With this last interpretation of $\mathfrak{D}_{ij;k}$, we can express it directly in terms of $\mathfrak{D}_k$ and $\mathfrak{D}_{ik}$; for, according to equation (5), the tensor potential $\mathfrak{B}_{ij}$ is derivable from the gravitational potentials $\mathfrak{B}$ (induced by the basic density distribution ρ) and $\mathfrak{D}_i$ (induced by the basic density distribution *times* x_i); and, since the basic density distribution for $\mathfrak{D}_{ij;k}$ (considered as a tensor potential in its first two indices) is ρx_k, it is clear that $\mathfrak{D}_k$ and $\mathfrak{D}_{ik}$ play for it the roles of "$\mathfrak{B}$" and "$\mathfrak{D}_i$" for $\mathfrak{B}_{ij}$. Hence

$$\mathfrak{D}_{ij;k} = -x_i \frac{\partial \mathfrak{D}_k}{\partial x_j} + \frac{\partial \mathfrak{D}_{ik}}{\partial x_j} , \tag{15}$$

a relation which can also be derived directly from the definitions (1), (10), and (11) of the respective quantities. In analogy with equation (6), we can also infer the existence of a superpotential, χ_k, in terms of which $\mathfrak{D}_{ij;k}$ must be expressible in the form

$$\mathfrak{D}_{ij;k} = \delta_{ij}\mathfrak{D}_k + \frac{\partial^2 \chi_k}{\partial x_i \partial x_j} , \tag{16}$$

where (cf. eq. [4])

$$\chi_k = -G\int_V \rho(x') x_k' |x - x'| dx' . \tag{17}$$

III. THE COMPLETELY SYMMETRIC TENSOR $\mathfrak{B}_{ijkl}$

From the definitions of the respective quantities, we can readily derive the relation

$$-x_k \frac{\partial \mathfrak{B}_{ij}}{\partial x_l} + \frac{\partial \mathfrak{D}_{ij;k}}{\partial x_l} = 3\mathfrak{B}_{ijkl} - \delta_{il}\mathfrak{B}_{jk} - \delta_{jl}\mathfrak{B}_{ik} . \tag{18}$$

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Interchanging the indices k and l and remembering that $\mathfrak{V}_{ijkl}$ is symmetric in its indices, we obtain

$$-x_l \frac{\partial \mathfrak{V}_{ij}}{\partial x_k} + \frac{\partial \mathfrak{D}_{ij;l}}{\partial x_k} = 3\mathfrak{V}_{ijkl} - \delta_{ik}\mathfrak{V}_{jl} - \delta_{jk}\mathfrak{V}_{il} . \tag{19}$$

On the subtraction of one of equations (18) and (19) from the other, we obtain

$$\frac{\partial \mathfrak{D}_{ij;k}}{\partial x_l} - \frac{\partial \mathfrak{D}_{ij;l}}{\partial x_k} + \delta_{il}\mathfrak{V}_{jk} + \delta_{jl}\mathfrak{V}_{ik} - \delta_{ik}\mathfrak{V}_{jl} - \delta_{jk}\mathfrak{V}_{il} = x_k \frac{\partial \mathfrak{V}_{ij}}{\partial x_l} - x_l \frac{\partial \mathfrak{V}_{ij}}{\partial x_k} . \tag{20}$$

Substituting for the tensor potentials, on the left-hand side of equation (20) in terms of $\mathfrak{V}$ and χ (in accordance with eq. [6]), we find, after some further reductions, that

$$\begin{aligned} \frac{\partial}{\partial x_l}\left[\mathfrak{D}_{ij;k} + \left(x_i \frac{\partial}{\partial x_j} + x_j \frac{\partial}{\partial x_i}\right)\frac{\partial \chi}{\partial x_k}\right] - \frac{\partial}{\partial x_k}\left[\mathfrak{D}_{ij;l} + \left(x_i \frac{\partial}{\partial x_j} + x_j \frac{\partial !}{\partial x_i}\right)\frac{\partial \chi}{\partial x_l}\right] \\ = x_k \frac{\partial \mathfrak{V}_{ij}}{\partial x_l} - x_l \frac{\partial \mathfrak{V}_{ij}}{\partial x_k} . \end{aligned} \tag{21}$$

The left-hand side of this equation is manifestly the curl of the expression in the square brackets, considered as a vector with respect to the index after the semicolon; and the right-hand side is similarly the curl of $\mathfrak{V}_{ij}x_k$ with respect to the same index (k). Accordingly, we may write

$$\mathfrak{D}_{ij;k} + \left(x_i \frac{\partial}{\partial x_j} + x_j \frac{\partial}{\partial x_i}\right)\frac{\partial \chi}{\partial x_k} = x_k \mathfrak{V}_{ij} + \frac{\partial \chi_{ij}}{\partial x_k} , \tag{22}$$

where χ_{ij} is some symmetric tensor. Rewriting equation (22) in the manner

$$\mathfrak{D}_{ij;k} = \frac{\partial}{\partial x_k}\left[\chi_{ij} - \left(x_i \frac{\partial}{\partial x_j} + x_j \frac{\partial}{\partial x_i}\right)\chi\right] + \delta_{ik}\frac{\partial \chi}{\partial x_j} + \delta_{jk}\frac{\partial \chi}{\partial x_i} + x_k \mathfrak{V}_{ij} , \tag{23}$$

differentiating it with respect to x_l, and making further use of equation (6) relating the tensor potential and χ, we find

$$\begin{aligned} \frac{\partial \mathfrak{D}_{ij;k}}{\partial x_l} = \frac{\partial^2}{\partial x_k \partial x_l}\left[\chi_{ij} - \left(x_i \frac{\partial}{\partial x_j} + x_j \frac{\partial}{\partial x_i}\right)\chi\right] + \delta_{lk}\mathfrak{V}_{ij} + \delta_{ik}(\mathfrak{V}_{jl} - \mathfrak{V}\delta_{jl}) \\ + \delta_{jk}(\mathfrak{V}_{il} - \mathfrak{V}\delta_{il}) + x_k \frac{\partial \mathfrak{V}_{ij}}{\partial x_l} . \end{aligned} \tag{24}$$

Inserting this last result in equation (18), we find

$$\begin{aligned} 3\mathfrak{V}_{ijkl} = \frac{\partial^2}{\partial x_k \partial x_l}\left[\chi_{ij} - \left(x_i \frac{\partial}{\partial x_j} + x_j \frac{\partial}{\partial x_i}\right)\chi\right] + \delta_{ik}(\mathfrak{V}_{jl} - \mathfrak{V}\delta_{jl}) \\ + \delta_{jk}(\mathfrak{V}_{il} - \mathfrak{V}\delta_{il}) + \delta_{lk}\mathfrak{V}_{ij} + \delta_{il}\mathfrak{V}_{jk} + \delta_{jl}\mathfrak{V}_{ik} , \end{aligned} \tag{25}$$

and, adding to this equation the identity

$$0 = -\delta_{ij}\frac{\partial^2 \chi}{\partial x_k \partial x_l} + \delta_{ij}(\mathfrak{V}_{kl} - \delta_{kl}\mathfrak{V}) , \tag{26}$$

we obtain

$$\begin{aligned} 3\mathfrak{V}_{ijkl} = \frac{\partial^2}{\partial x_k \partial x_l}\left[\chi_{ij} - \left(x_i \frac{\partial}{\partial x_j} + x_j \frac{\partial}{\partial x_i}\right)\chi - \delta_{ij}\chi\right] + \mathfrak{V}_{ij}\delta_{kl} + \mathfrak{V}_{jk}\delta_{li} + \mathfrak{V}_{kl}\delta_{ij} \\ + \mathfrak{V}_{li}\delta_{jk} + \mathfrak{V}_{ik}\delta_{jl} + \mathfrak{V}_{jl}\delta_{ik} - (\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{jk}\delta_{li})\mathfrak{V} . \end{aligned} \tag{27}$$

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From the symmetry in all four indices of $\mathfrak{B}_{ijkl}$ and of the terms on the right-hand side in the tensor potential and in $\mathfrak{B}$, we conclude that

$$\chi_{ij} - \left(x_i \frac{\partial}{\partial x_j} + x_j \frac{\partial}{\partial x_i} \right) \chi - \delta_{ij} \chi = \frac{\partial^2 \Phi}{\partial x_i \partial x_j}, \tag{28}$$

where Φ is some scalar function. We can therefore write

$$3\mathfrak{B}_{ijkl} = \frac{\partial^4 \Phi}{\partial x_i \partial x_j \partial x_k \partial x_l} - (\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{jk}\delta_{li}) \mathfrak{B} + \mathfrak{B}_{ij}\delta_{kl} + \mathfrak{B}_{jk}\delta_{li} + \mathfrak{B}_{kl}\delta_{ij} + \mathfrak{B}_{li}\delta_{jk} + \mathfrak{B}_{ik}\delta_{jl} + \mathfrak{B}_{jl}\delta_{ik} . \tag{29}$$

It remains to find an equation which will determine Φ. The required equation can be obtained by contracting equation (29) with respect to k and l (say). We obtain

$$3\mathfrak{B}_{ij} = \frac{\partial^2}{\partial x_i \partial x} \nabla^2 \Phi - 5\delta_{ij}\mathfrak{B} + \delta_{ij}\mathfrak{B} + 3\mathfrak{B}_{ij} + 4\mathfrak{B}_{ij}, \tag{30}$$

or

$$0 = \frac{\partial^2}{\partial x_i \partial x_j} \nabla^2 \Phi + 4(\mathfrak{B}_{ij} - \delta_{ij}\mathfrak{B}) = \frac{\partial^2}{\partial x_i \partial x_j} (\nabla^2 \Phi + 4\chi) . \tag{31}$$

From equation (31) we may conclude, without loss of generality, that

$$\nabla^2 \Phi = -4\chi . \tag{32}$$

Thus Φ may be considered as the superpotential for χ.

With χ_{ij} given by equation (28), equation (23) for $\mathfrak{D}_{ij;k}$ becomes

$$\mathfrak{D}_{ij;k} = \frac{\partial^3 \Phi}{\partial x_i \partial x_j \partial x_k} + \delta_{ij} \frac{\partial \chi}{\partial x_k} + \delta_{jk} \frac{\partial \chi}{\partial x_i} + \delta_{ik} \frac{\partial \chi}{\partial x_j} + x_k \mathfrak{B}_{ij}, \tag{33}$$

or, expressing $\mathfrak{B}_{ij}$ in terms of $\mathfrak{B}$ and χ, we have

$$\mathfrak{D}_{ij;k} = \frac{\partial^2}{\partial x_i \partial x_j} \left(\frac{\partial \Phi}{\partial x_k} + x_k \chi \right) + \delta_{ij} \left(\frac{\partial \chi}{\partial x_k} + x_k \mathfrak{B} \right). \tag{34}$$

An equivalent expression for $\mathfrak{D}_{ij;k}$ is (cf. eq. [7])

$$\mathfrak{D}_{ij;k} = \delta_{ij}\mathfrak{D}_k + \frac{\partial^2}{\partial x_i \partial x_j} \left(\frac{\partial \Phi}{\partial x_k} + x_k \chi \right). \tag{35}$$

A comparison with equation (16) now shows that

$$\chi_k = \frac{\partial \Phi}{\partial x_k} + x_k \chi . \tag{36}$$

From equations (4), (17), and (37) we find

$$\frac{\partial \Phi}{\partial x_k} = G \int_V \rho(\boldsymbol{x}') (x_k - x_k') |\boldsymbol{x} - \boldsymbol{x}'| d\boldsymbol{x}', \tag{37}$$

or

$$\Phi = \tfrac{1}{3} G \int_V \rho(\boldsymbol{x}') |\boldsymbol{x} - \boldsymbol{x}'|^3 d'\boldsymbol{x} . \tag{38}$$

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IV. CONCLUDING REMARKS

It is evident that the tensors described in this and in the earlier paper (Chandrasekhar and Lebovitz 1962*a*) are only the first few in an entire hierarchy of tensors that one may define:

$\mathfrak{D}_{ijk\cdots}$ = the completely symmetric tensor which is the Newtonian potential induced by the "density" $\rho x_i x_j x_k \ldots$,

$\mathfrak{D}_{ij;kl\cdots}$ = the tensor potential "$\mathfrak{B}_{ij}$" induced by $\rho x_k x_l \ldots$,

$\mathfrak{D}_{ijkl;mn\cdots}$ = the tensor potential "$\mathfrak{B}_{ijkl}$" induced by $\rho x_m x_n \ldots$,

$\mathfrak{B}_{ijklmn\cdots}$ = the completely symmetric tensor of *even* order, which, on contraction with respect to any two of its indices, yields the corresponding tensor of two ranks lower; and the Newtonian potential is the completely contracted scalar of this tensor.

And associated with these tensors, we shall have a sequence of superpotentials of *odd* orders:

$$\chi^{(2n-1)} = \frac{(-1)^n G}{2n-1} \int_V \rho(x') \, | x - x' |^{2n-1} dx' \qquad (n = 1, 2, \ldots) . \quad (39)$$

It is clear, however, that these tensors of higher rank are not likely to be of much practical use as their "bookkeeping" will become excessively complicated. Indeed, even the tensors described in this paper (in contrast to those described in the earlier paper) are useful in only very special problems (cf. Chandrasekhar 1962).

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