Snippets of Physics

1. Potentials of Potatoes: A Surprise in Newtonian Gravity*

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* This is based on an article originally published by the author in *Physics Education*, Vol 22, No. 4, p.263, 2006.

Keywords
Newtonian gravity, gravitational force, cartesian coordinates, symmetries of mass distribution, asymmetric density distribution, Poisson equation.

It is well known in Newtonian gravity that a spherically symmetric density distribution produces a force outside it which falls as the square of the distance. Surprisingly, the converse is not true! Just because the force falls as inverse square outside a density distribution, you cannot conclude that it must be spherically symmetric. This article, first in a new series, explores this and related issues.

Think of a planet shaped like a diseased potato, distinctly non-spherical. Is it possible that the gravitational force exerted by that planet, everywhere outside it, falls as r^{-2}? Many physicists tend to think that this is impossible; there seems to be a belief that if a mass distribution produces a *strictly* $1/r^2$ force outside it, then such a distribution must be spherically symmetric. Nothing could be farther from the truth! It is possible to come up with completely unusual mass distributions which exert an inverse square law force on the outside world. I will describe several aspects of this issue in this installment.

To begin with, let us agree that there is no 'cheating' involved in this problem. We are not talking about gravitational field far away from the body which falls *approximately* as $1/r^2$. The result should be *exact* and must hold everywhere outside the the body, right from its surface. You also need not worry about silly things like looking at a spherically symmetric distribution in a strange coordinate system, etc. We are thinking of honest-to-god Cartesian coordinates with concepts like spherical symmetry having the usual meaning.

To understand the implications of the question clearly, let us review some basics of Newtonian gravity. The Newtonian gravitational field $\mathbf{F}$ can be expressed as the gradient of a potential ϕ which satisfies Poisson's equation. We have

$$\nabla^2\phi = 4\pi G\rho; \quad \mathbf{F} = -\nabla\phi, \tag{1}$$

where $\rho(\mathbf{x})$ is the matter density which is assumed to be either positive or zero everywhere. In these mathematical terms our problem translates to the following: Can you find a density distribution $\rho(\mathbf{x})$ which is *not* spherically symmetric (in some chosen coordinate system) and vanishes outside some compact region $\mathcal{R}$ around the origin, such that outside $\mathcal{R}$ the potential ϕ falls as $1/r$? Of course, any spherically symmetric $\rho(\mathbf{x})$ will produce such a potential but the key question is: Must it be spherically symmetric?

Can you find a density distribution ρ $(\mathbf{x})$ which is *not* spherically symmetric (in some chosen coordinate system) and vanishes outside some compact region $\mathcal{R}$ around the origin, such that outside $\mathcal{R}$ the potential ϕ falls as $1/r$?

Some thinking will convince you that there is no simple way of going about analysing this problem. Usually we are given some $\rho(\mathbf{x})$ and asked to find the $\phi(\mathbf{x})$. We are now interested in the inverse question, which – in a broader context – is the following: If we know the gravitational force in some *finite* region of space, how unique is the density distribution producing that force?

Let me give you some surprising instances wherein completely different density distributions can produce the *same* gravitational field in some finite region. This will be a good warm up for the original question we want to attack.

One example, beaten to death in standard textbooks, is the field produced by an infinite, plane sheet of matter of surface mass density σ. You might not have learnt it in the context of gravity but I am sure you would have encountered it in some electrostatics course. Translating it to gravity, you see that such infinite planes with constant surface density produce a gravitational force

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$\mathbf{F} = -2\pi G\sigma\hat{\mathbf{n}}$ which is constant everywhere and directed towards the sheet. (Here $\hat{\mathbf{n}}$ is the unit vector in the direction perpendicular to the sheet.)

We now ask the question: Is it possible to come up with a density distribution which is not plane-symmetric but will produce constant gravitational field in some compact region of space $\mathcal{S}$? The answer is "yes"; and some of you might have even worked it out without quite realizing its importance.

The configuration is shown in *Figure* 1. We take a sphere, of radius R and constant density ρ_0, centered at the origin. Inside it we carve out another spherical region of radius L centered at the point $\mathbf{l}$. Consider the force on a particle located inside the cavity at $(\mathbf{l}+\mathbf{r})$. The force due to a constant density sphere is $\mathbf{F} = -(4/3)\pi G\rho\mathbf{x}$ (so that $-\nabla.\mathbf{F} = 4\pi G\rho$). Hence, the force we want is

$$\mathbf{F} = \mathbf{F}_{sph} - \mathbf{F}_{hole} = -\frac{4}{3}\pi G\rho[\mathbf{l}+\mathbf{r}] + \frac{4}{3}\pi G\rho\mathbf{r} = -\frac{4\pi}{3}G\rho\mathbf{l}, \tag{2}$$

which is clearly a constant inside the hole. Thus a spherical hole inside a sphere is a region with constant gravitational force! Suppose you measure the gravitational

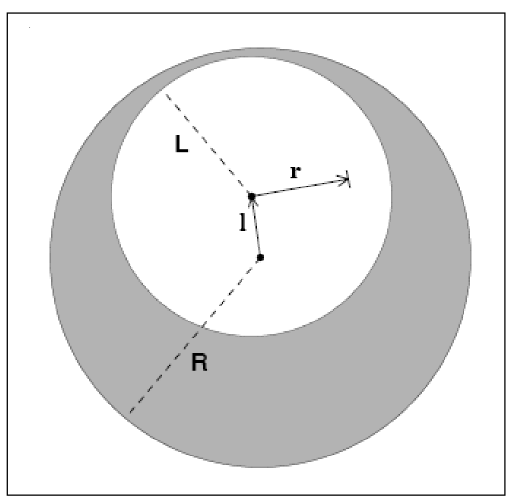

Figure 1.

field in some finite region $\mathcal{S}$ and find it to be strictly constant. Can you say anything about the mass distribution which is producing this force? Of course not. It could have been produced by an infinite plane sheet or you could be living inside a hole-in-a-sphere; these are just two of infinitely many possibilities. Most of these mass distributions, which produce a constant gravitational field, will not have any specific symmetry. (If you are not convinced, think of the superposition of the any two such configurations.)

Is it possible to come up with a completely asymmetric distribution of matter which exerts zero gravitational force in some region?

The hole-in-the-sphere example can be twisted around to lead to another interesting conclusion. You must have again learnt, while studying Newtonian gravity, that a spherical shell of matter exerts no gravitational force on a particle inside it. (This is just a special case of equation (2); when $\mathbf{l} = 0$, the force vanishes.) Is it possible to come up with a completely asymmetric distribution of matter which exerts zero gravitational force in some region? The answer is again "yes" and all you need to do is the following: Suppose you make two hole-in-the-sphere distributions with different values for the parameters – one with density ρ_1, radius R_1, hole radius L_1 and the centre of the hole located at $\mathbf{l}_1$ with respect to the centre of the sphere; the second one with density ρ_2, radius R_2, etc. We superpose the spheres such that: (i) $\mathbf{l}_1$ and $\mathbf{l}_2$ are pointing in the opposite directions; (ii)$\rho_1 l_1 = \rho_2 l_2$; and (iii) part of the two spherical cavities overlap. The resulting density distribution is clearly not spherically symmetric. But in the region of the cavity which is common to the holes of both spheres, the gravitational force is strictly zero. This is because each sphere produces an equal and opposite force in the cavity when $\rho_1 l_1 = \rho_2 l_2$.

The moral of the story is worth remembering. Just knowing the symmetries of the gravitational force in some finite region does not allow you to conclude about the *symmetries* of the mass distribution. This itself is a

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source of surprise for many since we are so accustomed to assuming the same symmetries for the field and its source.

All these must have convinced you that it is quite possible to have completely asymmetric density distributions producing highly regular gravitational fields. We are now ready to tackle the question we originally started with: Are there density distributions which are not spherically symmetric but produce an inverse square force?

Let us first consider this problem in the case of electrostatics. Is it possible to have a charge distribution which is not spherically symmetric but produces an inverse square electric field? Incredibly enough, you already know such a distribution from your regular electrostatics course! Remember the problem of a point charge and a conducting sphere which is solved by the method of images? We start with a conducting sphere of radius a and a point charge $+Q$ located outside the sphere at a distance L from the centre of the sphere. The charge $+Q$ induces a surface charge distribution on the conducting sphere and the net electric field at any point is the sum of the electric fields due to the surface charge distribution σ and the point charge $+Q$. This problem is solved by showing that it is equivalent to that of two point charges: The real charge $+Q$ and an 'image' charge $q = -(a/L)Q$ placed at a distance $l = (a^2/L)$ inside the sphere in the line joining the centre of the sphere to the charge $+Q$. The fields outside this sphere produced by the point charges Q and q are identical to those due to the point charge $+Q$ and the charge distribution σ. It follows that the charge distribution σ produces a field which is equivalent to that of a point charge q! Of course this distribution σ is far from spherically symmetric since the induced charge on the side nearer to $+Q$ will be differently distributed compared to the induced charge on the farther side. We have thus come up with a charge distribution which is not spherically symmetric but produces an inverse square law force outside a finite region.

The key difference between electrostatics and gravity is that in electrostatics, the charge density need not be positive definite while in gravity the mass density has to be positive definite. So you may think that the above example is fine as far as electrostatics goes but may not help in matters of gravity. Actually that is not true. Given any electrostatic configuration which produces an inverse square law force, it is possible to construct an everywhere-positive charge density which also produces such a force. Take the electrostatic configuration in the previous example and find out where the charge density σ is most negative. Suppose the lowest value of the charge density is $-|\sigma_0|$. We now take a spherically symmetric charge distribution made of positive charge density ρ_0 with $\rho_0 > |\sigma_0|$ and put it on top of the original charge distribution and centre it on the location of the image charge. We now have a charge distribution which is: (i) everywhere positive definite, (ii) produces an inverse square law force in the outside region and (iii) is not spherically symmetric. The property (ii) follows from the fact that both pieces of charge distribution individually produced an inverse square law force and the property (iii) is obvious from the fact that you are superposing a spherically symmetric distribution with a non-spherical one. So there you are; we have a non-spherical, everywhere positive, density distribution producing a strictly inverse square force outside a finite region.

Given any electrostatic configuration which produces an inverse square law force, it is possible to construct an everywhere-positive charge density which also produces such a force.

If you are still shaking your head in disbelief, let me assure you that there is no black magic involved. It is quite possible to have such distributions, and – in fact – there are infinitely many such configurations. Those of you who are mathematically inclined might like the following construction of some such distributions using a property of Poisson equation known as 'inversion'. In-

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version is a mathematical operation under which you associate to any point $\mathbf{x}$, another point $\mathbf{x}_{inv} \equiv (a^2/x^2)\mathbf{x}$ where a is the radius of the 'inverting sphere'. From this definition it immediately follows that points inside a sphere of radius a are mapped to points outside and vice-versa.

There is an interesting connection between inversion and the solutions to Poisson equation. Suppose $\phi[\mathbf{x};\rho(\mathbf{x})]$ is the gravitational potential at a point $\mathbf{x}$ due to a density distribution $\rho(\mathbf{x})$. Consider now a new density distribution $\rho'(\mathbf{x}) = (a/x)^5\rho(\mathbf{x}_{inv})$ obtained by taking the original density at the inverted point $\mathbf{x}_{inv} \equiv (a^2/x^2)\mathbf{x}$ and multiplying by $(a/x)^5$. It can be shown that the gravitational potential due to $\rho'(\mathbf{x})$ is given by $\phi'(\mathbf{x}) = (a/x)\phi(\mathbf{x}_{inv})$. That is, the new gravitational potential at any given point is the old gravitational potential at the inverted point $\mathbf{x}_{inv}$ multiplied by (a/x).

We can use this result to produce strange looking mass distributions with strictly inverse square law force. We start with the result, obtained earlier, that one can have very asymmetric density distributions which can produce zero gravitational force inside an empty compact region of space $\mathcal{A}$. In *Figure* 2, we assume that there are sources outside of $\mathcal{A}$ (which are not shown) that produce a constant gravitational potential inside the region of space $\mathcal{A}$. The exact shape of this region is immaterial for our discussion.

Let $\mathcal{C}$ be an imaginary spherical surface of radius a with centre somewhere inside $\mathcal{A}$. Let us now invert the surface of the region $\mathcal{A}$ using the inverting sphere $\mathcal{C}$ and obtain the surface $\mathcal{A}'$. In this process, the region inside $\mathcal{A}$ gets mapped to region outside $\mathcal{A}'$. Since the region inside $\mathcal{A}$ was originally empty, the region outside $\mathcal{A}'$ will be empty in the inverted configuration; all the sources which were originally outside $\mathcal{A}$ are now mapped to the region inside $\mathcal{A}'$. Consider now the gravitational poten-

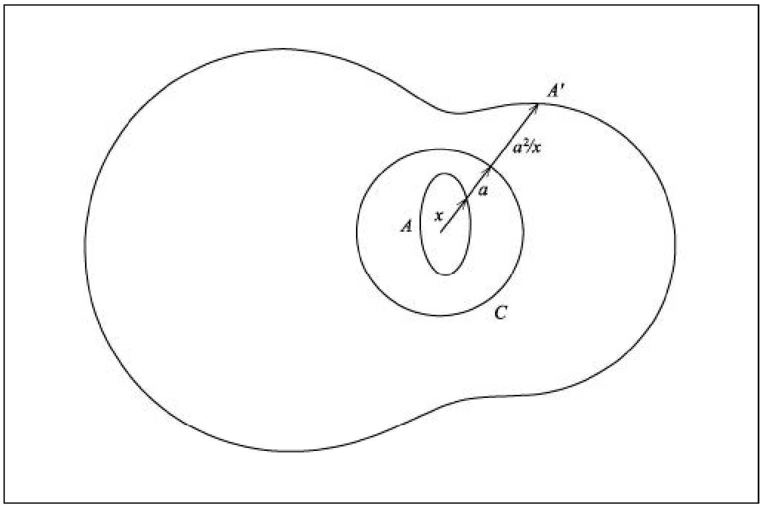

Figure 2. Schematic picture showing the effect of inversion under which a given point **x** *is mapped to another point* ($\mathbf{x}_{inv}a^2/x^2$)**x**. *When the points in surface of a compact region $\mathcal{A}$ is inverted using the inverting sphere $\mathcal{C}$, we obtain the surface $\mathcal{A}'$. In this process, the region inside A gets mapped to region outside $\mathcal{A}'$ and vice versa.*

tial outside $\mathcal{A}'$ due to this source ρ' which is now inside $\mathcal{A}'$. This potential is obtained by taking the potential due to the inverted point inside $\mathcal{A}$ and multiplying it by (a/x). But since the potential everywhere inside $\mathcal{A}$ is a constant it follows that the potential outside $\mathcal{A}'$ falls as $|\mathbf{x}|^{-1}$. We now have a region $\mathcal{A}'$ outside which the gravitational force is strictly inverse square and the density distribution producing this force is far from spherical!

To the extent I can figure out, this problem was first raised and answered by Lord Kelvin. It seems that Newton never worried about this question. (So I am told by experts on Newton.) This is somewhat surprising since Newton worried a lot about the original problem, viz., whether a spherically symmetric mass distribution will produce a force as though all its mass is concentrated at the origin.

Suggested Reading

[1] Some of these issues are discussed in classical, geometrical, style in older books on potential theory, like W Thompson (Lord Kelvin) and P G Tait, *Principles of Mechanics and Dynamics*, Dover, 1962.

[2] A simple proof for the inversion theorem can be found in T Padmanabhan, *Cosmology and Astrophysics - Through Problems*, Cambridge University Press, p.151, 1996.

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