Kolmogorov-Arnold-Moser Theorem

Can Planetary Motion be Stable?

Govindan Rangarajan

The contribution of Kolmogorov to classical mechanics is illustrated through the famous Kolmogorov-Arnold-Moser (KAM) theorem. This theorem solves a long-standing problem regarding stability in non-linear Hamiltonian dynamics. Various concepts required to understand the KAM theorem are also developed.

Govindan Rangarajan obtained his Ph D from University of Maryland, College Park and has been with the Department of Mathematics and Centre for Theoretical Studies, Indian Institute of Science since 1992. His areas of interest include non-linear dynamics, fractals and lie algebraic perturbation theory.

Introduction

Kolmogorov was a versatile mathematical genius who made important contributions to several areas of mathematics. One such contribution was the solution to a long-standing problem in classical mechanics. The problem concerns long-term stability in non-linear Hamiltonian systems (more on Hamiltonian systems later). Its solution is relevant to such important issues as the stability of our solar system etc.

The genesis of the problem can be traced back to Newton. Newton was able to solve the equations that determine the motion of two bodies (say, the sun and the earth) interacting with each other through the gravitational force. However, when he added a third body (say, the moon), he was unable to solve the corresponding equations determining the simultaneous motion of all three bodies. This is the (in)famous 3-body problem.

We now rephrase the 3-body problem in a manner more suitable for our purposes. As noted above, the equations of motion for the 2-body system can be solved analytically (the system is said to be 'integrable'). When a third body is added, this can be considered as a 'perturbation' to the original, integrable 2-body system. We are interested in determining whether solutions to this 3-body system exist and whether they are close to the 2-body solutions if the perturbation is small enough. This problem can be generalised to consider small perturbations to any 'integrable' system (not necessarily the 2-body system).

Kolmogorov was the first to provide a solution to the above general problem in a theorem formulated in 1954 (see Suggested Reading). However, he provided only an outline of the proof. The actual proof (with all the details) turned to be quite difficult and was provided by Arnold and Moser (see Suggested Reading). The result they obtained is now popularly known as the Kolmogorov-Arnold-Moser (KAM) theorem. The significance of Kolmogorov's contribution is best summarised by the following comment by Arnold (see Suggested Reading): 'One of the most remarkable of A N Kolmogorov's mathematical achievements is his work on classical mechanics of 1954. A simple and novel idea, the combination of very classical and essentially modern methods, the solution of a 200-year old problem, a clear geometrical picture and great breadth of outlook – these are the merits of the work'.

Needless to say, we will not even attempt to give the proof of the KAM theorem in this article. We will be content with merely stating the theorem and motivating the various conditions that appear in it. To do even this, we first need to understand various concepts like Hamiltonian systems, canonical transformations, integrable systems, action-angle variables etc. We will briefly delve into each of these topics in the run up to the KAM theorem.

Hamiltonian Systems

Hamiltonian systems form an important class of systems in classical mechanics. Our solar system is a prime example of a Hamiltonian system. In fact, any mechanical system without friction can be described as a Hamiltonian system.

A Hamiltonian system with N degrees of freedom is characterised by a single function $H(q, p)$, called the Hamiltonian, which is a function of the (generalised) coordinates $q = (q_1, q_2, \ldots, q_N)$ and (generalised) momenta $p = (p_1, p_2, \ldots, p_N)$. We have restricted ourselves to Hamiltonians which do not depend explicitly on time for the sake of simplicity. The coordinates and momenta together constitute the $2N$-dimensional 'phase space' of the system. The evolution of these coordinates and momenta with time is given by the following Hamilton's equations of motion:

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i},\ i=1, 2, \ldots, N, \tag{1}$$

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i},\ i=1, 2, \ldots, N. \tag{2}$$

Thus we have a set of $2N$ first order ordinary differential equations that have to be solved to obtain the desired time evolution of q and p. This can be done in principle once the so-called initial conditions (the values of q and p at time $t=0$) are specified.

Using the equations of motion, it can be easily seen that H is constant in time for our time-independent Hamiltonian. One says that H is a conserved quantity. Quite often the value of the Hamiltonian corresponds to the total energy of the system[1]. In such cases, the conservation of H is equivalent to conservation of energy.

[1] This is true if the equations defining the (generalised) coordinates do not depend on time explicitly and if the forces in the system are derivable from a conservative potential.

As a simple example, let us consider the following one degree of freedom Hamiltonian describing a particle of mass m moving in a potential V:

$$H(q_1, p_1) = \frac{p_1^2}{2m} + V(q_1).$$

Applying the Hamilton's equations of motion we get:

$$\frac{dq_1}{dt} = \frac{\partial H}{\partial p_1} = \frac{p_1}{m} \tag{3}$$

$$\frac{dp_1}{dt} = -\frac{\partial H}{\partial q_1} = -\frac{\partial V}{\partial q_1}. \tag{4}$$

The reader may recall from high school physics that $-(\partial V/\partial q_1)$ corresponds to the force F acting on the particle. Combining the above two equations we get

$$\frac{dp_1}{dt} = m\frac{d^2 q_1}{dt^2} = ma = F,$$

where a denotes the acceleration. Thus, we have recovered Newton's second law of motion.

We see that the Hamiltonian formalism does give expected results in cases such as the one described above. However, in more general settings, the formalism is a more powerful and elegant tool. In particular, it is very useful in developing perturbation theories (which we will come to soon). Furthermore, it provides some of the basic language used in constructing quantum mechanics and statistical mechanics.

We now dig deeper into Hamiltonian mechanics. From the Hamilton's equations of motion it is easy to see that if H is a complicated function of q and p, then the differential equations also become quite complicated and difficult to solve. Therefore, it may be worthwhile to transform to a new set of coordinates and momenta (denoted by Q and P) such that the corresponding Hamiltonian is a simple function of this new set. However, one can not make any arbitrary transformation to new variables. The transformation that we make should respect the Hamiltonian structure i.e. equations of motion in the new variables should have the same functional form as before:

$$\frac{dQ_i}{dt} = \frac{\partial H'}{\partial P_i}, \; i = 1, 2, \ldots, N, \tag{5}$$

$$\frac{dP_i}{dt} = -\frac{\partial H'}{\partial Q_i}, \; i = 1, 2, \ldots, N. \tag{6}$$

Here H' denotes the transformed Hamiltonian in the new variables. Transformations which respect the Hamiltonian structure are called 'canonical transformations'. Henceforth, all variable transformations that we make would be restricted to such canonical transformations.

Coming back to our problem, we would like to make a canonical transformation from (q, p) to (Q, P) such that H' is a simple function of Q and P. It should be so simple that one can solve the resultant equations of motion trivially. For some special Hamiltonian systems, the above goal can be achieved. One can canonically transform to a special set of variables called the 'action-angle' variables (I,θ) such that the transformed Hamiltonian is a function only of the N action variables I_i. In this case, the equations of motion are easily solved:

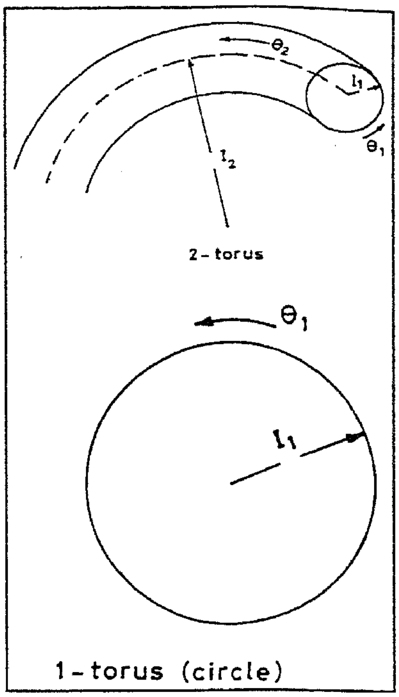

Figure 1.

$$\frac{d\theta_i}{dt} = \frac{\partial H'}{\partial I_i} \equiv \omega_i(I)\ i=1,2,\ldots,N. \quad (7)$$

$$\frac{dI_i}{dt} = -\frac{\partial H'}{\partial \theta_i} = 0,\ i = 1, 2, \ldots, N. \quad (8)$$

Here ω_i s are called the characteristic frequencies of the system. Solving the above equations we get

$$\theta_i(t) = \theta_i(0) + \omega_i(I)\,t,\ i=1, 2, \ldots, N, \quad (9)$$
$$I_i(t) = I_i(0),\ i=1,2, \ldots, N. \quad (10)$$

Thus the actions are constant in time and are said to be invariant. Furthermore, the motion does not occupy the full $2N$-dimensional phase space but is restricted to the N-dimensional surface of an N-torus and θ_i s are nothing but angles along the N independent loops[2] on this torus (see *Figure 1*). The ω_i s are the frequencies of rotation around these N loops and I_i s are related to the radii of these loops. Note that once the initial conditions $q_i(0)$ and $p_i(0)$ are specified in the original variables, the canonical transformation completely fixes $I_i(0)$ and $\theta_i(0)$. Once $I_i(0)=I_i(t)$s are fixed, the torus on which the motion is

[2] The N independent loops correspond to N closed paths that cannot be deformed into one another or to a point.

[3] For a mathematically more rigorous definition of integrable Hamiltonian systems, see V. I. Arnold, *Mathematical Methods of Classical Mechanics* (Springer, 1978).

restricted to, is fixed. The torus is called an invariant torus. The frequencies of motion on this invariant torus are also fixed since these frequencies depend only on the invariant actions. The actual trajectory followed on this torus is specified by the values of $\theta_i(0)$. The special Hamiltonian systems for which all of the above is realised are called 'integrable Hamiltonian systems' [3]. One can show that all one degree of freedom Hamiltonian systems are integrable.

Kolmogorov-Arnold-Moser (KAM) Theorem

A natural question that arises at this point is whether by making an appropriate canonical transformation (provided one is clever enough!) one can make all non-linear Hamiltonian systems with more than one degree of freedom integrable. By a non-linear Hamiltonian, we refer to a Hamiltonian which gives rise to non-linear equations of motion.

We start by restricting ourselves to the simplest case of a Hamiltonian system 'close' to an integrable Hamiltonian system in the following sense: The Hamiltonian H can be written as a sum of a Hamiltonian H_0 known to be integrable and a small additional piece $\varepsilon\, H_1$ (where ε is a dimensionless number assumed to be small)

$$H(q, p) = H_0(q, p) + \varepsilon\, H_1(q, p).$$

Such Hamiltonians are called 'near-integrable' Hamiltonians. Since H_0 is integrable, one can make a canonical transformation to the action-angle variables (I, θ) such that H_0' is a function only of actions. However, H_1' would still depend on both I and θ. In these variables, the transformed Hamiltonian H' is given by

$$H'(I, \theta) = H'_0(I) + \varepsilon\, H'_1(I, \theta).$$

If the perturbation is zero (i.e. $\varepsilon=0$), we are left with only H_0'. Since this is integrable, the motion is restricted to the surface of an invariant torus for given initial conditions. If we now 'turn on' the perturbation ($\varepsilon \neq 0$), we want to investigate how the motion changes. Provided ε is small enough, we would expect the motion to lie on a slightly distorted version of the original invariant torus. To show this, we have to make a canonical transformation to (I', θ') such that the Hamiltonian H'' in these new variables is a function only of I'. Using the perturbation theory approach, we will attempt to do this only to first order in ε i.e. we will neglect all terms that contain higher powers of ε. As is shown in the Appendix, even this is not possible. We encounter small denominators and the desired canonical transformation diverges and becomes ill-defined. Consequently, we can not make the transformed Hamiltonian a function only of the actions.

In light of the above, the situation appears hopeless as far as saying anything about the behaviour of even near-integrable Hamiltonian systems is concerned. It took Kolmogorov's genius to snatch a partial victory from the jaws of apparent defeat. Kolmogorov realized that the above roadblock was a result of trying to solve the problem for all possible initial conditions. If one excludes some problematic initial conditions, one can obtain precise albeit qualitative conclusions about the behaviour of the majority of trajectories. In particular, the problem occurs (refer to the Appendix) under the following conditions: Consider a torus (invariant under the integrable H_0) whose characteristic frequencies ω_i satisfy the so-called 'resonant condition' $\mathbf{m} \cdot \omega = 0$ (for some integer vector $\mathbf{m}$) [4]. If one tries to investigate how this torus deforms when one turns on the perturbation, the canonical transformation diverges (since $\mathbf{m} \cdot \omega$ occurs in the denominator of the relevant expression cf. sidebar). So we exclude all such torii (or equivalently, the corresponding initial conditions). Divergence problems can occur even if $\mathbf{m} \cdot \omega$ is not exactly zero, but close to it (the precise condition will be given in the statement of the KAM theorem below). So, such torii also have to be excluded. Fortunately, the size of the set of all such excluded torii is very small. Kolmogorov was able to show that the remaining 'non-resonant' torii only get slightly deformed when the perturbation is turned on provided the perturbation is

[4] The resonant condition is easier to understand for 2 degrees of freedom. In this case, we have $m_1\omega_1 + m_2\omega_2 = 0$ for some integers m_1 and m_2. This is equivalent to the condition that the ratio of the characteristic frequencies ω_1 and ω_2 is a rational number. Therefore, invariant torii for which this condition is satisfied are sometimes called 'rational torii'.

small. There is still one unsolved problem. Till now, we have kept only terms that are first order in ε (terms proportional to ε^2 etc. were discarded). What happens when we put back the higher order terms? By formulating a new type of perturbation theory called the 'super-convergent' perturbation theory, Kolmogorov was able to show that no further divergence problems occur even in this case.

We are now in a position to state the Kolmogorov-Arnold-Moser (KAM) theorem.

Kolmogorov-Arnold-Moser (KAM) Theorem: Consider an analytic N degrees of freedom Hamiltonian $H(I,\theta)$ and let $H=H_0(I)+\varepsilon H_1(I,\theta)$ with

$$\det\left|\frac{\partial^2 H_0}{\partial I_i \partial I_j}\right| \neq 0.$$

Then the torii of the unperturbed ($\varepsilon=0$) integrable Hamiltonian H_0 which satisfy the following inequality

$$|\mathbf{m}\cdot\omega| > \frac{K(\varepsilon)}{|\mathbf{m}|^{(N+1)}},\ |\mathbf{m}| = |m_1|+|m_2| + \ldots + |m_N|,$$

survive the perturbation (they merely get deformed)[5]. The set of torii not satisfying the above inequality is small and its size tends to zero as $\varepsilon \to 0$.

We will now motivate the various conditions that appear in the KAM theorem. We will restrict ourselves to 2 degrees of freedom for the sake of simplicity. We require H to be analytic since this is a key factor in ensuring the convergence of the series defining the canonical transformation [6]. The condition on the determinant ensures that the characteristic frequencies ω_1 and ω_2 are not independent of both the actions I_1 and I_2. This has the following consequence: At resonance ($m_1\omega_1 + m_2\omega_2 = 0$) we have already seen that we get into problems. In such a case, the

[5] For a given value of the perturbation strength ε, the coefficient K is fixed. As we increase N we are testing for resonances of higher and higher orders. The inequality allows us to approach resonance as N increases, but not too closely. K tends to zero as $\varepsilon \to 0$.

[6] Moser was able to relax this condition. His proof requires the existence of only a finite number of derivatives of H.

amplitude of motion starts increasing (i.e. I starts changing). If the ω_is are independent of I_is, the system continues to remain at resonance and the amplitude increases without limit. However, if the frequencies depend on the actions (as in our case), the frequencies start changing as I changes and the resonance condition $m_1\omega_1 + m_2\omega_2 = 0$ is no longer satisfied. The system 'drops off' the resonance. Hence the excursion in I is limited. Therefore the bad effects of the resonant torii do not extend to the whole space.

Next, we consider the condition $|m_1\omega_1+m_2\omega_2| > K(\varepsilon)/(|m_1|+|m_2|)^3$. Consider the torii whose characteristic frequencies do not satisfy this condition. In such cases, it can be shown using number theory that the ratio ω_1/ω_2 is well approximated by rational numbers [7]. Moreover, the number of torii which do not satisfy this condition (and hence are excluded by the KAM theorem) is proportional to K. Further, K goes to zero as $\varepsilon \rightarrow 0$. Thus, for a small ε (i.e. if the perturbation strength is small), the majority of the torii are merely deformed. What happens to the torii excluded by the KAM theorem? Using another theorem (the Poincaré-Birkhoff theorem), it can be shown that they get destroyed by the perturbation and 'chaotic motion' develops in their vicinity. In chaotic motion, two trajectories that start close to one another diverge exponentially and the motion is not integrable. As the perturbation increases in magnitude, more and more torii get destroyed. Often, the torus whose ratio of characteristic frequencies is most badly approximated by rational numbers is destroyed the last! This ratio corresponds to the golden mean $(5^{1/2} - 1)/2$ which is the most irrational number in the sense that its continued fraction expansion converges most slowly. In fact, the golden mean is also used by painters to determine the most aesthetically pleasing placement of the horizon in their paintings. Thus the most irrational number according to number theory is the most aesthetically pleasing ratio according to painters and also corresponds to the most stable configuration according to non-linear Hamiltonian dynamics!

[7] It is well approximated in the sense that the continued fraction expansion of the ratio either terminates after a finite number of terms or converges quite rapidly.

Finally, we are in a position to answer the question with which we started this Section. From the above discussion it is clear that most near-integrable Hamiltonian systems are not completely integrable since motion is chaotic near resonant torii and therefore not integrable. However, the motion for the majority of initial conditions is integrable as a consequence of KAM theorem. Thus the phase space is a mixture of integrable and non-integrable regions with the former being in the majority. On the other hand, if ε is large, KAM theorem no longer applies and most of the phase space could be non-integrable. In extreme cases, the whole of phase space is dominated by chaotic motion.

To summarise, we hope that we have provided an example of Kolmogorov's enormous contributions to mathematics by studying the KAM theorem. Through this theorem, we have also attempted to give a glimpse of the field of non-linear Hamiltonian dynamics.

Appendix: Canonical Perturbation Theory

Consider the following Hamiltonian

$$H'(I,\theta) = H_0'(I) + \varepsilon H_1'(I,\theta)$$

where H_0' is the integrable part. We would like to perform a canonical transformation to a new set of action-angle variables such that the transformed Hamiltonian H'' is a function only of the new actions I' to first order in ε. The canonical transformations are usually performed using a 'generating function'. We will use the following generating function:

$$S(I',\theta) = I' \cdot \theta + \varepsilon S_1(I',\theta).$$

Here S_1 is yet to be determined.

The relations between old and new variables are given by

$$I = \frac{\partial S}{\partial \theta}, \ \theta' = \frac{\partial S}{\partial I'}.$$

Thus,

$$I = I' + \varepsilon \frac{\partial S_1}{\partial \theta}, \quad \theta' = \theta + \varepsilon \cdot \frac{\partial S}{\partial I'}.$$

We now substitute for (I, θ) in H' in terms of the new variables (I', θ') using the above relations. Since we want the transformed Hamiltonian to be dependent only on I', we fix S_1 by requiring that all θ' dependent terms in H' sum to zero (to first order in ε). To obtain an explicit expression for S_1, we express H_1' and S_1 as a Fourier series in the angle variables θ_i:

$$H_1' = \sum_{\mathbf{m}} H_{1,\mathbf{m}}(I') \exp(i\mathbf{m} \cdot \theta), \tag{11}$$

$$S_1 = \sum_{\mathbf{m}} S_{1,\mathbf{m}}(I') \exp(i\mathbf{m} \cdot \theta), \tag{12}$$

where $\mathbf{m}$ is an N-component vector of integers. Using these series we obtain

$$S_1 = i \sum_{\mathbf{m}} \frac{H_{1,\mathbf{m}}(I')}{\mathbf{m} \cdot \omega} \exp(i\mathbf{m} \cdot \theta)$$

Here ω is the N-dimensional vector of the characteristic frequencies which characterises an invariant torus of the integrable H_0'. If the infinite sum in the above expression for S_1 converges, then we are done since the desired canonical transformation is given by the generating function $S = S_0 + \varepsilon S_1$. However, the factor $\mathbf{m} \cdot \omega$ in the denominator gives rise to problems. For any ω, we can always find a $\mathbf{m}$ such that $\mathbf{m} \cdot \omega$ is equal to zero or very close to zero. If this happens that particular term 'blows up' and the sum will not converge. Since the sum in the expression for S_1 runs over all values of $\mathbf{m}$, we will always encounter such 'small denominators'. Thus the desired canonical transformation can not be carried out.

Suggested Reading

- A N Kolmogorov. Preservation of conditionally periodic movements with small change in the Hamilton function, Los Alamos National Laboratory translation LA-TR-71-67 by Helen Dahlby of *Akad. Nauk. S.S.S.R., Doklady.* 98. 527,1954.
- The general theory of dynamical systems and classical mechanics, *International Congress of Mathematicians*, Amsterdam. 1954. vol 1, 315.
- V I Arnold. Small denominators and problems of stability of motion in classical and celestial mechanics. *Russ. Math. Surveys.* 18.6. 85, 1963.
- J Moser. *Stable and Random Motions in Dynamical Systems*. Princeton University Press, 1973.

Address for Correspondence
Govindan Rangarajan
Department of Mathematics and Centre for Theoretical Studies
Indian Institute of Science
Bangalore 560 012, India