Snippets of Physics

5. Why are Black Holes Hot?*

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One of the celebrated results in black hole physics is that black holes have a temperature and they emit a thermal spectrum of radiation. Though a rigorous derivation of this result requires quantum field theory, a flavour of the essential ideas can be provided at an elementary level as indicated here.

In classical general relativity, material can fall into a black hole but nothing can come out of it. In the early seventies, Bekenstein argued that this asymmetry can lead to violation of second law of thermodynamics unless we associate an entropy with the black hole which is proportional to its area. Thus black holes were attributed entropy and energy (equal to Mc^2 where M is the mass of the black hole) but it was not clear whether they have a temperature. If a black hole has a non-zero temperature, then it has to radiate a thermal spectrum of particles and this seemed to violate the classical notion that 'nothing can come out of a black hole'.

In the mid-seventies, Hawking discovered that black holes, when viewed in a quantum mechanical perspective, do have a temperature. (For a brief taste of history related to this, see *Box* 1). A black hole which forms due to collapse of matter will emit – at late times – radiation which is characterized by this temperature. The rigorous derivation of this result requires a fair knowledge of quantum field theory but I will present, in this installment, a simplified derivation which captures its essence.

Let us start with a simple problem in special relativity

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Box 1. A Little History

Entropy (which means, 'inherent tendency') is a familiar and important concept in thermodynamics. The three laws of thermodynamics (nicely summarized as 'You can't win', 'You can't break even' and 'You can't quit playing') revolve around entropy. In particular we know that irreversible processes (like pouring cold milk on hot tea) always increase the entropy of the universe. Around 1971, John Wheeler (see *Box* 2) posed the following question to Jacob Bekenstein, then a graduate student at Princeton. He remarked to Jacob Bekenstein that when a process like mixing hot and cold teas takes place leading to a common temperature, it conserves the world's energy but increases the world's entropy. There is no way to erase or undo it. But let a black hole swim by and let us drop the hot tea and the cold tea into it. 'Then is not all evidence of my crime erased forever?', asked Wheeler. Soon Bekenstein came up with the answer. He told Wheeler that you can not remove entropy from the universe by throwing it into a black hole. Instead, he claimed, the black hole already has an entropy, and you only increase it when you drop tea into it.

In fact, prior to this, Stephen Hawking had proved that in any classical interaction the surface area of a black hole's horizon could only increase and never decrease. Bekenstein used this and claimed that the entropy of a black hole is proportional to its area. Hawking, however, did not agree with this! In fact, he felt Bekenstein had misused his discovery of the increase of the area of the event horizon. After all, Hawking had already noticed and rejected the area-entropy idea on quite solid grounds: If we attribute entropy and energy to a black hole, it will also have a non-zero temperature – but black holes cannot have a temperature, because they cannot radiate. This was indeed the stand taken by the established physicists – especially, Hawking, Bardeen and Carter – in 1972 Les Houches meeting on black holes. Over the summer of 1972, the three of them worked out the four laws of black hole mechanics which identifies mathematically the surface gravity with the temperature. But right up front, the paper makes it clear that the laws are 'similar to, but distinct from' those of thermodynamics and the temperature, entropy should not be thought of as 'real'!

It is a curious twist of fate that – a few years later, while still attempting to disprove Bekenstein's ideas categorically – it was Hawking who ended up discovering that black holes do have a temperature, they do radiate and they have a real entropy as predicted by Bekenstein. And John Wheeler could not have got away pouring the tea down the black hole!

but analyze it in a slightly unconventional way. Consider an inertial reference frame S and an observer who is moving at a speed v along the x-axis in this frame. If her trajectory is $x = vt$, then the clock she is carrying will show the proper time $\tau = t/\gamma$, where $\gamma = (1-v^2/c^2)^{-1/2}$. Combining these results we can write her trajectory in parametrized form as

$$t(\tau) = \gamma\tau; \qquad x(\tau) = \gamma v\tau. \tag{1}$$

Box 2. John A Wheeler

While this issue was being processed we came to know of the sad demise of John Archibald Wheeler (1911–2008), one of the pioneering physicists of recent times. He began his career with atomic and particle physics, and introduced important concepts like S-matrix, and helped build one of the first models of atomic nucleus along with Neils Bohr. Later he turned his attention to general relativity and helped revive the interest of physicists in this topic in 1960s, and as a matter of fact, coined the now household terms like 'black hole' and 'wormhole'. He was for many years at the University of Princeton and taught many famous physicists including Richard Feynman. His book *Gravitation* on general relativity coauthored with his students C Misner and K Thorne is considered one of the definitive texts in the subject, and another book *Spacetime Physics* written along with E Taylor is one of the most original and best introductory books on relativity.

– Editor

These equations give us her position in the space-time when her clock reads τ.

Let us suppose that a monochromatic plane wave exists at all points in the inertial frame. We represent it by the function $\phi(t, x) = \exp -i- (t - x/c)$. This is clearly a plane wave of unit amplitude – as you will see soon, that we don't care about the amplitude – and frequency - propagating along the positive x-axis. At any given x, it oscillates with time as $e^{-i\ \ t}$, so - is the frequency as measured in S. Our moving observer, of course, will measure how the ϕ changes with respect to *her* proper time. This is easily obtained by substituting the trajectory $t(\tau) = \gamma\tau; x(\tau) = \gamma v\tau$ into the function $\phi(t, x)$ obtaining $\phi[\tau] \equiv \phi[t(\tau), x(\tau)]$. A simple calculation gives

$$\phi[t(\tau), x(\tau)] = \phi[\tau] = \exp\left[-i\tau\text{-}\ \gamma(1 - v/c)\right] = \exp -i\left[\tau\text{-}\ \sqrt{\frac{1 - v/c}{1 + v/c}}\right]. \tag{2}$$

Clearly, the observer sees the wave changing over time with a frequency

$$\text{-}\,' \equiv \text{-}\ \sqrt{\frac{1 - v/c}{1 + v/c}}. \tag{3}$$

So an observer moving with uniform velocity will perceive a monochromatic wave as a monochromatic wave but with a Doppler shifted frequency; this is, of course, a standard result in special relativity derived in a slightly different manner.

The real fun begins when we use the same procedure for a uniformly *accelerated* observer along the x-axis. If we know the trajectory $t(\tau), x(\tau)$ of a uniformly accelerated observer, in terms of the proper time τ shown by the clock she carries, then we can determine $\phi[t(\tau), x(\tau)] = \phi[\tau]$ and answer this question. So we first need to determine the trajectory $t(\tau), x(\tau)$ of a uniformly accelerated observer in terms of the proper time τ. Remembering that the equation of motion in special relativity is $\mathrm{d}(m\gamma\mathbf{v})/\mathrm{d}\mathbf{t} = \mathbf{F}$, we can write the equation of motion for an observer moving with constant acceleration g along the x-axis as

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{v}{\sqrt{1-v^2/c^2}} = g. \tag{4}$$

This equation is trivial to integrate since g is a constant. Solving for $v = \mathrm{d}x/\mathrm{d}t$ and integrating once again, we can get the trajectory to be a hyperbola

$$x^2 - c^2t^2 = c^4/g^2 \tag{5}$$

with suitable choices for initial conditions. We also know from special relativity that when a stationary clock registers a time interval $\mathrm{d}t$, the moving clock will show a smaller proper time interval $\mathrm{d}\tau = \mathrm{d}t[1-(v^2(t)/c^2)]^{1/2}$, where $v(t)$ is the instantaneous speed of the clock[1]. Determining $v(t)$ from (5), one can determine the relation between the proper time τ shown in a clock carried by the accelerated observer and t by:

$$\tau = \int_0^t \mathrm{d}t' \sqrt{1-\frac{v^2(t')}{c^2}} = \frac{c}{g}\sinh^{-1}\left(\frac{gt}{c}\right). \tag{6}$$

Inverting this relation one can find t as a function of τ. Using (5) we can then express x in terms of τ and get the trajectory of the uniformly accelerated observer to be

$$x(\tau) = \frac{c^2}{g}\cosh\left(\frac{g\tau}{c}\right); \quad t(\tau) = \frac{c}{g}\sinh\left(\frac{g\tau}{c}\right). \tag{7}$$

This is exactly in the same spirit as the trajectory in (1) for an inertial observer except that we are now talking about a uniformly accelerated observer. You should be able to fill the gaps in the algebra!

[1] This formula is valid for clocks in arbitrary state of motion, including accelerated motion. I stress this because students sometimes think this result is valid only for inertial motion of the clock.

We can now proceed exactly in analogy with (2) to figure out how the *accelerated* observer will view the monochromatic wave. We get:

$$\phi[t(\tau), x(\tau)] = \phi[\tau] = \exp i\frac{c}{g}\left[-\ \ \exp\left(-\frac{g\tau}{c}\right)\right] = \exp i\theta(\tau). \tag{8}$$

Unlike the case of uniform velocity, we now find that the phase $\theta(\tau)$ of the wave itself is decreasing exponentially with time! Since the instantaneous frequency of the wave is the time derivative of the phase, $\omega(\tau) = -\mathrm{d}\theta/\mathrm{d}\tau$, we find that an accelerated observer will see the wave with an instantaneous frequency that is getting exponentially redshifted:

$$\omega(\tau) = -\ \ \exp - \left(\frac{g\tau}{c}\right). \tag{9}$$

Since this is not a monochromatic wave at all, the next best thing is to ask for the power spectrum of this wave which will tell us how it can be built out of monochromatic waves of different frequencies[2]. We will take the power spectrum of this wave to be $P(\nu) = |f(\nu)|^2$, where $f(\nu)$ is the Fourier transform of $\phi(t)$ with respect to t:

[2] This is what an engineer would have done to analyse a time dependent signal!

$$\phi(t) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\nu}{2\pi} f(\nu)\mathrm{e}^{i\nu t}, \tag{10}$$

Evaluating this Fourier transform is a nice exercise in complex analysis and you can do it by changing to the variable $-\ \ \exp[-(gt/c)] = z$ and analytically continuing to Im z. You will then find that:

$$f(\nu) = (c/g)(-\ \ )^{-i\nu g/c}\Gamma(i\nu c/g)\mathrm{e}^{-\pi\nu c/2g}, \tag{11}$$

where Γ is the standard Gamma function. Taking the modulus $|f(\nu)|^2$ using the identity $\Gamma(x)\Gamma(-x) = -\pi/x\sin(\pi x)$, we get

$$\nu|f(\nu)|^2 = \frac{\beta}{\mathrm{e}^{\beta h\nu} - 1}; \quad \beta \equiv \frac{1}{k_{\mathrm{B}}T} = \frac{2\pi c}{\hbar g}. \tag{12}$$

This leads to the the remarkable result that the power, per logarithmic band in frequency, is a Planck spectrum with temperature $k_{\rm B}T = (\hbar g/2\pi c)$. The characteristic wavelength corresponding to this frequency is c^2/g which happens to be about 1 light year for Earth's gravity – so the scope of experimental detection of this result is slim[3]. Also note that though $f(\nu)$ in (11) depends on - , the power spectrum $|f(\nu)|^2$ is independent of - . It does not matter what the frequency of the original wave was!

[3] Incidentally, this gives a relation between earth's gravity and its orbital period around the sun; one of the cosmic coincidences which does not seem to have any deep significance.

The moral of the story is simple: An exponentially redshifted complex wave will have a power spectrum which is thermal with a temperature proportional to the acceleration – which causes the exponential redshift in the first place. This is the key to a quantum field theory result, due to Unruh, that a thermometer which is uniformly accelerated will behave as though it is immersed in a thermal bath.

There are two issues I have glossed over to get the correct result. First, I defined the Fourier transform in (10) with $\mathrm{e}^{i\nu t}$ while the frequency of the original wave was $\mathrm{e}^{-i\;\;t}$. So one is actually talking about the negative frequency component of a wave which has a positive frequency in the inertial frame. Second – and closely related issue – is that I have been working with complex wave modes, not just the real parts of them. Both these can be justified by a more rigorous analysis when these modes actually describe the vacuum fluctuations in the inertial frame rather than some real wave. But the essential idea – and even the essential maths – is captured by this analysis.

So what about the temperature of black holes? Well, black holes produce an exponential redshift on the waves which propagate from close to the gravitational radius to infinity. To make the connection we need to recall two results from a previous article of this series[4]: First,

[4] T Padmanabhan, Schwarzschild Metric at a Discounted Price, *Resonance*, Vol.13, No.4, p.312, 2008.

A thermometer which is uniformly accelerated will behave as though it is immersed in a thermal bath.

the line element of a black hole is

$$\mathrm{d}s^2 = \left(1 - \frac{2GM}{c^2 r}\right) c^2 \mathrm{d}t^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} \mathrm{d}r^2 - r^2 \left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right). \qquad (13)$$

Second, if $\omega(r)$ is the frequency of radiation emitted by a body of radius r and ω_∞ is the frequency with which this radiation is observed at large distances, then $\omega_\infty = \omega(r)(1 - 2GM/c^2 r)^{1/2}$. Consider now a wave packet of radiation emitted from a radial distance r_e at time t_e and observed at a large distance r at time t. The trajectory of the wave packet is, of course, given by $\mathrm{d}s^2 = 0$ in (13) which, when we use $\mathrm{d}\theta = \mathrm{d}\phi = 0$, is easy to integrate. We get

$$c(t - t_e) = r - r_e + \frac{2GM}{c^2} \ln\left(\frac{1 - 2GM/c^2 r}{1 - 2GM/c^2 r_e}\right)$$

$$= r - r_e + \frac{4GM}{c^2} \ln\left(\frac{\omega_e}{\omega(r)}\right) \qquad (14)$$

for $r_e \gtrsim 2GM/c^2$, $r \gg 2GM/c^2$. This gives the frequency of radiation measured by an observer at infinity to be exponentially redshifted:

$$\omega(t) \propto \exp -(c^3 t/4GM) \equiv K \exp -(gt/c), \qquad (15)$$

where K is a constant (which turns out to be unimportant) and we have introduced the quantity

$$g = c^4/4GM = GM/(2GM/c^2)^2 \qquad (16)$$

which gives the gravitational acceleration GM/r^2 at the Schwarzschild radius $r = 2GM/c^2$ and is called the *surface gravity*. Once you have the exponential redshift, the rest of the analysis proceeds as before. An observer detecting the exponentially redshifted radiation at late times ($t \to \infty$), originating from a region close to $r = 2GM/c^2$, will attribute to this radiation a Planckian power spectrum given by (12) which becomes:

$$k_{\rm B}T = \frac{\hbar g}{2\pi c} = \frac{\hbar c^3}{8\pi GM}. \qquad (17)$$

This result lies at the foundation of associating a temperature with a black hole.

Once again, the extra (nontrivial) issues are related to the question of what is the origin of the complex wave mode in the case of a black hole. The answer is the same as in the case of an accelerated observer we discussed earlier with one interesting twist. Think of a spherical body surrounded by vacuum. In quantum theory, this vacuum will have a pattern of fluctuations which can be described in terms of complex wave modes. Suppose the body now collapses to form a black hole. The collapse upsets the delicate balance between the wave modes in the vacuum and manifests – at late times – as thermal radiation propagating to infinity.

Given the expression in (17) for the temperature $T(M)$ of the black hole and the energy (Mc^2), one can formally integrate the relation $\mathrm{d}S = \mathrm{d}E/T$ to obtain the entropy of the black hole:

$$\frac{S}{k_B} = \int_0^M \frac{\mathrm{d}(\bar{M}c^2)}{T(\bar{M})} = \pi\left(\frac{2GM}{c^2}\right)^2\left(\frac{G\hbar}{c^3}\right)^{-1} = \frac{1}{4}\frac{4\pi r_H^2}{L_P^2}, \qquad (18)$$

where $r_{\rm H} = 2GM/c^2$ is the horizon radius of the black hole and $L_{\rm P} = (G\hbar/c^3)^{1/2}$ is the so-called Planck length. The entropy (which should be dimensionless in sensible units with $k_{\rm B} = 1$) is just one quarter of the area of the horizon in units of Planck length. Getting this factor 1/4 is a holy grail in models for quantum gravity – but that is another story.

Suggested Reading

[1] T Padmanabhan, *An Invitation to Astrophysics*, World Scientific, Chapter 5, 2006.

[2] K S Thorne, *Black holes and time warps*, W W Norton, pp.422–435, 1994.

[3] J A Wheeler, *A journey into gravity and spacetime*, Freeman, p.221, 1990.

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