HYDROMAGNETIC OSCILLATIONS OF A FLUID SPHERE WITH INTERNAL MOTIONS*

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Received June 19, 1956

ABSTRACT

A stationary solution of the equations of hydromagnetics for an incompressible inviscid fluid of infinite electrical conductivity is given by $\boldsymbol{v} = \boldsymbol{H}/\sqrt{(4\pi\rho)}$, where $\boldsymbol{v}$ denotes the velocity, $\boldsymbol{H}$ the magnetic field, and ρ the density. The case is considered in which $\boldsymbol{v}$ and $\boldsymbol{H}$ have symmetry about an axis, and convenient forms for the equations governing axisymmetric oscillations about the stationary solution are obtained. These latter equations take their simplest forms when the initial magnetic field and fluid motions have no poloidal components. This case is considered in some detail.

I. INTRODUCTION

Since Babcock's discovery of the magnetic variables, the oscillations of a fluid sphere with a prevalent magnetic field have been the subject of investigations by several authors (Schwarzschild 1949; Gjellestad 1950, 1952; Cowling 1952, 1955; Ferraro and Memory 1952; Chandrasekhar and Limber 1954; Ferraro and Plumpton 1955). These investigations have been inconclusive, in part because of a lack of definiteness in characterizing the initial equilibrium configuration. The indefiniteness consists in the following: If one supposes, as has hitherto been customary in the subject, that the initial configuration is one of hydrostatic equilibrium, then there are immediate restrictions on the possible magnetic fields which can prevail. Thus, in media of uniform density, the assumption of hydrostatic equilibrium requires that the magnetic field, $\boldsymbol{H}$, which is present, be such that curl $\boldsymbol{H} \times \boldsymbol{H}$ is the gradient of a scalar function. In the investigations on hydromagnetic oscillations to which reference has been made, the restrictions on $\boldsymbol{H}$ arising from such considerations are not included; in particular, the magnetic fields which have been assumed to be present are generally incompatible with a spherical shape, which is also assumed (cf. Chandrasekhar and Fermi 1953, and several later investigations along the same lines). However, Chandrasekhar and Prendergast (1956) have recently characterized in an explicit way magnetic fields which have symmetry about an axis and which satisfy the requirements of hydrostatic equilibrium in media of uniform density; and Prendergast (1956*a*) has constructed a model which satisfies these requirements and is a true equilibrium configuration. Prendergast (1957) has also examined the periods of oscillation of his model by a variational method and related them to the condition for dynamical stability arising from the virial theorem (Chandrasekhar and Fermi 1953; Chandrasekhar and Limber 1954). Prendergast's investigations represent the first really self-consistent treatment of the hydromagnetic oscillations of a fluid sphere. Nevertheless, these latter investigations emphasize that if magnetic variability in nature is to be interpreted as variability about an initial state of static equilibrium, then the magnetic fields which prevail must be of very special kinds—a conclusion which is perhaps not altogether satisfying (cf. Chandrasekhar 1956*e*). One need not suffer these limitations if one does not restrict one's self (arbitrarily?) to static configurations with no internal motions. Thus, by assuming internal motions and sup-

* The research reported in this paper has been supported in part by the Geophysics Research Directorate of the Air Force Cambridge Research Center, Air Research and Development Command, under Contract AF 19(604)-299 with the University of Chicago.

posing that the velocity ($\boldsymbol{v}$) at each point is related to the magnetic field at that point by (cf. Chandrasekhar 1956*b*)

$$\boldsymbol{v} = \frac{\boldsymbol{H}}{\sqrt{(4\pi\rho)}}, \tag{1}$$

we can satisfy the equations of motion governing $\boldsymbol{v}$ and $\boldsymbol{H}$ identically; and we need not restrict $\boldsymbol{v}$ and $\boldsymbol{H}$ in any way except that they both be solenoidal. By thus relinquishing the condition of static equilibrium, we gain an enormous freedom in the choice of the possible magnetic fields. Indeed, this circumstance makes one wonder whether magnetic fields, when they occur on the cosmic scale, are not always accompanied by correlated motions. Regardless of one's views on this general matter, it is of interest that, in the framework of solution (1), we can construct a large class of non-static, but stationary, equilibrium configurations which have simple properties. It is the object of this paper to construct one such model and study its oscillations.

II. THE EQUATIONS OF THE PROBLEM

We shall restrict ourselves to magnetic fields and fluid motions which have symmetry about an axis. Convenient forms for the corresponding equations for $\boldsymbol{H}$ and $\boldsymbol{v}$ have recently been written and discussed (Chandrasekhar 1956*c;* this paper will be referred to hereafter as "Paper I"). We shall begin by quoting the relevant formulae.

Letting

$$\boldsymbol{h} = \frac{\boldsymbol{H}}{\sqrt{(4\pi\rho)}}, \tag{2}$$

we express $\boldsymbol{h}$ and $\boldsymbol{v}$ (following Lüst and Schlüter 1954) as superpositions of toroidal and poloidal fields in terms of four scalars P, T, U, and V in the following manner (Paper I, eqs. [15] and [16]):

$$\boldsymbol{h} = \boldsymbol{1}_z \times \boldsymbol{r}T + \text{curl}\,(\boldsymbol{1}_z \times \boldsymbol{r}P) \tag{3}$$

and

$$\boldsymbol{v} = \boldsymbol{1}_z \times \boldsymbol{r}V + \text{curl}\,(\boldsymbol{1}_z \times \boldsymbol{r}U), \tag{4}$$

where $\boldsymbol{1}_z$ is a unit vector in the z-direction (which is assumed to be the axis of symmetry). The equations governing P, T, U, and V, then, are (Paper I, eqs. [35]–[38])

$$\varpi^3 \frac{\partial P}{\partial t} = -[\varpi^2 P, \varpi^2 U], \tag{5}$$

$$\varpi \frac{\partial T}{\partial t} = -[T, \varpi^2 U] + [V, \varpi^2 P], \tag{6}$$

$$\varpi^3 \frac{\partial V}{\partial t} = -[\varpi^2 V, \varpi^2 U] + [\varpi^2 T, \varpi^2 P], \tag{7}$$

and

$$\varpi \Delta_5 \frac{\partial U}{\partial t} = [\Delta_5 P, \varpi^2 P] - [\Delta_5 U, \varpi^2 U] + \varpi \frac{\partial}{\partial z}(T^2 - V^2), \tag{8}$$

where, for the sake of brevity, we have used the notation

$$[f, g] = \frac{\partial(f, g)}{\partial(z, \varpi)} = \frac{\partial f}{\partial z}\frac{\partial g}{\partial \varpi} - \frac{\partial f}{\partial \varpi}\frac{\partial g}{\partial z}, \tag{9}$$

and Δ_5 denotes, as usual, the Laplacian operator for axisymmetric functions in five-dimensional Euclidean space.

a) The Stationary Solution

Equations (5)–(8) clearly admit the solutions

$$P = U = X(z, \varpi) \quad \text{(say)}$$

and

$$T = V = Y(z, \varpi) \quad \text{(say)} . \tag{10}$$

This represents a stationary solution of the equations of motion and corresponds under conditions of axisymmetry to solution (1), which the equations of hydromagnetics generally allow.

b) The Equations Governing the Oscillations about the Stationary Solution

It has been shown quite generally (Chandrasekhar 1956*b*) that solution (1) represents a stable solution of the basic equations. For a discussion of axisymmetric oscillations about the stationary solution (10), it will be convenient to have the general perturbation equations for the toroidal and the poloidal scalars.

Writing

$$P = X + P_1, \quad U = X + U_1, \quad T = Y + T_1, \quad \text{and} \quad V = Y + V_1 , \tag{11}$$

we readily find from equations (5)–(8) that the corresponding perturbation equations are

$$\varpi^3 \frac{\partial P_1}{\partial t} = - [\varpi^2 \phi, \varpi^2 X] , \tag{12}$$

$$\varpi \frac{\partial T_1}{\partial t} = - [\psi, \varpi^2 X] + [Y, \varpi^2 \phi] , \tag{13}$$

$$\varpi^3 \frac{\partial V_1}{\partial t} = + [\varpi^2 \psi, \varpi^2 X] + [\varpi^2 Y, \varpi^2 \phi] , \tag{14}$$

and

$$\varpi \Delta_5 \frac{\partial U_1}{\partial t} = [\Delta_5 X, \varpi^2 \phi] + [\Delta_5 \phi, \varpi^2 X] + 2\varpi \frac{\partial}{\partial z} Y\psi , \tag{15}$$

where we have used the abbreviations

$$\phi = P_1 - U_1 \quad \text{and} \quad \psi = T_1 - V_1 . \tag{16}$$

Rewriting equation (14) in the form

$$\varpi \frac{\partial V_1}{\partial t} = [\psi, \varpi^2 X] + [Y, \varpi^2 \phi] - 2\varpi\psi \frac{\partial X}{\partial z} - 2\varpi Y \frac{\partial \phi}{\partial z} \tag{17}$$

and combining it with equation (13), we have

$$\varpi \frac{\partial \psi}{\partial t} = - 2 [\psi, \varpi^2 X] + 2\varpi Y \frac{\partial X}{\partial z} + 2\varpi Y \frac{\partial \phi}{\partial z} . \tag{18}$$

Similarly, by combining equations (12) and (15), we have

$$-\varpi \Delta_5 \frac{\partial \phi}{\partial t} = \varpi \Delta_5 \left\{ \frac{1}{\varpi^3} [\varpi^2 \phi, \varpi^2 X] \right\} + [\Delta_5 X, \varpi^2 \phi] + [\Delta_5 \phi, \varpi^2 X] + 2\varpi \frac{\partial}{\partial z} Y\psi . \tag{19}$$

Equations (18) and (19) represent a pair of simultaneous equations for ϕ and ψ. Once these equations have been solved, equations (12) and (13) will complete the solution by determining P_1 and T_1 in terms of ϕ and ψ.

III. HYDROMAGNETIC OSCILLATIONS OF A FLUID SPHERE WITH TOROIDAL FIELDS AND ROTATIONAL MOTIONS ONLY

In this paper we shall restrict ourselves to a particularly simple case of the general equations of the preceding section. We shall suppose that in the stationary state the motions and the magnetic fields are purely toroidal, i.e.,

$$P = U = X = 0 . \tag{20}$$

Now the boundary conditions on a toroidal magnetic field are that it vanishes on the bounding sphere and is free of singularity at the origin. Thus

$$Y\ (= T = V) \quad \text{must vanish on} \quad r = R\,, \tag{21}$$

where R denotes the radius of the sphere. Accordingly, Y can be expressed as a linear combination of the fundamental toroidal functions (Chandrasekhar 1956*d*, eq. [36]; this paper will be referred to hereafter as "Paper II"):

$$T_{n,\,j} = \frac{J_{n+3/2}\,(\alpha_{j,\,n}\,r)}{r^{3/2}}\,C_n^{3/2}\,(\mu)\,, \tag{22}$$

where r is now measured in units of R; $\alpha_{j,\,n}$ is the jth zero of the Bessel function $J_{n+3/2}(x)$; and $C_n^{3/2}\,(\mu)$ denotes the Gegenbauer polynomial.

When $X = 0$, equations (18) and (19) reduce to

$$\frac{\partial \psi}{\partial t} = 2\,Y\,\frac{\partial \phi}{\partial z} \tag{23}$$

and

$$\Delta_5\,\frac{\partial \phi}{\partial t} = -\,2\,\frac{\partial}{\partial z}\,Y\psi\,, \tag{24}$$

while equations (12) and (13) become

$$\frac{\partial P_1}{\partial t} = 0 \tag{25}$$

and

$$\varpi\,\frac{\partial T_1}{\partial t} = [\,Y,\,\varpi^2\phi]\;. \tag{26}$$

From equation (25) it follows that

$$P_1 \equiv 0\;. \tag{27}$$

Thus *during the oscillation no poloidal magnetic field will be generated; however, meridional motions, as well as rotational motions and toroidal magnetic fields, will be generated.*

Eliminating ψ between equations (23) and (24), we have

$$\Delta_5\,\frac{\partial^2 \phi}{\partial t^2} = -\,4\,\frac{\partial}{\partial z}\left(Y^2\,\frac{\partial \phi}{\partial z}\right). \tag{28}$$

Separating the time dependence by writing

$$\phi(z,\,\varpi)e^{i\sigma t} \quad \text{in place of} \quad \phi(z,\,\varpi,\,t)\,, \tag{29}$$

we obtain the equation

$$\sigma^2\Delta_5\phi = 4\,\frac{\partial}{\partial z}\left(Y^2\,\frac{\partial \phi}{\partial z}\right). \tag{30}$$

The boundary conditions which we should impose on ϕ can be inferred from equation (26). In spherical polar co-ordinates this equation is

$$\frac{\partial T_1}{\partial t}=e^{i\sigma t}\left\{\frac{1-\mu^2}{r^2}\frac{\partial}{\partial r}(r^2\phi)\frac{\partial Y}{\partial \mu}-\frac{\partial Y}{\partial r}\frac{\partial}{\partial \mu}[(1-\mu^2)\phi]\right\}. \tag{31}$$

Since T_1, like Y, must vanish on $r=1$, it is clearly necessary that

$$\phi=0 \quad \text{on} \quad r=1\,. \tag{32}$$

Accordingly, we must seek solutions of equation (30) which vanish on $r=1$ and are free of singularities at the origin. This is a characteristic value problem for σ^2.

a) The Variational Method

Let ϕ_j be a solution belonging to a characteristic value σ_j^2. Multiplying the equation governing ϕ_j by $\varpi^3\phi_k$ (belonging to σ_k^2) and integrating over the volume of the sphere, we have

$$\sigma_j^2\int\int\phi_k\Delta_5\phi_j\varpi^3 d\varpi\, d z=4\int\int\phi_k\frac{\partial}{\partial z}\left(Y^2\frac{\partial\phi_j}{\partial z}\right)\varpi^3 d\varpi\, d z\,. \tag{33}$$

This equation can be rewritten in the form

$$\sigma_j^2\int\int\phi_k\left\{\frac{\partial}{\partial\varpi}\left(\varpi^3\frac{\partial\phi_j}{\partial\varpi}\right)+\varpi^3\frac{\partial^2\phi_j}{\partial z^2}\right\}d\varpi\, d z=4\int\int\phi_k\frac{\partial}{\partial z}\left(Y^2\frac{\partial\phi_j}{\partial z}\right)\varpi^3 d\varpi\, d z\,. \tag{34}$$

Integrating by parts the integrals on both sides of this equation and remembering that ϕ vanishes on the boundary, we obtain

$$\sigma_j^2\int\int \operatorname{grad}\phi_j\cdot\operatorname{grad}\phi_k\varpi^3 d\varpi\, d z=4\int\int Y^2\frac{\partial\phi_j}{\partial z}\frac{\partial\phi_k}{\partial z}\varpi^3 d\varpi\, d z\,. \tag{35}$$

From this equation it follows that

$$\int\int\operatorname{grad}\phi_j\cdot\operatorname{grad}\phi_k\varpi^3 d\varpi\, d z=0 \qquad \text{if} \qquad j\neq k\,, \tag{36}$$

and, when $j=k$, we have (on further suppressing the subscripts)

$$\sigma^2=\frac{4\int\int Y^2(\partial\phi/\partial z)^2\varpi^3 d\varpi\, d z}{\int\int|\operatorname{grad}\phi|^2\varpi^3 d\varpi\, d z}\,. \tag{37}$$

From equations (36) and (37) it is evident that the characteristic value problem associated with equations (30) and (32) is a self-adjoint one. Accordingly, equation (37) can be made the basis of a variational method for determining σ^2.

b) Case when Initial Magnetic Field and Fluid Motions Are Both Described by (0, 1) Toroidal Mode

We shall apply the formulae of the preceding section to the case when the magnetic field and the fluid motions in the initial stationary state both belong to the (0, 1) toroidal mode, i.e., when

$$Y=Y_0\frac{J_{3/2}(\alpha_{1,\,0}r)}{r^{3/2}}, \tag{38}$$

where Y_0 is a constant of the dimension sec $^{-1}$. To determine the corresponding lowest modes of oscillation, we expand ϕ in terms of the basic toroidal functions, $T_{n,\,j}$, and determine the coefficients in the expansion by minimizing the expression for σ^2 given by equation (37).

Thus, letting

$$\phi = \sum_{n,\, j} A_{n,\, j} T_{n,\, j} \tag{39}$$

and making use of the properties (Paper II, eqs. [32], [34], and [39])

$$\Delta_5 T_{n,\, j} = -\alpha_{j,\, n}^2 T_{n,\, j} \tag{40}$$

and

$$\int_0^1 \int_{-1}^{+1} T_{n,\, j} T_{m,\, k} r^4 (1-\mu^2)\, d r\, d\mu = \frac{(n+1)(n+2)}{2n+3} [J'_{n+3/2}(\alpha_{j,\, n})]^2 \delta_{n,\, m} \delta_{j,\, k}, \tag{41}$$

we find that the denominator in the expression for σ^2 becomes

$$\begin{aligned} \iint |\operatorname{grad} \phi|^2 \varpi^3 d\varpi\, d z &= -\iint \phi \Delta_5 \phi \varpi^3 d\varpi\, d z \\ &= \int_0^1 \int_{-1}^{+1} \Big(\sum_{n,\, j} A_{n,\, j} T_{n,\, j}\Big)\Big(\sum_{m,\, k} A_{m,\, k} \alpha_{k,\, m}^2 T_{m,\, k}\Big) r^4 (1-\mu^2)\, d r\, d\mu \qquad (42) \\ &= \sum_{n,\, j} \frac{(n+1)(n+2)}{2n+3} [J'_{n+3/2}(\alpha_{j,\, n})]^2 \alpha_{j,\, n}^2 A_{n,\, j}^2 . \end{aligned}$$

Considering, next, the numerator in equation (37), we first observe that the recurrence relations satisfied by the Gegenbauer polynomials (Paper II, eqs. [40] and [41]) and the Bessel functions enable us to write

$$\frac{\partial \phi}{\partial z} = \frac{1}{r^{5/2}} \sum_{n,\, j} A_{n,\, j} \left(F_{n,\, j} C_{n+1}^{3/2} + G_{n,\, j} C_{n-1}^{3/2}\right), \tag{43}$$

where

$$F_{n,\, j} = \frac{n+1}{(2n+3)^2} \{ (\alpha_{j,\, n} r) [n J_{n+1/2}(\alpha_{j,\, n} r) - (n+3) J_{n+5/2}(\alpha_{j,\, n} r)] - n(2n+3) J_{n+3/2}(\alpha_{j,\, n} r) \} \tag{44}$$

and

$$G_{n,\, j} = \frac{n+2}{(2n+3)^2} \{ (\alpha_{j,\, n} r) [n J_{n+1/2}(\alpha_{j,\, n} r) - (n+3) J_{n+5/2}(\alpha_{j,\, n} r)] + (n+3)(2n+3) J_{n+3/2}(\alpha_{j,\, n} r) \} . \tag{45}$$

With $\partial\phi/\partial z$ given by equation (43), we have

$$\begin{aligned} 4\int_0^1 \int_{-1}^{+1} Y^2 \left(\frac{\partial \phi}{\partial z}\right)^2 r^4 (1-\mu^2)\, d r\, d\mu &= 4 Y_0^2 \int_0^1 \frac{d r}{r^4} [J_{3/2}(\alpha_{1,\, 0} r)]^2 \\ &\times \int_{-1}^{+1} d\mu (1-\mu^2) \Big\{ \sum_n C_n^{3/2}(\mu) \Big[\sum_j (A_{n-1,j} F_{n-1,\, j} + A_{n+1,\, j} G_{n+1,\, j}) \Big] \Big\}^2 . \end{aligned} \tag{46}$$

On making use of the orthogonality properties of the Gegenbauer polynomials (Paper II, eq. [34]), we can reduce the foregoing to the form

$$\begin{aligned} 4\int_0^1 \int_{-1}^{+1} Y^2 \left(\frac{\partial \phi}{\partial z}\right)^2 r^4 (1-\mu^2)\, d r\, d\mu &= 4 Y_0^2 \sum_n \frac{2(n+1)(n+2)}{2n+3} \\ &\times \int_0^1 \frac{d r}{r^4} [J_{3/2}(\alpha_{1,\, 0} r)]^2 \Big\{ \sum_j (A_{n-1,\, j} F_{n-1,\, j} + A_{n+1,\, j} G_{n+1,\, j}) \Big\}^2 . \end{aligned} \tag{47}$$

From equation (47) it is clear that the solutions for ϕ which include in their expansions only the even, or only the odd, orders of the Gegenbauer polynomials fall into noncombining groups. The various modes of oscillation can therefore be classified as even or odd—a fact which could also have been inferred, directly, from equation (30).

For the form of ϕ assumed, the variational expression for σ^2 is given by the ratio of the quantities on the right-hand sides of equations (42) and (47). In practice one includes only a few terms in expansion (39) and minimizes the expression for σ^2 with respect to the $A_{n,j}$'s. The results, summarized in Tables 1 and 2, were derived in this manner.

TABLE 1

$\sigma^2/Y_0^2 = S_0^2$ FOR THE LOWEST EVEN MODE

No	Modes Included in the Variational Calculation	S_0^2	No	Modes Included in the Variational Calculation	S_0^2
1	(0, 1) and (0, 2)	0 1461	5	(0, 1), (0, 2), (0, 3), and (0, 4)	0 0408
2	(0, 1), (0, 2), and (0, 3)	07070	6	(0, 1), (0, 2), (0, 3), (0, 4), (2, 1), and (2, 2)	0 03842
3	(0, 1), (0, 2), (2, 1), and (2, 2)	06980			
4	(0, 1), (0, 2), (0, 3), (2, 1), and (2, 2)	0 06223			

TABLE 2

$\sigma^2/Y_0^2 = S_1^2$ FOR THE LOWEST ODD MODE

No	Modes Included in the Variational Calculation	S_1^2
1	(1, 1) and (1, 2)	0 9697
2 .	(1, 1), (1, 2), and (1, 3)	0 5383

An inspection of Tables 1 and 2 shows that the sequence of trial functions used has not yet produced a satisfactory convergence toward the true characteristic values. This is also evident from the following list of the coefficients in the trial function (No. 6 in Table 1), including the modes (0, 1), (0, 2), (0, 3), (0, 4), (2, 1), and (2, 2), which minimizes the expression for σ^2:

$$A_{0,1} = 1;\ A_{0,2} = -0.93723;\ A_{0,3} = +0.52133;\ A_{0,4} = -0.16927;\quad A_{2,1} = -0.18525;\ \text{and}\ A_{2,2} = +0.0635\,. \tag{48}$$

This lack of convergence is somewhat surprising. It must originate in the basic equation's not being separable into the variables r and μ and in the problem's being so inextricably two-dimensional that it is impossible to approximate ϕ by a linear combination of the first few toroidal functions. A further fact which may be relevant in this connection is that equation (30) must change its character—hyperbolic or elliptic—within the unit sphere. This can be seen as follows:

Rewriting equation (30) in the form

$$\frac{\partial^2\phi}{\partial\varpi^2}+\left(1-4\,\frac{Y^2}{\sigma^2}\right)\frac{\partial^2\phi}{\partial z^2}+\frac{3}{\varpi}\frac{\partial\phi}{\partial\varpi}-\frac{4}{\sigma^2}\frac{\partial\phi}{\partial z}\frac{\partial Y^2}{\partial z}=0\,, \tag{49}$$

we observe that, since $Y = 0$ on $r = 1$, the equation is certainly elliptic on and near the boundary. If it were elliptic throughout, then we could infer from the fundamental theorem on elliptic equations (cf. Courant and Hilbert 1937) that there exists no non-zero solution of equation (30) which vanishes on the boundary of the sphere. Hence a necessary condition for the existence of a solution of equation (30) which satisfies the boundary condition (32) is that σ^2 be less than Y^2 in at least some parts of the unit sphere; in these parts the equation will be hyperbolic. Thus the equation must have this mixed character of being hyperbolic in some parts and elliptic in others.

Returning to the formula (cf. Tables 1 and 2)

$$\sigma^2 = Y_0^2 S_1^2 , \tag{50}$$

we shall express Y_0^2 in terms of more easily interpretable quantities: The magnetic energy, $\mathfrak{M}$, in the undisturbed sphere is given quite generally by

$$\mathfrak{M} = \tfrac{1}{2}\rho R^5 \int_0^1 \int_{-1}^{+1} \int_0^{2\pi} Y^2 r^4 (1-\mu^2)\, dr\, d\mu\, d\phi ; \tag{51}$$

and, for Y of the chosen form (eq. [38]), we readily find that

$$\mathfrak{M} = \tfrac{2}{3}\pi\rho R^5 \,[J'_{3/2}(\alpha_{1,0})]^2 Y_0^2 . \tag{52}$$

Defining $\langle H^2 \rangle_{\text{av}}$ by

$$\mathfrak{M} = \frac{\langle H^2 \rangle_{\text{av}}}{8\pi} \left(\tfrac{4}{3}\pi R^3\right) , \tag{53}$$

we can write

$$Y_0^2 = \frac{V_A^2}{R^2} [J'_{3/2}(\alpha_{1,0})]^{-2} , \tag{54}$$

where V_A denotes the Alfvén velocity for the root-mean-square field. Inserting this expression for Y_0^2 in equation (50), we obtain

$$\sigma^2 = S^2 [J'_{3/2}(\alpha_{1,0})]^{-2} \frac{V_A^2}{R^2} = 7.408 S^2 \frac{V_A^2}{R^2} . \tag{55}$$

Thus the period of oscillation, $2\pi/\sigma$, apart from a numerical factor, is given by the time taken for the Alfvén wave to travel once around the circumference of the sphere. For the lowest modes considered,

$$\sigma_0^2 < 0.28 \frac{V_A^2}{R^2} \quad \text{and} \quad \sigma_1^2 < 0.50 \frac{V_A^2}{R^2} . \tag{56}$$

IV. CONCLUDING REMARKS

This paper makes hardly more than a beginning in the study of hydromagnetic oscillations of models with internal motions. There are a great many problems in this general field which can be considered by the methods of this paper and whose solutions might be expected to contribute toward our understanding of hydromagnetic phenomena. While it is clearly premature to relate the results of this paper to particular aspects of known phenomena, it is perhaps worth pointing out that, by considering the hydromagnetic oscillations of a fluid sphere with rotational motions, we have removed a sharp distinction between the two models which have been proposed to interpret the magnetic variables—the model based on the oscillations of an initial static equilibrium configuration and the model based on a rotating star (cf. Deutsch 1956).

In conclusion I wish to express my thanks to Miss Donna Elbert for valuable assistance with the numerical calculations.

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